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Monotonicity with respect to p of the best constants associated with Sobolev immersions of type $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ when $q \in \{1, p, \infty\}$

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In memory of our good friend and collaborator Prof. Csaba Varga

Abstract. The goal of this paper is to collect some known results on the monotonicity with respect to p of the best constants associated with Sobolev immersions of type $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ when $q \in \{1, p, \infty\}$. More precisely, letting

$$\lambda(p, q; \Omega) := \inf_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \| \, |\nabla u|_D \, \|_{L^p(\Omega)} \|u\|_{L^q(\Omega)}^{-1} ,$$

we recall some monotonicity results related with the following functions

$$\begin{aligned} (1, \infty) \ni p &\quad \mapsto \quad |\Omega|^{p-1} \lambda(p, 1; \Omega)^p , \\ (1, \infty) \ni p &\quad \mapsto \quad \lambda(p, p; \Omega)^p , \\ (D, \infty) \ni p &\quad \mapsto \quad \lambda(p, \infty; \Omega)^p , \end{aligned}$$

when $\Omega \subset \mathbb{R}^D$ is a given open, bounded and convex set with smooth boundary.

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1. Introduction

1.1. Goal of the paper

For each open and bounded set $\Omega \subset \mathbb{R}^D$ ($D \geq 1$) the following continuous Sobolev immersions hold true $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, (see, e.g. H. Brezis [9, pp. 284-285 & 212-213]) for each $p \in [1, \infty)$ and each q that satisfies the following restrictions

$$q \in \begin{cases} \left[1, \dfrac{Dp}{D-p}\right], & \text{if } p \in [1, D) \ \& \ D \geq 2 , \\ [1, \infty), & \text{if } p = D \geq 2 , \\ [1, \infty], & \text{if } p \in (D, \infty) \ \& \ D \geq 2 \quad \text{or} \quad p \in [1, \infty] \ \& \ D = 1 . \end{cases}$$

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It follows that, for each Ω, p and q as above there exists a constant $c(p,q;\Omega)>0$ such that

$$c(p,q;\Omega)\|u\|_{L^q(\Omega)} \leq \|\,|\nabla u|_D\,\|_{L^p(\Omega)}, \quad \forall\, u\in W_0^{1,p}(\Omega)\,.$$

Let $\lambda(p,q;\Omega)$ be the best constant in the above inequality, namely

$$\lambda(p,q;\Omega) := \inf_{u\in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\|\,|\nabla u|_D\,\|_{L^p(\Omega)}}{\|u\|_{L^q(\Omega)}}\,. \tag{1.1}$$

The goal of this paper is to recall certain results concerning some monotonicity properties of $\lambda(p,q;\Omega)$ with respect to p when $q\in\{1,p,\infty\}$ and Ω are fixed. More precisely, we will present some monotonicity results related with the following functions

$$(1,\infty)\ni p\mapsto |\Omega|^{p-1}\lambda(p,1;\Omega)^p\,, \tag{1.2}$$

$$(1,\infty)\ni p\mapsto \lambda(p,p;\Omega)^p\,, \tag{1.3}$$

$$(D,\infty)\ni p\mapsto \lambda(p,\infty;\Omega)^p\,, \tag{1.4}$$

when $\Omega\subset\mathbb{R}^D$ is a given open, bounded and convex set with smooth boundary.

1.2. Notations

For each positive integer $D\geq 2$ denot by $|\cdot|_D$ the Euclidean norm on $\mathbb{R}^D$. For each subset $\Omega\subset\mathbb{R}^D$, let $\partial\Omega$ be its boundary and denote by $|\partial\Omega|$ and $|\Omega|$, the $(D-1)$-dimensional Lebesgue perimeter of $\partial\Omega$ and the D-dimensional Lebesgue volume of Ω, respectively. Next, for each positive integer $D\geq 1$ define

$$\mathbb{P}^D := \{\Omega\subset\mathbb{R}^D : \Omega \text{ is an open, bounded, convex set with smooth boundary } \partial\Omega\}\,,$$

and for each $\Omega\in\mathbb{P}^D$ let δ_Ω be the distance function to the boundary of Ω, i.e.

$$\delta_\Omega(x) := \inf_{y\in\partial\Omega} |x-y|_D, \quad \forall\, x\in\Omega\,.$$

Denote by R_Ω the inradius of Ω (that is the radius of the largest ball which can be inscribed in Ω, or, $R_\Omega=\|\delta_\Omega\|_{L^\infty(\Omega)}$). Further, let $\delta:\mathbb{P}^D\to[0,\infty)$ denote the average integral of δ_Ω, that is

$$\delta(\Omega) := \frac{1}{|\Omega|}\int_\Omega \delta_\Omega(x)\,dx\,,$$

and let $h:\mathbb{P}^D\to[0,\infty)$ denote the Cheeger constant of Ω, that is

$$h(\Omega) := \inf_{\omega\subset\Omega}\frac{|\partial\omega|}{|\omega|}\,, \tag{1.5}$$

where the quotient $\frac{|\partial\omega|}{|\omega|}$ is taken among all smooth subdomains $\omega\subset\Omega$. We recall that $h(\Omega)$ also has the equivalent definition

$$h(\Omega) := \inf_{u\in W_0^{1,1}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|_D dx}{\displaystyle\int_\Omega |u|dx}\,, \tag{1.6}$$

and, consequently, by relation (1.1) with $p=q=1$ we have $h(\Omega)=\lambda(1,1;\Omega)$.

1.3. A simple observation regarding the monotonicity of the functions (1.2), (1.3) and (1.4)

A simple application of Hölder's inequality leads to the following monotonicity results regarding functions (1.2), (1.3) and (1.4) when $\Omega \in \mathbb{P}^D$ is an arbitrary but fixed set (see, C. Enache and the first author of this paper [14, relation (2.1)], P. Lindqvist [27, Theorem 3.2], G. Ercole & G.A. Pereira [16, Lemma 3.1])

$$h(\Omega) \le |\Omega|^{(p-1)/p}\lambda(p,1;\Omega) \le |\Omega|^{(q-1)/q}\lambda(q,1;\Omega), \quad \forall\, 1<p<q<\infty\,,$$

$$h(\Omega) \le p\lambda(p,p;\Omega) \le q\lambda(q,q;\Omega), \quad \forall\, 1<p<q<\infty\,,$$

$$|\Omega|^{-1/p}\lambda(p,\infty;\Omega) \le |\Omega|^{-1/q}\lambda(q,\infty;\Omega), \quad \forall\, D<p<q<\infty\,.$$

However, these results cannot offer any direct information in relation with the monotonicity of the functions (1.2), (1.3) and (1.4).

2. Monotonicity of the function $(1,\infty) \ni p \mapsto |\Omega|^{p-1}\lambda(p,1;\Omega)^p$

2.1. A connection with the p-torsion problem

By relation (1.1) with $q=1$ we have that

$$\lambda(p,1;\Omega)^p := \inf_{u\in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\|\,|\nabla u|_D\,\|^p_{L^p(\Omega)}}{\|u\|^p_{L^1(\Omega)}}, \quad \forall\, p\in(1,\infty)\,.$$

It follows that

$$|\Omega|^{p-1}\lambda(p,1;\Omega)^p = \inf_{u\in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{|\Omega|^{-1}\displaystyle\int_\Omega |\nabla u|_D^p\,dx}{\left(|\Omega|^{-1}\displaystyle\int_\Omega |u|\,dx\right)^p}, \quad \forall\, p\in(1,\infty)\,.$$

It is standard to check that for each $p\in(1,\infty)$ there exists a nonnegative minimizer of $|\Omega|^{p-1}\lambda(p,1;\Omega)^p$ in $W_0^{1,p}(\Omega)\setminus\{0\}$. Moreover, it is well-known (see, e.g. L. Brasco [7, pp. 320-321]) that if $u_p \in W_0^{1,p}(\Omega)\setminus\{0\}$ is a nonnegative minimizer of $|\Omega|^{p-1}\lambda(p,1;\Omega)^p$ then

$$v_p(x) := \left(\int_\Omega |\nabla u_p(y)|_D^p\,dy\right)^{-1/(p-1)}\left(\int_\Omega u_p(y)\,dy\right)^{1/(p-1)} u_p(x)\,,$$

gives the unique (weak) solution of the *p-torsion problem*, namely

$$\begin{cases} -\Delta_p v = 1, & \text{in } \Omega,\\ v=0, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\Delta_p v := \operatorname{div}(|\nabla v|_D^{p-2}\nabla v)$ stands for the p-Laplace operator. Conversely, if v_p is the unique (weak) solution of problem (2.1) then it is a positive minimizer of $|\Omega|^{p-1}\lambda(p,1;\Omega)^p$.

On the other hand, we recall that the *p-torsional rigidity* on Ω is defined as follows

$$T_p(\Omega) := \int_\Omega v_p\,dx,$$

and it has the following variational characterization (see, e.g., F. Della Pietra, N. Gavitone, & S. Guarino Lo Bianco [13, relations (18) and (19)])

$$T_p(\Omega)^{p-1} = \sup_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\left(\displaystyle\int_\Omega |u|\,dx\right)^p}{\displaystyle\int_\Omega |\nabla u|_D^p\,dx}\,.$$

Consequently, we can relate function (1.2) with the p-torsional rigidity by the following formula

$$|\Omega|^{p-1}\lambda(p,1;\Omega)^p = |\Omega|^{p-1}T_p(\Omega)^{1-p}\,. \tag{2.2}$$

2.2. The case of a ball

In the particular case when $\Omega = B_R$ (that is a ball of radius R, centered at the origin) $v_p \in W_0^{1,p}(B_R)$, the unique solution of problem (2.1), can be explicitly computed (see, B. Kawohl [24, relation (3.8)]),

$$v_p(x) = \frac{D(p-1)}{p}\left[\left(\frac{R}{D}\right)^{\frac{p}{p-1}} - \left(\frac{|x|_D}{D}\right)^{\frac{p}{p-1}}\right], \quad \forall\, x \in B_R\,.$$

Therefore

$$T_p(B_R) = \int_{B_R} v_p\,dx = \frac{\omega_D}{D^{\frac{p}{p-1}}\left(D+\dfrac{p}{p-1}\right)}R^{D+\frac{p}{p-1}}, \tag{2.3}$$

where $\omega_D = |\partial B_1|$ (that is the area of the unit ball in $\mathbb{R}^D$), and, by relation (2.2), we get

$$|B_R|^{p-1}\lambda(p,1;B_R)^p = |B_R|^{p-1}T_p(B_R)^{1-p} = D\left(D+\frac{p}{p-1}\right)^{p-1}R^{-p},$$

where in the last relation we used the fact that $|B_R| = \dfrac{\omega_D R^D}{D}$. Consequently, in this particular case our problem reduces to the analysis of the monotonicity of the function

$$(1,\infty) \ni p \mapsto D\left(D+\frac{p}{p-1}\right)^{p-1}R^{-p}\,.$$

2.3. The case when $\Omega \in \mathbb{P}^D$ is a general set

In the general case explicit formulas for the quantity $|\Omega|^{p-1}\lambda(p,1;\Omega)^p$ are not available in the literature and, consequently, the analysis of the monotonicity of the function given in relation (1.2), i.e.

$$(1,\infty) \ni p \mapsto |\Omega|^{p-1}\lambda(p,1;\Omega)^p\,,$$

is not trivial. However, a hint regarding the possible monotonicity of the above function can be easily obtained by recalling the following asymptotic formula (see L.E. Payne & G.A. Philippin [32])

$$\lim_{p\to\infty}\int_\Omega v_p\,dx = \int_\Omega \delta_\Omega\,dx\,. \tag{2.4}$$

Note that, actually, v_p converges uniformly over $\overline{\Omega}$ to δ_Ω, as $p \to \infty$ (see, T. Bhattacharya, E. DiBenedetto, & J.J. Manfredi [2] and B. Kawohl [24]). Combining (2.4) with (2.2) we deduce that

$$\lim_{p\to\infty} |\Omega|^{\frac{p-1}{p}} \lambda(p,1;\Omega) = \delta(\Omega)^{-1},$$

which further implies

$$\lim_{p\to\infty} |\Omega|^{p-1}\lambda(p,1;\Omega)^p = \begin{cases} +\infty & \text{if } \delta(\Omega) < 1, \\ 0 & \text{if } \delta(\Omega) > 1\,. \end{cases}$$

Consequently, if the map given in relation (1.2) has a certain monotonicity then it should be increasing if $\delta(\Omega) < 1$ and decreasing if $\delta(\Omega) > 1$. The precise result concerning the monotonicity of function (1.2) was obtained by C. Enache and the authors of this paper in [14] and [15]. More exactly, by [14, Theorem 2] and [15, Remark 2] we have the following result.

Theorem 2.1. *For each $D \geq 1$ there exists a constant $T \in [(2D)^{-1}, 1]$ such that for each set $\Omega \in \mathbb{P}^D$ with $\delta(\Omega) \leq T$ the map given in relation (1.2) is increasing. Moreover, for any $s > T$ there exists a set $\Omega \in \mathbb{P}^D$, with $\delta(\Omega) = s$, for which the map given in relation (1.2) is not monotone.*

Remark 1. We note that, actually, we can give a better lower bound for the constant T from the above theorem. Indeed, by [10, Proposition 6.1] we have that

$$\delta(\Omega) \geq \frac{R_\Omega}{D+1}, \quad \forall\, \Omega \in \mathbb{P}^D\,.$$

It follows that for each $\Omega \in \mathbb{P}^D$ with $\delta(\Omega) < (D+1)^{-1}$ we have $R_\Omega < 1$ which by [14, Lemma 1] implies that

$$T(p;\Omega) < T(q;\Omega), \quad \forall\, 1 < p < q < \infty,\ \forall\, \Omega \in \mathbb{P}^D \text{ with } \delta(\Omega) < (D+1)^{-1}\,.$$

This observation combined with the proof of [14, Proposition 1] implies that $T \geq (D+1)^{-1}$. Consequently, in the conclusion of Theorem 2.1 we have $T \in [(D+1)^{-1}, 1]$ which improves the older bounds for T, namely $T \in (0, 1]$ (obtained in [14, Theorem 2]) and $T \in [(2D)^{-1}, 1]$ (obtained in [15, Remark 2]).

2.3.1. Open problems related to the monotonicity of function (1.2).

Problem 1. Note that by [14, Proposition 2] we have that for each ball B_R with $R > D + 1$ (and consequently $\delta(B_R) > 1$) the map given in relation (1.2) is not monotone. Consequently, for each real number $s > 1$ a set $\Omega \in \mathbb{P}^D$ with $\delta(\Omega) = s$ for which the map given in relation (1.2) is not monotone could be chosen to be a ball. However, in general, the question if for any set $\Omega \in \mathbb{P}^D$ with $\delta(\Omega) > T$ the map given in relation (1.2) is not monotone is open.

Problem 2. Another open problem related with the result from Theorem 2.1 is the following: if $D \geq 2$ does the number T given by Theorem 2.1 satisfy $T = 1$ or can the situation $T < 1$ occur? Moreover, if the case $T < 1$ holds true, then does T depend on D (the dimension of the Euclidean space) or not?

2.4. An alternative variational characterization for $\lambda(p,1;\Omega)$ on sets with small $\delta(\Omega)$

The monotonicity result from Theorem 2.1 allow us to obtain an alternative variational characterization of the constant $\lambda(p,1;\Omega)$ on domains $\Omega\in\mathbb{P}^D$ with $\delta(\Omega)\le T$ (where T is the constant given by Theorem 2.1). More precisely, if for any $\Omega\in\mathbb{P}^D$ and each $p\in(1,\infty)$ let us define

$$\Lambda(p,1;\Omega):=\inf_{v\in X_0\setminus\{0\}}\frac{|\Omega|^{-1}\displaystyle\int_\Omega(\exp(|\nabla v|_D^p)-1)\,dx}{\exp\left(\left(|\Omega|^{-1}\displaystyle\int_\Omega|v|\,dx\right)^p\right)-1}\,, \tag{2.5}$$

where $X_0:=W^{1,\infty}(\Omega)\cap\left(\cap_{q>1}W_0^{1,q}(\Omega)\right)$. Then by [14, Theorem 3] we have the following result:

Theorem 2.2. *Let $D\ge 1$ be an integer and $\Omega\in\mathbb{P}^D$ be a set. If $\|\delta_\Omega\|_{L^\infty(\Omega)}\le 1$, then $\Lambda(p,1;\Omega)>0$, for all $p\in(1,\infty)$, while if $\delta(\Omega)>1$, then $\Lambda(p,1;\Omega)=0$, for all $p\in(1,\infty)$. Moreover, if $\delta(\Omega)\le T$, where T is the constant given by Theorem 2.1, then $\lambda(p,1;\Omega)=|\Omega|^{\frac{1-p}{p}}\Lambda(p,1;\Omega)^{1/p}$, for all $p\in(1,\infty)$.*

Remark 2. Note that the fact that $\|\delta_\Omega\|_{L^\infty(\Omega)}\le 1$ implies $\delta(\Omega)\le 1$. However, the fact that for any $\Omega\in\mathbb{P}^D$ with $\delta(\Omega)\le 1$ it holds $\Lambda(p,1;\Omega)>0$, for all $p\in(1,\infty)$ is an open problem. This problem would be solved for instance if one can show that $T=1$.

2.5. Monotonicity of the p-torsional rigidity

Another monotonicity result that can be related with the above discussion is that of the function

$$(1,\infty)\ni p\to T_p(\Omega)\,, \tag{2.6}$$

when $\Omega\in\mathbb{P}^D$ is given. Note that by relation (2.2) this is equivalent with the monotonicity of the map

$$(1,\infty)\ni p\to\lambda(p,1;\Omega)^{-p/(p-1)}\,.$$

2.5.1. The case of a ball. The discussion of the particular case when Ω is a ball, say $\Omega=B_R$ consists in the investigation of the monotonicity of the function given by relation (2.3), namely

$$(1,\infty)\ni p\mapsto\frac{\omega_D}{D^{\frac{p}{p-1}}\left(D+\dfrac{p}{p-1}\right)}R^{D+\frac{p}{p-1}}\,.$$

By [15, Theorem 3] we have the following result.

Theorem 2.3. (a) *If $R\ge De^{\frac{1}{D+1}}$ then $(1,\infty)\ni p\to T_p(B_R)$ is decreasing on the entire interval $(1,\infty)$.*

(b) *If $R\in(D,De^{\frac{1}{D+1}})$ then $(1,\infty)\ni p\to T_p(B_R)$ is decreasing on $\left(1,\frac{1-D\ln\left(\frac{R}{D}\right)}{1-(D+1)\ln\left(\frac{R}{D}\right)}\right)$ and increasing on $\left(\frac{1-D\ln\left(\frac{R}{D}\right)}{1-(D+1)\ln\left(\frac{R}{D}\right)},\infty\right)$.*

(c) *If* $R \leq D$ *then* $(1, \infty) \ni p \to T_p(B_R)$ *is increasing on the entire interval* $(1, \infty)$.

2.5.2. The case when $\Omega \in \mathbb{P}^D$ is a general set. The analysis of the monotonicity of the map given by relation (2.6) on a general set $\Omega \in \mathbb{P}^D$ is, as in the case of the function (1.2), more difficult since we do not have explicit formulas for $T_p(\Omega)$. However, a hint can be given in this case if we take into account an asymptotic formula which can be found in one of the papers by H. Bueno & G. Ercole [11, Theorem 2, relation (16)] or H. Bueno, G. Ercole, & S. S. Macedo [12, relation (1.10)]), namely

$$\lim_{p \to 1^+} T_p(\Omega)^{1-p} = h(\Omega)\,, \tag{2.7}$$

where $h(\Omega)$ stands for the Cheeger constant of Ω given by relations (1.5) and (1.6). It follows that

$$\lim_{p \to 1^+} T_p(\Omega) = \begin{cases} 0, & \text{if } h(\Omega) > 1, \\ \infty, & \text{if } h(\Omega) < 1. \end{cases} \tag{2.8}$$

Consequently, if the function given in relation (2.6) has a certain monotonicity then it should be increasing if $h(\Omega) > 1$ and decreasing if $h(\Omega) < 1$. However, the analysis from [15] shows that it is more useful to work with the quotient $\frac{|\partial\Omega|}{|\Omega|}$ than with the Cheeger constant, $h(\Omega)$, when we analyse the monotonicity of the p-torsional rigidity with respect to $p \in (1, \infty)$. According to relation (1.5) that fact is not unexpected even if $h(\Omega) \leq \frac{|\partial\Omega|}{|\Omega|}$. Note that in the particular case when $\Omega = B_R$ the result from Theorem 2.3 is consistent with the above discussion since it is well-known that $\frac{|\partial B_R|}{|B_R|} = h(B_R) = \frac{D}{R}$ and then we observe that function $(1, \infty) \ni p \to T_p(B_R)$ is increasing if $h(B_R) = \frac{D}{R} > 1$ and decreasing if $h(B_R) = \frac{D}{R} \leq e^{-1/(D+1)} < 1$.

The general result concerning the monotonicity of function (2.6) was obtained by C. Enache and the authors of this paper in [15, Theorem 2]. We recall this result below.

Theorem 2.4. *Assume* $D \geq 2$. *Then there exist two real numbers* $A_1 \in \left[\frac{1}{2}, e^{\frac{-1}{D+1}}\right]$ *and* $A_2 \in [1, D]$ *such that*

(i) for each $\Omega \in \mathbb{P}^D$ *with* $\frac{|\partial\Omega|}{|\Omega|} \leq A_1$ *the map given in relation (*2.6*) is decreasing on the entire interval* $(1, \infty)$*;*

(ii) for each $\Omega \in \mathbb{P}^D$ *with* $\frac{|\partial\Omega|}{|\Omega|} \geq A_2$ *the map given in relation (*2.6*) is increasing on the entire interval* $(1, \infty)$*;*

(iii) for each real number $s \in (A_1, A_2)$ *there exists* $\Omega \in \mathbb{P}^D$ *with* $\frac{|\partial\Omega|}{|\Omega|} = s$ *such that the map given in relation (*2.6*) is not monotone on* $(1, \infty)$.

2.5.3. Open problems related to the monotonicity of function (2.6).

Problem 1. Note that by Theorem 2.3 we have that for each ball B_R with $R \in (D, De^{\frac{1}{D+1}})$ the map given in relation (2.6) is not monotone. Consequently, for each real number $s \in (e^{\frac{-1}{D+1}}, 1)$ a set $\Omega \in \mathbb{P}^D$ with $\frac{|\partial\Omega|}{|\Omega|} = s$ for which the map given in relation (2.6) is not monotone could be chosen to be a ball. However, in general, the question if for any set $\Omega \in \mathbb{P}^D$ with $\frac{|\partial\Omega|}{|\Omega|} \in (A_1, A_2)$ the map given in relation (2.6) is not monotone is open.

Problem 2. Another open problem related with the result from Theorem 2.3 is the following: do the numbers A_1 and A_2 given by Theorem 2.3 satisfy $A_1 = e^{\frac{-1}{D+1}}$ and $A_2 = 1$ or can the situations $A_1 < e^{\frac{-1}{D+1}}$ and $A_2 > 1$ occur? Further, if the case $A_2 > 1$ holds true, then does A_2 depend on D (the dimension of the Euclidean space) or not?

3. Monotonicity of the function $(1,\infty) \ni p \mapsto \lambda(p,p;\Omega)^p$

3.1. A connection with the eigenvalue problem of the p-Laplace operator

By relation (1.1) with $q = p$ we have that

$$\begin{aligned} \lambda(p,p;\Omega)^p \quad &:= \inf_{u\in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\|\, |\nabla u|_D \,\|^p_{L^p(\Omega)}}{\|u\|^p_{L^p(\Omega)}} \\ &= \inf_{u\in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|_D^p \, dx}{\displaystyle\int_\Omega |u|^p \, dx}, \quad \forall\, p \in (1,\infty)\,. \end{aligned}$$

It is well-known that this minimization problem is related to the eigenvalue problem for the p-Laplace operator, namely

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.1}$$

in the sense that $\lambda(p,p;\Omega)^p$ represents the lowest eigenvalue of the problem (3.1), also known as the *principal frequency* of the p-Laplace operator (see, e.g. P. Lindqvist [26]). (We recall that by an eigenvalue of problem (3.1) we understand a parameter λ for which the problem possesses a nontrivial (weak) solution.)

3.2. The case $D = 1$

In the particular case when $D = 1$, if $\Omega \in \mathbb{P}^1$ then there exists a, $b \in \mathbb{R}$ with $a < b$ such that $\Omega = (a,b)$. It is well-known (see, e.g. P. Lindqvist [28]) that

$$\lambda(p,p;(a,b))^p = (p-1)\left(\frac{2}{b-a}\right)^p \left(\frac{\pi/p}{\sin(\pi/p)}\right)^p, \quad \forall\, p \in (1,\infty)\,.$$

Consequently, in this particular case our problem reduces to the analysis of the monotonicity of the function

$$(1,\infty) \ni p \mapsto (p-1)\left(\frac{2}{b-a}\right)^p \left(\frac{\pi/p}{\sin(\pi/p)}\right)^p.$$

The corresponding investigation was carried on by R. Kajikiya, M. Tanaka, & S. Tanaka in [23, Theorem 1.1]. More precisely, they proved the following result.

Theorem 3.1. *If $\frac{b-a}{2} \le 1$ then the map $p \mapsto \lambda(p,p;(a,b))^p$ is increasing on the entire interval $(1,\infty)$. If $\frac{b-a}{2} > 1$ then there exists $p^\star = p^\star\left(\frac{b-a}{2}\right) \in (1,\infty)$ such that $p \mapsto \lambda(p,p;(a,b))^p$ is increasing on $(1,p^\star)$ and decreasing on $(p^\star,\infty)$.*

3.3. The case $D \geq 2$

In the general case, when $D \geq 2$, there is no explicit formula of $\lambda(p,p;\Omega)^p$ when $p \in (1,\infty)\setminus\{2\}$, not even on simple domains such as balls or squares. This fact makes the study of the monotonicity of the map given in relation (1.3), i.e.

$$(1,\infty) \ni p \mapsto \lambda(p,p;\Omega)^p\,,$$

more complicated. However, a hint regarding its monotonicity comes from the following asymptotic formula due to P. Juutinen, P. Lindqvist, & J. J. Manfredi [21, Lemma 1.5] and N. Fukagai, M. Ito, & K. Narukawa [18, Corollaries 3.2 and 4.5]

$$\lim_{p\to\infty} \lambda(p,p;\Omega) = R_\Omega^{-1}\,,$$

which yields

$$\lim_{p\to\infty} \lambda(p,p;\Omega)^p = \begin{cases} +\infty & \text{if } R_\Omega < 1,\\ 0 & \text{if } R_\Omega > 1\,. \end{cases}$$

Consequently, if the map given in relation (1.3) has a certain monotonicity then it should be increasing if $R_\Omega < 1$ and decreasing if $R_\Omega > 1$.

A first result concerning the monotonicity of the map (1.3) when $D \geq 2$ can be found in a paper by V. Bobkov & M. Tanaka, namely [3, Proposition 9], where the following theorem was proved.

Theorem 3.2. *Assume that $D \geq 2$ is an integer and $\Omega \subset \mathbb{R}^D$ is a domain satisfying*

$$B_r \subset \Omega \subset B_R\,,$$

where r, $R \in (1,e)$ are two real numbers such that

$$\max\{1, e\ln R\} < r \leq R < e\,,$$

and B_r, B_R stand for two balls having radii r and R, respectively. Then the map given by relation (1.3) is not monotone on $(1,\infty)$.

This result was complemented and improved by M. Bocea and the first author of this paper in [5, Theorem 1]. The precise result is formulated in the following theorem.

Theorem 3.3. *Let $D \geq 2$ be a given integer. Then there exists a real number $M \in [e^{-1},1]$ such that for each $\Omega \in \mathbb{P}^D$ with $R_\Omega \leq M$ the map given by relation (1.3) is increasing on the entire interval $(1,\infty)$. Moreover, for each $s > M$ there exists a domain $\Omega \in \mathbb{P}^D$ with $R_\Omega = s$ for which the map given by relation (1.3) is not monotone on the entire interval $(1,\infty)$.*

3.3.1. Open problems related to the monotonicity of function (1.3).

Problem 1. The following open problem can be formulated in relation with the above result: if $D \geq 2$ and M is the number given by Theorem 3.3 is it true that for all $\Omega \in \mathbb{P}^D$ with $R_\Omega > M$ the map given by relation (1.3) is not monotone on the entire interval $(1,\infty)$? In [5, Proposition 1 & Theorem 1] the authors proved the existence of such kind of domains by using as main argument a result due to R. Kajikiya [22, Proposition 2.3] (see also L. Brasco [8, Theorem 1.1] for a similar result). Moreover, the first author of this paper complemented the result by proving in [30, Theorem 1 (d)] that when Ω is a ball, say B_R, with the radius strictly larger than 1, ($R > 1$), then the map given by relation (1.3) is not monotone on the entire interval $(1, \infty)$. (That time the main argument was based on some estimates of the principal frequency due to J. Benedikt & P. Drábek [1, Theorem 2].) Excepting these particular investigations the general case is an open problem.

Problem 2. Another open problem related with the result from Theorem 3.3 is the following: if $D \geq 2$ does the number M given by Theorem 3.3 satisfy $M = 1$ or can the situation $M < 1$ occur? Moreover, if the case $M < 1$ holds true, then does M depend on D (the dimension of the Euclidean space) or not?

3.3.2. Monotonicity results for similar eigenvalue problems. In this section we recall certain papers where similar results with those formulated in Theorem 3.3 can be found.

Monotonicity results for variational eigenvalues of the Dirichlet p-Laplace operator. Since $\lambda(p, p; \Omega)^p$ represents the lowest eigenvalue of the problem (3.1) it is natural to ask if similar results hold true for other eigenvalues of the problem. In that context, firstly, we need to recall the well known fact that the description of the entire set of eigenvalues of problem (3.1), when $p \neq 2$, is still an open question. However, for the sequence of variational eigenvalues produced by using the Ljusternik-Schnirelman theory (see, e.g. P. Lindqvist [29] or A. Lê [25] for the description of the set of variational eigenvalues of problem (3.1)) similar results as those given in Theorem 3.3 were obtained by the first author of this paper in [30, Theorem 1].

Monotonicity results for the principal frequency on an annulus. A similar result with those from Theorem 3.3 was obtained when Ω is an annulus (i.e., Ω is the difference of two concentric balls), and consequently $\Omega \notin \mathbb{P}^D$, by A. Grecu and the first author of this paper in [19, Theorem 1].

Monotonicity of the first positive eigenvalue of the Neumann p-Laplace operator. Similar investigations with those from Theorem 3.3 were considered in the context of the first positive eigenvalue of the p-Laplace operator under the homogeneous Neumann boundary condition by the first author of this paper in collaboration with J. D. Rossi in [31, Theorem 1.1].

Monotonicity results for the principal frequency of the anisotropic p-Laplace operator. M. Bocea in collaboration with the authors of this paper discussed the monotonicity of the principal frequency of the anisotropic p-Laplace operator in [6, Theorem 1].

3.4. An alternative variational characterization for $\lambda(p, p; \Omega)$ on sets with small inradius

We note that combining the monotonicity result from Theorem 3.3 with those obtained by M. Bocea and the first author of this paper in [4, Theorem 2] we deduce that for each set $\Omega \in \mathbb{P}^D$ with $R_\Omega \in (0, M]$, where M is given by Theorem 3.3, we have the following alternative variational characterization of $\lambda(p,p;\Omega)^p$ when $p\in(1,\infty)$, namely

$$\lambda(p,p;\Omega)^p=\inf_{u\in X_0\setminus\{0\}}\frac{\displaystyle\int_\Omega \Phi_p(|\nabla u|)\,dx}{\displaystyle\int_\Omega \Phi_p(|u|)\,dx},$$

where, $X_0:=W^{1,\infty}(\Omega)\cap\left(\cap_{q>1}W_0^{1,q}(\Omega)\right)$ and $\Phi_p(t)$ can be taken to be either one of the functions $t\mapsto\sinh(|t|^p)$, $t\mapsto\cosh(|t|^p)-1$, or $t\mapsto\exp(|t|^p)-1$. It is interesting that this variational characterization fails to hold true when $\Omega\in\mathbb{P}^D$ with $R_\Omega\in(1,\infty)$ since in that case the above infimum vanishes (see, [4, Theorem 2]).

4. Monotonicity of the function $(D,\infty)\ni p\mapsto\lambda(p,\infty;\Omega)^p$

4.1. A connection between $\lambda(p,\infty;\Omega)^p$ and an eigenvalue problem

By relation (1.1) with $q=\infty$ we have that

$$\begin{aligned}\lambda(p,\infty;\Omega)^p &:= \inf_{u\in W_0^{1,p}(\Omega)\setminus\{0\}}\frac{\|\,|\nabla u|_D\,\|^p_{L^p(\Omega)}}{\|u\|^p_{L^\infty(\Omega)}}\\ &= \inf_{u\in W_0^{1,p}(\Omega)\setminus\{0\}}\frac{\displaystyle\int_\Omega|\nabla u|_D^p\,dx}{\|u\|^p_{L^\infty(\Omega)}},\quad\forall\,p\in(D,\infty)\,.\end{aligned}$$

Note that $\lambda(p,\infty;\Omega)^p$ is also known as the *best constant in Morrey's inequality*, that is the largest constant $C>0$ for which the following inequality holds true

$$C\|u\|^p_{L^\infty(\Omega)}\le\int_\Omega|\nabla u(x)|_D^p\,dx,\quad\forall\,u\in W_0^{1,p}(\Omega)\,.$$

It is well known (see, e.g. G. Ercole & G. A. Pereira [16, Theorem 2.5] or R. Hynd & E. Lindgren [20]) that for each $p\in(D,\infty)$ there exists a nonnegative minimizer of $\lambda(p,\infty;\Omega)^p$, say $u_p\in W_0^{1,p}(\Omega)$, such that

$$\|u_p\|_{L^\infty(\Omega)}=1\quad\text{and}\quad\|\,|\nabla u_p|_D\,\|^p_{L^p(\Omega)}=\lambda(p,\infty;\Omega)^p\,.$$

Moreover, there exists a unique point $x_p\in\Omega$ such that

$$u_p(x_p)=\|u_p\|_{L^\infty(\Omega)}=1\,,$$

and the following equation is satisfied in the sense of distributions

$$\begin{cases}-\mathrm{div}(|\nabla u_p(x)|_D^{p-2}\nabla u_p(x))=\lambda(p,\infty;\Omega)^p|u_p(x_p)|^{p-2}u_p(x_p)\delta_{x_p}(x), & \text{if } x\in\Omega,\\ u_p(x)=0, & \text{if } x\in\partial\Omega,\end{cases}$$

where by δ_{x_p} the Dirac mass concentrated at x_p was denoted.

4.1.1. The case of a ball. In the particular case when $\Omega = B_R$ (that is a ball of radius R, centered at the origin) then

$$\lambda(p,\infty;B_R)^p = \frac{Dv_D}{R^{p-D}}\left(\frac{p-D}{p-1}\right)^{p-1}, \tag{4.1}$$

where $v_D = |B_1|$ denotes the volume of the unit ball in $\mathbb{R}^D$, (see, e.g., [16, relation (1.9)]). Consequently, in this particular case our problem reduces to the analysis of the monotonicity of the function

$$(D,\infty) \ni p \mapsto \frac{Dv_D}{R^{p-D}}\left(\frac{p-D}{p-1}\right)^{p-1}.$$

The precise result in this case is the following (see [17, Theorem 1.2])

Theorem 4.1. *For every integer $D \geq 1$ if $R \leq 1$ then the map $p \mapsto \lambda(p,\infty;B_R)^p$ is increasing on the entire interval (D,∞), while, if $R > 1$ then the map $p \mapsto \lambda(p,\infty;B_R)^p$ is not monotone on (D,∞).*

4.1.2. The case when $\Omega \in \mathbb{P}^D$ is a general set. In the general case an explicit formula for the quantity $\lambda(p,\infty;\Omega)^p$ is not available in the literature and, consequently, the analysis of the monotonicity of the map given in relation (1.4), i.e.

$$(1,\infty) \ni p \mapsto \lambda(p,\infty;\Omega)^p,$$

is more complicated. However, a hint regarding its monotonicity comes from the following asymptotic formula (see, e.g. [16, Theorem 3.2])

$$\lim_{p\to\infty} \lambda(p,\infty;\Omega) = R_\Omega^{-1},$$

where $R_\Omega = \|\delta_\Omega\|_{L^\infty(\Omega)}$ denotes the inradius of Ω, which yields

$$\lim_{p\to\infty} \lambda(p,\infty;\Omega)^p = \begin{cases} +\infty & \text{if } R_\Omega < 1, \\ 0 & \text{if } R_\Omega > 1. \end{cases}$$

Consequently, if the map given in relation (1.4) has a certain monotonicity then it should be increasing if $R_\Omega < 1$ and decreasing if $R_\Omega > 1$.
On the other hand, by [16, Corollary 2.7] for each $\Omega \in \mathbb{P}^D$ and each $p \in (D,\infty)$ the following inequalities hold

$$\lambda\left(p,\infty;B_{\sqrt[D]{|\Omega|/v_D}}\right)^p \leq \lambda(p,\infty;\Omega)^p \leq \lambda\left(p,\infty;B_{R_\Omega}\right)^p. \tag{4.2}$$

Combining relations (4.1) and (4.2) we deduce that

$$\lim_{p\to D^+} \lambda(p,\infty;\Omega) = 0, \quad \forall\, \Omega \in \mathbb{P}^D.$$

The above pieces of information show that the map given in relation (1.4) is not monotone on (D,∞) for any set $\Omega \in \mathbb{P}^D$ with $R_\Omega > 1$. The general result on the monotonicity of the map given in relation (1.4) was obtained by M. Fărcăşeanu in collaboration with the first author of this paper in [17, Theorem 1.2] and is given in the following theorem.

Theorem 4.2. *For every integer $D \geq 2$ there exists $L \in [e^{-1}, 1]$ such that for each $\Omega \in \mathbb{P}^D$ with $R_\Omega \in (0, L]$ the map given in relation (1.4) is increasing on (D, ∞) while for each $\Omega \in \mathbb{P}^D$ with $R_\Omega > 1$ the map given in relation (1.4) is not monotone on (D, ∞).*

4.1.3. An open problem related to the monotonicity of function (1.4). The following open problem can be formulated in relation with the above result: if $D \geq 2$ does the number L given by Theorem 4.2 satisfy $L = 1$ or can the situation $L < 1$ occur? Moreover, if the case $L < 1$ holds true, then does L depend on D (the dimension of the Euclidean space) or not?

4.2. An alternative variational characterization for $\lambda(p, \infty; \Omega)$ on sets with small inradius

The monotonicity results from Theorems 4.1 and 4.2 allow us to obtain an alternative variational characterization of the constant $\lambda(p, \infty; \Omega)$ on domains $\Omega \in \mathbb{P}^D$ with $R_\Omega \leq L$ (where L is the constant given by Theorem 4.2). More precisely, if for any $\Omega \in \mathbb{P}^D$ and each $p \in (1, \infty)$ we define

$$\Lambda(p, \infty; \Omega) := \inf_{u \in X_0 \setminus \{0\}} \frac{\displaystyle\int_\Omega (\exp(|\nabla u|_D^p) - 1)\, dx}{\exp\left(\|u\|_{L^\infty(\Omega)}^p\right) - 1}, \tag{4.3}$$

where $X_0 := W^{1,\infty}(\Omega) \cap \left(\cap_{q>1} W_0^{1,q}(\Omega)\right)$, then by [17, Theorem 1.3] we have the following result.

Theorem 4.3. *Let $D \geq 1$ be an integer and $\Omega \in \mathbb{P}^D$ be a set. If $R_\Omega < 1$, then $\Lambda(p, \infty; \Omega) > 0$, for all $p \in (D, \infty)$, while if $R_\Omega > 1$, then $\Lambda(p, \infty; \Omega) = 0$, for all $p \in (D, \infty)$. Moreover, if $R_\Omega \leq L$, with L the constant given by Theorem 4.2, then $\Lambda(p, \infty; \Omega) = \lambda(p, \infty; \Omega)^p$, for all $p \in (D, \infty)$. In the particular case when $\Omega = B_R$ (i.e., Ω is a ball of radius R from $\mathbb{R}^D$) then $\Lambda(p, \infty; B_R) = 0$, for all $p \in (D, \infty)$ if $R > 1$ and $\Lambda(p, \infty; B_R) = \lambda(p, \infty; B_R)^p$, for all $p \in (D, \infty)$ if $R \in (0, 1]$.*

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