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Boundary value problems for fractional differential inclusions with Hadamard type derivatives in Banach spaces

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Abstract. The authors establish sufficient conditions for the existence of solutions to boundary value problems for fractional differential inclusions involving the Hadamard type fractional derivative of order $\alpha \in (1, 2]$ in Banach spaces. Their approach uses Mönch's fixed point theorem and the Kuratowski measure of noncompacteness.

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1. Introduction

In this paper we are concerned with the existence of solutions to boundary value problems (BVP for short) for fractional order differential inclusions. In particular, we consider the boundary value problem

$$^{H}D^{r}y(t) \in F(t, y(t)), \text{ for a.e. } t \in J = [1, T], \ 1 < r \leq 2, \tag{1.1}$$

$$y(1) = 0, \ y(T) = y_T, \tag{1.2}$$

where $^{H}D^{r}$ is the Hadamard fractional derivative, $(E, |\cdot|)$ is a Banach space, $\mathcal{P}(E)$ is the family of all nonempty subsets of E, $F : [1, T] \times E \to \mathcal{P}(E)$ is a multivalued map, and $y_T \in \mathbb{R}$.

Differential equations of fractional order are valuable in modeling phenomena in various fields of science and engineering. They can be found in viscoelasticity, electrochemistry, control, porous media, electromagnetism, etc. The monographs of Hilfer [18], Kilbas *et al.* [19], Podlubny [23], and Momani *et al.* [21] are very good sources on the background mathematics and various applications of fractional derivatives. The literature on Hadamard-type fractional differential equations has not undergone as much development as it has for the Caputo and Riemann-Liouville fractional derivatives; see, for example, the papers of Ahmed and Ntouyas [2], Benhamida, Graef, and Hamani [10], and Thiramanus, Ntouyas, and Tariboon [24].

The fractional derivative that Hadamard [16] introduced in 1892 differs from other fractional derivatives in the sense that the kernel of the integral in the definition of the Hadamard derivative contains a logarithmic function with an arbitrary exponent. A detailed description of the Hadamard fractional derivative and integral can be found in [11, 12, 13].

In this paper, we present existence results for the problem (1.1)-(1.2) in the case where the right hand side is convex valued. This result relies on the set-valued analog of Mönch's fixed point theorem combined with the technique of measure of noncompactness. Recently, this has proved to be a valuable tool in studying fractional differential equations and inclusions in Banach spaces; for additional details, see the papers of Laosta *et al.* [20], Agarwal *et al.* [1], and Benchohra *et al.* [7, 8, 9]. Our results here extend to the multivalued case some previous results in the literature and constitutes what we hope is an interesting contribution to this emerging field. We include an example to illustrate our main results.

2. Preliminaries

This section contains definitions, concepts, lemmas, and preliminary facts that will be used in the remainder of this paper. Let $C(J, E)$ be the Banach space of all continuous functions from J into E with the norm

$$\|y\|_\infty = \sup\{|y(t)| : t \in J\},$$

and let $L^1(J, E)$ be the Banach space of Lebesgue integrable functions $y : J \to E$ with the norm

$$\|y\|_{L^1} = \int_1^T |y(t)|dt.$$

The space $AC^1(J, E)$ is the space of functions $y : J \to E$ that are absolutely continuous and have an absolutely continuous first derivative.

For any Banach space X, we set

$P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$,
$P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$,
$P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, and
$P_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$.

A multivalued map $G : X \to \mathcal{P}(X)$ is *convex (closed)* valued if $G(X)$ is convex (closed) for all $x \in X$. We say that G is *bounded on bounded sets* if $G(B) = \cup_{x\in B} G(x)$ is bounded in X for all $B \in P_b(X)$ (i.e., $\sup_{x\in B}\{\sup\{|y| : y \in G(x)\}\}$ is bounded).

The mapping G is *upper semi-continuous (u.s.c)* on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and for each open set N of X containing $G(x_0)$, there exists an open neighborhood N_0 of x_0 such that $G(N_0) \subset N$. A map G is said to be *completely continuous* if $G(B)$ is relatively compact for every $B \in P_b(X)$.

If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c if and only if G has a closed graph (i.e., $x_n \to x_*$, $y_n \to y_*$,

$y_n \in G(x_n)$ imply $y_* \in G(x_*)$). The mapping $G : X \to \mathcal{P}(X)$ has a fixed point if there exists $x \in X$ such that $x \in G(x)$. The set of fixed points of the multivalued operator G will be denoted by $Fix\, G$. A multivalued map $G : J \to P_{cl}(X)$ is said to be measurable if for every $y \in X$, the function

$$t \to d(y, G(t)) = inf\{|y - z| : z \in G(t)\}$$

is measurable.

Definition 2.1. A multivalued map $F : J \times E \to \mathcal{P}(E)$ is said to be Carathéodory if:
(1) $t \to F(t, u)$ is measurable for each $u \in E$;
(2) $u \to F(t, u)$ is upper semicontinuous for a.e. $t \in J$.

For each $y \in AC^1(J, E)$, define the set of selections of F by

$$S_{F,y} = \{v \in L^1(J, E) : v(t) \in F(t, y(t)) \text{ a.e. } t \in J\}.$$

Let (X, d) be a metric space induced from the normed space $(X, |\cdot|)$. The function $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{\infty\}$ given by

$$H_d(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\}$$

is known as the Hausdorff-Pompeiu metric.

For more details on multivalued maps, see the books of Aubin and Cellina [4], Aubin and Frankowska [5], Castaing and Valadier [14], and Deimling [15].

For convenience, we first recall the definitions of the Kuratowski measure of noncompacteness and summarize the main properties of this measure.

Definition 2.2. ([3, 6]) Let E be a Banach space and let Ω_E be the bounded subsets of E. The Kuratowski measure of noncompactness is the map $\beta : \Omega_E \to [0, \infty)$ defined by

$$\beta(B) = \inf\{\epsilon > 0 : B \subset \bigcup_{j=1}^{m} B_j \text{ and } diam(B_j) \le \epsilon\}.$$

Properties: The Kuratowski measure of noncompactness satisfies the following properties (for more details see [3, 6]):

(P_1) $\beta(B) = 0$ if and only if $\overline{B}$ is compact (B is relatively compact).

(P_2) $\beta(B) = \beta(\overline{B})$.

(P_3) $A \subset B$ implies $\beta(A) \le \beta(B)$.

(P_4) $\beta(A + B) \le \beta(A) + \beta(B)$.

(P_5) $\beta(cB) = |c|\beta(B), \;\; c \in \mathbb{R}$.

(P_6) $\beta(convB) = \beta(B)$.

Here $\overline{B}$ and $conv\, B$ denote the closure and the convex hull of the bounded set B, respectively.

For a given set V of functions $u : J \to E$, we set

$$V(t) = \{u(t) : u \in V\}, \, t \in J,$$

and

$$V(J) = \{u(t) : u \in V(t),\, t \in J\}.$$

Theorem 2.3. ([17], [22, Theorem 1.3]) *Let E be a Banach space and let C be a countable subset of $L^1(J,E)$ such that there exists $h \in L^1(J,\mathbb{R}_+)$ with $|u(t)| \leq h(t)$ for a.e. $t \in J$ and every $u \in C$. Then the function $\varphi(t) = \beta(C(t))$ belongs to $L^1(J,\mathbb{R}_+)$ and satisfies*

$$\beta\left(\left\{\int_0^T u(s)ds : u \in C\right\}\right) \leq 2\int_0^T \beta(C(s))ds.$$

Lemma 2.4. ([20, Lemma 2.6]) *Let J be a compact real interval, F be a Carathéodory multivalued map, and let θ be a linear continuous map from $L^1(J,E) \to C(J,E)$. Then the operator*

$$\theta \circ S_{F,y} : L^1(J,E) \to P_{cp,c}(C(J,E)), \quad y \to (\theta \circ S_{F,y})(y) = \theta(S_{F,y})$$

is a closed graph operator in $L^1(J,E) \times C(J,E)$.

In what follows, $\log(\cdot) = \log_e(\cdot)$, and $n = [r]+1$ where $[r]$ denotes the integer part of r.

Definition 2.5. ([19]) The Hadamard fractional integral of order r for a function $h : [1,+\infty) \to \mathbb{R}$ is defined by

$$I^r h(t) = \frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1}\frac{h(s)}{s}ds, \ r > 0,$$

provided the integral exists.

Definition 2.6. ([19]) For a function h on the interval $[1,+\infty)$, the Hadamard fractional derivative of h of order r is defined by

$$({}^H D^r h)(t) = \frac{1}{\Gamma(n-r)}\left(t\frac{d}{dt}\right)^n \int_1^t \left(\log\frac{t}{s}\right)^{n-r-1}\frac{h(s)}{s}ds, \ n-1 < r < n, \ n = [r]+1.$$

Let us now recall Mönch's fixed point theorem.

Theorem 2.7. ([22, Theorem 3.2]) *Let K be a closed and convex subset of a Banach space E, U be a relatively open subset of K, and $N : \overline{U} \to \mathcal{P}(K)$. Assume that graph N is closed, N maps compact sets into relatively compact sets, and for some $x_0 \in U$, the following two conditions are satisfied:*

(i) *$M \subset \overline{U}$, $M \subset conv(x_0 \cup N(M))$, $\overline{M} = \overline{C}$, with C a countable subset of M, implies $\overline{M}$ is compact;*
(ii) *$x \notin (1-\lambda)x_0 + \lambda N(x)$ for all $x \in \overline{U} \setminus U$, $\lambda \in (0,1)$.*

Then there exists $x \in \overline{U}$ with $x \in N(x)$.

3. Main results

Let us start by defining what we mean by a solution of the problem (1.1)-(1.2).

Definition 3.1. A function $y \in AC^1(J,E)$ is said to be a solution of (1.1)-(1.2) if there exist a function $v \in L^1(J,E)$ with $v(t) \in F(t,y(t))$ for a.e. $t \in J$, such that ${}^H D^\alpha y(t) = v(t)$ on J, and the conditions $y(1) = 0$ and $y(T) = y_T$ are satisfied.

Lemma 3.2. *Let $h : J \to E$ be a continuous function. A function y is a solution of the fractional integral equation*

$$y(t) = \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} h(s)\frac{ds}{s} + \frac{(\log t)^{r-1}}{(\log T)^{r-1}} \left[y_T - \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} h(s)\frac{ds}{s}\right] \tag{3.1}$$

if and only if y is a solution of the fractional BVP

$$ {}^H D^r y(t) = h(t), \text{ for a.e. } t \in J = [1,T], \quad 1 < r \le 2, \tag{3.2}$$

$$ y(1) = 0, \;\; y(T) = y_T. \tag{3.3}$$

Proof. Applying the Hadamard fractional integral of order r to both sides of (3.2), we obtain

$$y(t) = c_1(\log t)^{r-1} + c_2(\log t)^{r-2} + {}^H I^r h(t). \tag{3.4}$$

From (3.3), we have $c_2 = 0$ and

$$c_1 = \frac{1}{(\log T)^{r-1}}[y_T - {}^H I^r h(T)].$$

Hence, we obtain (3.1). Conversely, it is clear that if y satisfies equation (3.1), then (3.2)-(3.3) hold. □

Theorem 3.3. *Let $R > 0$, $B = \{x \in E : \|x\| \le R\}$, $U = \{x \in C(J,E) : \|x\| \le R\}$, and assume that:*

(H1) *$F : J \times E \to \mathcal{P}_{cp,p}(E)$ is a Carathéodory multi-valued map;*

(H2) *For each $R > 0$, there exists a function $p \in L^1(J,E)$ such that*

$$\|F(t,u)\|_{\mathcal{P}} = \sup\{|v| : v(t) \in F(t,y)\} \le p(t)$$

for each $(t,y) \in J \times E$ with $|y| \ge R$, and

$$\liminf_{R\to\infty} \frac{\int_0^T p(t)dt}{R} = \delta < \infty;$$

(H3) *There exists a Carathéodory function $\psi : J \times [1,2R] \to \mathbb{R}_+$ such that*

$$\beta(F(t,M)) \le \psi(t,\beta(M)) \text{ a.e. } t \in J \text{ and each } M \subset B;$$

(H4) *The function $\varphi = 0$ is the unique solution in $C(J,[1,2R])$ of the inequality*

$$\varphi(t) \le 2\left\{ \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \psi(s,\varphi(s))\frac{ds}{s} + \frac{(\log t)^{r-1}}{(\log T)^{r-1}} \left[y_T + \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} \psi(s,\varphi(s))\frac{ds}{s}\right]\right\} \text{ for } t \in J. \tag{3.5}$$

Then the BVP (1.1)*-*(1.2) *has at least one solution in* $C(J,B)$*, provided that*

$$\delta < \frac{\Gamma(r+1)}{(\log T)^r}. \tag{3.6}$$

Proof. We wish to transform the problem (1.1)-(1.2) into a fixed point problem, so consider the multivalued operator

$$\begin{aligned} N(y) = \Bigg\{ h \in C(J,\mathbb{R}) : h(t) = {} & \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} v(s) \frac{ds}{s} \\ & + \frac{(\log t)^{r-1}}{(\log T)^{r-1}} \left[y_T - \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} v(s) \frac{ds}{s} \right], \quad v \in S_{F,y} \Bigg\}. \end{aligned}$$

Clearly, from Lemma 3.2, the fixed points of N are solutions to (1.1)-(1.2). We shall show that N satisfies the assumptions of Mönch's fixed point theorem. The proof will be given in several steps. First note that $\overline{U} = C(J,B)$.

Step 1: $N(y)$ *is convex for each* $y \in C(J,B)$*.*

Take h_1, $h_2 \in N(y)$; then there exist v_1, $v_2 \in S_{F,y}$ such that for each $t \in J$, we have

$$\begin{aligned} h_i(t) = {} & \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} v_i(s) \frac{ds}{s} \\ & + \frac{(\log t)^{r-1}}{(\log T)^{r-1}} \left[y_T - \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} v_i(s) \frac{ds}{s} \right] \end{aligned}$$

for $i = 1$, 2. Let $0 \le d \le 1$; then for each $t \in J$,

$$\begin{aligned} (dh_1 + (1-d)h_2)(t) = {} & \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} [dv_1 + (1-d)v_2] \frac{ds}{s} \\ & + \frac{(\log t)^{r-1}}{(\log T)^{r-1}} \left[y_T - \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} [dv_1 + (1-d)v_2] \frac{ds}{s} \right]. \end{aligned}$$

Since $S_{F,y}$ is convex (because F has convex values), we have

$$dh_1 + (1-d)h_2 \in N(y).$$

Step 2: $N(M)$ *is relatively compact for each compact* $M \subset \overline{U}$*.*

Let $M \subset \overline{U}$ be a compact set and let $\{h_n\}$ be any sequence of elements of $N(M)$. We will show that $\{h_n\}$ has a convergent subsequence by using the Arzelà-Ascoli criterion of compactness in $C(J,B)$. Since $h_n \in N(M)$, there exist $y_n \in M$ and $v_n \in S_{F,y}$ such that

$$\begin{aligned} h_n(t) = {} & \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} v_n(s) \frac{ds}{s} \\ & + \frac{(\log t)^{r-1}}{(\log T)^{r-1}} \left[y_T - \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} v_n(s) \frac{ds}{s} \right] \end{aligned}$$

for $n \geq 1$. Using Theorem 2.3 and the properties of the Kuratowski measure of noncompactness, we have

$$\beta(\{h_n(t)\}) \leq 2\left\{\frac{1}{\Gamma(r)}\int_1^t \beta\left(\left\{\left(\log\frac{t}{s}\right)^{r-1}\frac{v_n(s)}{s} : n \geq 1\right\}\right)ds\right.$$
$$\left. + \frac{(\log t)^{r-1}}{(\log T)^{r-1}}\left[y_T + \frac{1}{\Gamma(r)}\int_1^T \beta\left(\left\{\left(\log\frac{T}{s}\right)^{r-1}\frac{v_n(s)}{s} : n \geq 1\right\}\right)ds\right]\right\}. \tag{3.7}$$

On the other hand, since $M(s)$ is compact in E, the set $\{v_n(s) : n \geq 1\}$ is compact. Consequently, $\beta(\{v_n(s) : n \geq 1\}) = 0$ for a.e. $s \in J$. Furthermore,

$$\beta\left(\left\{\left(\log\frac{t}{s}\right)^{r-1}\frac{v_n(s)}{s}\right\}\right) = \left(\log\frac{t}{s}\right)^{r-1}\frac{1}{s}\beta(\{v_n(s) : n \geq 1\}) = 0$$

and

$$\beta\left(\left\{\left(\log\frac{T}{s}\right)^{r-1}\frac{v_n(s)}{s}\right\}\right) = \left(\log\frac{T}{s}\right)^{r-1}\frac{1}{s}\beta(\{v_n(s) : n \geq 1\}) = 0$$

for a.e. t, $s \in J$. Hence, from this and (3.7), $\{h_n(t) : n \geq 1\}$ is relatively compact in B for each $t \in J$. In addition, for each t_1, $t_2 \in J$ with $t_1 < t_2$, we have

$$\begin{aligned}|h_n(t_2) - h_n(t_1)| &= \left|\frac{1}{\Gamma(\alpha)}\int_1^{t_1}\left[\left(\log\frac{t_2}{s}\right)^{\alpha-1} - \left(\log\frac{t_1}{s}\right)^{\alpha-1}\right]\frac{v_n(s)}{s}ds\right.\\ &\quad \left. + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\alpha-1}\frac{v_n(s)}{s}ds\right|\\ &\leq \frac{p(t)}{\Gamma(\alpha)}\int_1^{t_1}\left[\left(\log\frac{t_2}{s}\right)^{\alpha-1} - \left(\log\frac{t_1}{s}\right)^{\alpha-1}\right]\frac{ds}{s}\\ &\quad + \frac{p(t)}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\alpha-1}\frac{ds}{s}.\end{aligned}$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero. This shows that $\{h_n : n \geq 1\}$ is equicontinuous. Consequently, $\{h_n : n \geq 1\}$ is relatively compact in $C(J, B)$.

Step 3: *N has a closed graph.*

Let $y_n \to y_*$, $h_n \in N(y_n)$, and $h_n \to h_*$. We need to show that $h_* \in N(y_*)$. Now $h_n \in N(y_n)$ means that there exists $v_n \in S_{F,y}$ such that, for each $t \in J$,

$$\begin{aligned}h_n(t) &= \frac{1}{\Gamma(r)}\int_1^t\left(\log\frac{t}{s}\right)^{r-1}v_n(s)\frac{ds}{s}\\ &\quad + \frac{(\log t)^{r-1}}{(\log T)^{r-1}}\left[y_T - \frac{1}{\Gamma(r)}\int_1^T\left(\log\frac{T}{s}\right)^{r-1}v_n(s)\frac{ds}{s}\right].\end{aligned}$$

Consider the continuous linear operator $\theta : L^1(J,E) \to C(J,E)$ defined by

$$\theta(v)(t) \to h_n(t) = \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} v_n(s)\frac{ds}{s} + \frac{(\log t)^{r-1}}{(\log T)^{r-1}} \left[y_T - \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} v_n(s)\frac{ds}{s} \right].$$

Clearly, $\|h_n(t) - h_*(t)\| \to 0$ as $n \to \infty$. From Lemma 2.4 it follows that $\theta \circ S_F$ is a closed graph operator. Moreover, $h_n(t) \in \theta(S_{F,y_n})$. Since $y_n \to y$, Lemma 2.4 implies

$$h(t) = \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} v(s)\frac{ds}{s} + \frac{(\log t)^{r-1}}{(\log T)^{r-1}} \left[y_T - \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} v(s)\frac{ds}{s} \right].$$

Step 4: *M is relatively compact in $C(J,B)$.*

Suppose $M \subset \overline{U}$, $M \subset conv(\{0\} \cup N(M))$, and $\overline{M} = \overline{C}$ for some countable set $C \subset M$. Using an argument similar to the one used in Step 2 shows that $N(M)$ is equicontinuous. Then, since $M \subset conv(\{0\} \cup N(M))$, we see that M is equicontinuous as well. To apply the Arzelà-Ascoli theorem, it remains to show that $M(t)$ is relatively compact in E for each $t \in J$. Since $C \subset M \subset conv(\{0\} \cup N(M))$ and C is countable, we can find a countable set $H = \{h_n : n \geq 1\} \subset N(M)$ with $C \subset conv(\{0\} \cup H)$. Then, there exist $y_n \in M$ and $v_n \in S_{F,y_n}$ such that

$$h_n(t) = \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} v_n(s)\frac{ds}{s} + \frac{(\log t)^{r-1}}{(\log T)^{r-1}} \left[y_T - \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} v_n(s)\frac{ds}{s} \right].$$

Since $M \subset C \subset conv(\{0\} \cup H))$, from the properties of the Kuratowski measure of noncompactness, we have

$$\beta(M(t)) \leq \beta(C(t)) \leq \beta(H(t)) = \beta(\{h_n(t) : n \geq 1\}).$$

Using (3.7) and the fact that $v_n(s) \in M(s)$, we obtain

$$\begin{aligned}
\beta(M(t)) \leq 2 \Bigg\{ & \frac{1}{\Gamma(r)} \int_1^t \beta\left(\left\{\left(\log\frac{t}{s}\right)^{r-1}\frac{v_n(s)}{s} : n \geq 1\right\}\right) ds \\
& + \frac{(\log t)^{r-1}}{(\log T)^{r-1}}\left[y_T + \frac{1}{\Gamma(r)}\int_1^T \beta\left(\left\{\left(\log\frac{T}{s}\right)^{r-1}\frac{v_n(s)}{s} : n \geq 1\right\}\right) ds\right]\Bigg\} \\
\leq 2 \Bigg\{ & \frac{1}{\Gamma(r)} \int_1^t \left(\log\frac{t}{s}\right)^{r-1} \beta(M(s))\frac{ds}{s} \\
& + \frac{(\log t)^{r-1}}{(\log T)^{r-1}}\left[y_T + \frac{1}{\Gamma(r)}\int_1^T \left(\log\frac{T}{s}\right)^{r-1} \beta(M(s))\frac{ds}{s}\right]\Bigg\} \\
\leq 2 \Bigg\{ & \frac{1}{\Gamma(r)} \int_1^t \left(\log\frac{t}{s}\right)^{r-1} \psi(s, \beta(M(s)))\frac{ds}{s} \\
& + \frac{(\log t)^{r-1}}{(\log T)^{r-1}}\left[y_T + \frac{1}{\Gamma(r)}\int_1^T \left(\log\frac{T}{s}\right)^{r-1} \psi(s, \beta(M(s)))\frac{ds}{s}\right]\Bigg\}.
\end{aligned}$$

We also have that the function φ given by $\varphi(t) = \beta(M(t))$ belongs to $C(J, [1, 2R])$. Consequently, by $(H4)$, $\varphi = 0$; that is, $\beta(M(t)) = 0$ for all $t \in J$. Now, by the Arzelà-Ascoli theorem, M is relatively compact in $C(J, B)$.

Step 5: Let $h \in N(y)$ with $y \in U$. We claim that $N(U) \subset U$. If this were not the case, then in view of $(H2)$, there exists functions $v \in S_{F,y}$ and $p \in L^1(J, E)$ such that

$$\begin{aligned}
h(t) = & \frac{1}{\Gamma(r)} \int_1^t \left(\log\frac{t}{s}\right)^{r-1} v(s)\frac{ds}{s} \\
& + \frac{(\log t)^{r-1}}{(\log T)^{r-1}}\left[y_T - \frac{1}{\Gamma(r)}\int_1^T \left(\log\frac{T}{s}\right)^{r-1} v(s)\frac{ds}{s}\right],
\end{aligned}$$

and

$$\begin{aligned}
R < \|N(y)\|_{\mathcal{P}} \leq & \frac{1}{\Gamma(r)} \int_1^t \left(\log\frac{t}{s}\right)^{r-1} |v(s)|\frac{ds}{s} \\
& + \frac{(\log t)^{r-1}}{(\log T)^{r-1}}\left[|y_T| + \frac{1}{\Gamma(r)}\int_1^T \left(\log\frac{T}{s}\right)^{r-1} |v(s)|\frac{ds}{s}\right] \\
\leq & \frac{(\log T)^r}{\Gamma(r+1)}\int_1^t p(s)ds + \frac{(\log T)^r}{\Gamma(r+1)}\int_1^T p(s)ds \\
\leq & 2\frac{(\log T)^r}{\Gamma(r+1)}\int_1^t p(s)ds.
\end{aligned}$$

Dividing both sides by R and taking the lim inf as $R \to \infty$, we have

$$2\left[\frac{(\log T)^r}{\Gamma(r+1)}\right]\delta \geq 1$$

which contradicts (3.6). Hence, $N(U) \subset U$.

As a consequence of Steps 1-5 and Mönch's Theorem (Theorem 2.7 above), N has a fixed point $y \in C(J, B)$ that in turn is a solution of problem (1.1)-(1.2). $\square$

4. An example

We conclude this paper with an example to illustrate our main result, namely, Theorem 3.3 above.

Consider the fractional differential inclusion

$$^{H}D^{\alpha}y(t) \in F(t, y(t)), \text{ for a.e. } t \in J = [1, e], \ 0 < \alpha \leq 1, \tag{4.1}$$

$$y(1) = 0, \ y(e) = 1. \tag{4.2}$$

Here, $F : [1, e] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map satisfying

$$F(t, y) = \{v \in \mathbb{R} : f_1(t, y) \leq v \leq f_2(t, y)\},$$

where $f_1, \ f_2 : [1, e] \times \mathbb{R} \to \mathbb{R}$, $f_1(t, \cdot)$ is lower semi-continuous (i.e., the set $\{y \in \mathbb{R} : f_1(t, y) > \mu\}$ is open for each $\mu \in \mathbb{R}$), and $f_2(t, \cdot)$ is upper semi-continuous (i.e., the set the set $\{y \in \mathbb{R} : f_2(t, y) < \mu\}$ is open for each $\mu \in \mathbb{R}$). We assume that there is a function $p \in L^1(J, \mathbb{R})$ such that

$$\begin{aligned} \|F(t, u)\|_{\mathcal{P}} &= \sup\{|v|, v(t) \in F(t, y)\} \\ &= \max(|f_1(t, y)|, |f_2(t, y)|\} \leq p(t), \ t \in [1, e], \ y \in \mathbb{R}. \end{aligned}$$

It is clear that F is compact and convex valued, and is upper semi-continuous.
Choose $C(s)$ to be the space of linear functions and choose $\varphi(t) = \beta(C(t))$ such that

$$\beta(u(s)) = \frac{u(s)}{2}$$

with

$$u(s) = as, \ a > 0, \ \frac{2}{a} \leq s \leq \frac{4R}{a}.$$

For $(t, y) \in J \times \mathbb{R}$ with $|y| \geq R$, we have

$$\liminf_{R \to \infty} \frac{\int_0^e p(t)dt}{R} = \delta < \infty.$$

Finally, we assume that there exists a Carathéodory function $\psi : J[1, 2R] \to \mathbb{R}_+$ such that

$$\beta(F(t, M)) \leq \psi(t, \beta(M)) \ a.e. \ t \in J \text{ and each } M \subset B = \{x \in \mathbb{R} : |x| \leq R\},$$

and $\varphi = 0$ is the unique solution in $C(J, [1, 2R])$ of the inequality

$$\begin{aligned} \varphi(t) \leq 2 \Bigg\{ &\frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \psi(s, \varphi(s)) \frac{ds}{s} \\ &+ (\log t)^{r-1} \left[1 + \frac{1}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s}\right)^{r-1} \psi(s, \varphi(s)) \frac{ds}{s}\right] \Bigg\}. \end{aligned}$$

for $t \in J$.

Since all the conditions of Theorem 3.3 are satisfied, problem (4.1)-(4.2) has at least one solution y on $[1, e]$.

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