Starlike and convex properties for Poisson distribution series

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Abstract. In this paper, we find the necessary and sufficient conditions, inclusion relations for Poisson distribution series belonging to the classes $S^*(\alpha, \beta)$ and $C^*(\alpha, \beta)$. Further, we consider an integral operator related to Poisson Distribution series.

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1. Introduction

Let $A$ denote the class of functions of the form

$$ f(z) = z + \sum_{n=2}^{\infty} a_n z^n $$

which are analytic and univalent in the open disc $U = \{z : z \in \mathbb{C} \ | \ |z| < 1\}$. Let $S$ be a subclass of $A$ consisting of functions whose non-zero coefficients from second on is give by

$$ f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad z \in U. $$

In 2014, Porwal [4] introduced a power series whose coefficients are probabilities of Poisson distribution

$$ K(m, z) := z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} z^n, \quad z \in U, $$

where $m > 0$. By ratio test the radius of convergence of the above series is infinity. Further, Porwal [4] defined a series

$$ \mathcal{F}(m, z) = 2z - K(m, z) = z - \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} z^n, \quad z \in U. $$
Corresponding to the series $K(m, z)$ using the Hadamard product for $f \in A$, Porwal and Kumar [5] introduced a new linear operator $I(m) : A \to A$ defined by

$$I(m)f(z) = z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} a_n z^n, \quad z \in \mathbb{U},$$

where $*$ denotes the convolution (or Hadamard product) of two series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$.

Let $S^*(\alpha, \beta)$ be the subclass of $T$ consisting of functions which satisfy the condition:

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < \beta, \quad z \in \mathbb{U},$$

where $0 \leq \alpha < 1$ and $0 < \beta \leq 1$.

Also, let $C^*(\alpha, \beta)$ be the subclass of $T$ consisting of functions which satisfy the condition:

$$\left| \frac{zf''(z)}{f'(z)} + 2(1 - \alpha) \right| < \beta, \quad z \in \mathbb{U},$$

where $0 \leq \alpha < 1$ and $0 < \beta \leq 1$.

The classes $S^*(\alpha, \beta)$ and $C^*(\alpha, \beta)$, were introduced and studied by Gupta and Jain [2] (see [3]). Also, we note that for $\beta = 1$ the classes $S^*(\alpha, \beta)$ and $C^*(\alpha, \beta)$ reduce to the class of starlike and convex functions of order $\alpha (0 \leq \alpha < 1)$ (see [6]).

A function $f \in A$ is said to be in the class $R^\tau(A, B)$, $(\tau \in \mathbb{C} \setminus \{0\}, -1 \leq B < A \leq 1)$, if it satisfies the inequality

$$\left| \frac{f'(z) - 1}{(A-B)\tau-B[f'(z)-1]} \right| < 1, \quad z \in \mathbb{U}.$$ 

This class was introduced by Dixit and Pal [1].

**Lemma 1.1.** [2] A function $f(z)$ of the form (1.2) is in $S^*(\alpha, \beta)$ if and only if

$$\sum_{n=2}^{\infty} [n(1 + \beta) - 1 + \beta(1 - 2\alpha)] |a_n| \leq 2\beta(1 - \alpha). \quad (1.3)$$

**Lemma 1.2.** [2] A function $f(z)$ of the form (1.2) is in $C^*(\alpha, \beta)$ if and only if

$$\sum_{n=2}^{\infty} n[n(1 + \beta) - 1 + \beta(1 - 2\alpha)] |a_n| \leq 2\beta(1 - \alpha). \quad (1.4)$$

To obtain our main results, we need the following lemmas:
Lemma 1.3. [1] If \( f \in \mathcal{R}^\tau(A, B) \) is of the form (1.1), then
\[
|a_n| \leq (A - B) \left| \frac{\tau}{n} \right|, \quad n \in \mathbb{N} \setminus \{1\}. \tag{1.5}
\]

In the present investigation, inspired by the works of Porwal [4] and Porwal and Kumar [5], we find the necessary and sufficient conditions for \( F(m, z) \) belonging to the classes \( \mathcal{S}^*(\alpha, \beta) \) and \( \mathcal{G}^*(\alpha, \beta) \). Also, we obtain inclusion relations for aforecited classes with \( \mathcal{R}^\tau(A, B) \).

2. Necessary and sufficient conditions

Theorem 2.1. If \( m > 0, 0 \leq \alpha < 1 \) and \( 0 < \beta \leq 1 \), then \( F(m, z) \in \mathcal{S}^*(\alpha, \beta) \) if and only if
\[
e^m m(1 + \beta) \leq 2\beta(1 - \alpha). \tag{2.1}
\]

Proof. Since
\[
F(m, z) = z - \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} z^n,
\]
in view of Lemma 1.1, it is enough to show that
\[
\sum_{n=2}^{\infty} \left[ n(1 + \beta) - 1 + \beta(1 - 2\alpha) \right] \frac{m^{n-1}}{(n-1)!} e^{-m} \leq 2\beta(1 - \alpha).
\]

Let
\[
T_1 = \sum_{n=2}^{\infty} \left[ n(1 + \beta) - 1 + \beta(1 - 2\alpha) \right] \frac{m^{n-1}}{(n-1)!} e^{-m}.
\]

Now,
\[
T_1 = \sum_{n=2}^{\infty} \left[ n(1 + \beta) - 1 + \beta(1 - 2\alpha) \right] \frac{m^{n-1}}{(n-1)!} e^{-m} = e^{-m} \sum_{n=2}^{\infty} \left[ (n-1)(1 + \beta) + 2\beta(1 - \alpha) \right] \frac{m^{n-1}}{(n-1)!}
\]
\[
= e^{-m} \left[ (1 + \beta) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-2)!} + 2\beta(1 - \alpha) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} \right]
\]
\[
= e^{-m} \left[ (1 + \beta) me^{-m} + 2\beta(1 - \alpha)(e^{-m} - 1) \right]
\]
\[
= (1 + \beta)m + 2\beta(1 - \alpha)(1 - e^{-m}).
\]

But this last expression is bounded by \( 2\beta(1 - \alpha) \), if and only if (2.1) holds. This completes the proof of Theorem 2.1. \( \square \)

Theorem 2.2. If \( m > 0, 0 \leq \alpha < 1 \) and \( 0 < \beta \leq 1 \), then \( F(m, z) \in \mathcal{G}^*(\alpha, \beta) \) if and only if
\[
e^m \left[ (1 + \beta)m^2 + 2(1 + \beta(2 - \alpha))m \right] \leq 2\beta(1 - \alpha). \tag{2.2}
\]
Proof. Since
\[ \mathcal{F}(m, z) = z - \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} z^n, \]
in view of Lemma 1.2, it is enough to show that
\[ \sum_{n=2}^{\infty} n[n(1 + \beta) - 1 + \beta(1 - 2\alpha)] \frac{m^{n-1}}{(n-1)!} e^{-m} \leq 2\beta(1 - \alpha). \]
Let
\[ T_2 = \sum_{n=2}^{\infty} n[n(1 + \beta) - 1 + \beta(1 - 2\alpha)] \frac{m^{n-1}}{(n-1)!} e^{-m}. \]
Therefore,
\[
T_2 = e^{-m} \left[ \sum_{n=2}^{\infty} (n-1)(n-2)(1+\beta) \frac{m^{n-1}}{(n-1)!} \\
+ \sum_{n=2}^{\infty} (n-1)[3(1+\beta) - 1 + \beta(1-2\alpha)] \frac{m^{n-1}}{(n-1)!} + \sum_{n=2}^{\infty} 2\beta(1-\alpha) \frac{m^{n-1}}{(n-1)!} \right] \\
= e^{-m} \left[ (1+\beta) \sum_{n=3}^{\infty} \frac{m^{n-1}}{(n-3)!} \\
+ 2[1+\beta(2-\alpha)] \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-2)!} + 2\beta(1-\alpha) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} \right] \\
= e^{-m} \left[ (1+\beta)m^2 e^m + 2(1+\beta(2-\alpha)) me^m + 2\beta(1-\alpha)(e^m - 1) \right] \\
= (1+\beta)m^2 + 2(1+\beta(2-\alpha))m + 2\beta(1-\alpha)(1-e^{-m}).
\]
But this last expression is bounded by \(2\beta(1-\alpha)\), if and only if (2.2) holds. This completes the proof of Theorem 2.2. \(\square\)

3. Inclusion results

**Theorem 3.1.** Let \(m > 0\), \(0 \leq \alpha < 1\) and \(0 < \beta \leq 1\). If \(f \in \mathcal{R}^\tau(A, B)\), then \(\mathcal{I}(m)f \in \mathcal{I}^*(\alpha, \beta)\) if and only if
\[
(A - B)|\tau| \left[ (1+\beta)(1-e^{-m}) + \frac{(\beta(1-2\alpha) - 1)}{m} (1-e^{-m} - me^{-m}) \right] \leq 2\beta(1-\alpha). \tag{3.1}
\]

**Proof.** In view of Lemma 1.1, it suffices to show that
\[ P_1 = \sum_{n=2}^{\infty} [n(1+\beta) - 1 + \beta(1-2\alpha)] \frac{m^{n-1}}{(n-1)!} e^{-m} |a_n| \leq 2\beta(1-\alpha). \]
Since $f \in \mathcal{R}^\tau (A, B)$, then by Lemma 1.3, we have

$$|a_n| \leq \frac{(A - B)|\tau|}{n}.$$ 

Therefore,

$$P_1 \leq \sum_{n=2}^{\infty} n(1 + \beta) - 1 + \beta(1 - 2\alpha)] \frac{m^{n-1}}{(n-1)!} e^{-m} \frac{(A - B)|\tau|}{n}$$

$$= (A - B)|\tau| e^{-m} \sum_{n=2}^{\infty} n(1 + \beta) - 1 + \beta(1 - 2\alpha)] \frac{m^{n-1}}{n!}$$

$$= (A - B)|\tau| e^{-m} \left[ (1 + \beta) \sum_{n=2}^{\infty} m^{n-1} \frac{1}{(n-1)!} + \frac{\beta(1 - 2\alpha) - 1}{m} \sum_{n=2}^{\infty} \frac{m^n}{n!} \right]$$

$$= (A - B)|\tau| \left[ (1 + \beta)(e^m - 1) + \frac{\beta(1 - 2\alpha) - 1}{m}(e^m - 1 - m) \right]$$

$$= (A - B)|\tau| \left[ (1 + \beta)[1 - e^{-m}] + \frac{\beta(1 - 2\alpha) - 1}{m}(1 - e^{-m} - me^{-m}) \right].$$

But this last expression is bounded by $2\beta(1 - \alpha)$, if (3.1) holds. This completes the proof of Theorem 3.1. \hfill \square

**Theorem 3.2.** Let $m > 0$, $0 \leq \alpha < 1$ and $0 < \beta \leq 1$. If $f \in \mathcal{R}^\tau (A, B)$, then $\mathcal{F}(m)f \in \mathcal{C}^*(\alpha, \beta)$ if and only if

$$(A - B)|\tau| \left[ m(1 + \beta) + 2\beta(1 - \alpha)(1 - e^{-m}) \right] \leq 2\beta(1 - \alpha). \quad (3.2)$$

**Proof.** In view of Lemma 1.2, it suffices to show that

$$P_2 = \sum_{n=2}^{\infty} n[n(1 + \beta) - 1 + \beta(1 - 2\alpha)] \frac{m^{n-1}}{(n-1)!} e^{-m} |a_n| \leq 2\beta(1 - \alpha).$$

Since $f \in \mathcal{R}^\tau (A, B)$, then by Lemma 1.3, we have

$$|a_n| \leq \frac{(A - B)|\tau|}{n}.$$
Therefore,

\[
P_2 \leq \sum_{n=2}^{\infty} n[n(1 + \beta) - 1 + \beta(1 - 2\alpha)] \frac{m^{n-1}}{(n-1)!} e^{-m} \frac{(A - B)|\tau|}{n}
\]

\[
= (A - B)|\tau|e^{-m} \sum_{n=2}^{\infty} n[1 + \beta - 1 + \beta(1 - 2\alpha)] \frac{m^{n-1}}{(n-1)!}
\]

\[
= (A - B)|\tau|e^{-m} \sum_{n=2}^{\infty} [(n - 1)(1 + \beta) + 2\beta(1 - \alpha)] \frac{m^{n-1}}{(n-1)!}
\]

\[
= (A - B)|\tau|e^{-m} \left[ (1 + \beta) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-2)!} + 2\beta(1 - \alpha) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} \right]
\]

\[
= (A - B)|\tau|e^{-m} \left[ m^{\infty}+1 + 2\beta(1 - \alpha)(e^{m} - 1) \right].
\]

But this last expression is bounded by \(2\beta(1 - \alpha)\), if (3.2) holds. This completes the proof of Theorem 3.2. \(\square\)

4. An integral operator

Theorem 4.1. If \(m > 0\), \(0 \leq \alpha < 1\) and \(0 < \beta \leq 1\), then

\[
\mathcal{G}(m, z) = \int_{0}^{z} \frac{\mathcal{F}(m, t)}{t} dt
\]

is in \(\mathcal{G}^*(\alpha, \beta)\) if and only if inequality (2.1) is satisfied.

Proof. Since

\[
\mathcal{G}(m, z) = z - \sum_{n=2}^{\infty} \frac{e^{-m} m^{n-1}}{(n-1)!} \frac{z^n}{n} = z - \sum_{n=2}^{\infty} \frac{e^{-m} m^{n-1}}{n!} \frac{z^n}{n}
\]

by Lemma 1.2, we need only to show that

\[
\sum_{n=2}^{\infty} n[n(1 + \beta) - 1 + \beta(1 - 2\alpha)] \frac{m^{n-1}}{n!} e^{-m} \leq 2\beta(1 - \alpha).
\]

Let

\[
Q_1 = \sum_{n=2}^{\infty} n[n(1 + \beta) - 1 + \beta(1 - 2\alpha)] \frac{m^{n-1}}{n!} e^{-m}.
\]
Now,
\[
Q_1 = \sum_{n=2}^{\infty} [n(1 + \beta) - 1 + \beta(1 - 2\alpha)] \frac{m^{n-1}}{(n-1)!} e^{-m}
\]
\[
= e^{-m} \sum_{n=2}^{\infty} [(n-1)(1 + \beta) + 2\beta(1 - \alpha)] \frac{m^{n-1}}{(n-1)!}
\]
\[
= e^{-m} \left[ (1 + \beta) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-2)!} + 2\beta(1 - \alpha) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} \right]
\]
\[
= e^{-m} [(1 + \beta)m e^m + 2\beta(1 - \alpha)(e^m - 1)]
\]
\[
= (1 + \beta)m + 2\beta(1 - \alpha)(1 - e^{-m}).
\]
But this last expression is bounded by \(2\beta(1 - \alpha)\), if and only if (2.1) holds. This completes the proof of Theorem 4.1. \(\square\)

**Theorem 4.2.** If \(m > 0\), \(0 \leq \alpha < 1\) and \(0 < \beta \leq 1\), then
\[
G(m, z) = \int_0^z \frac{F(m,t)}{t} dt
\]
is in \(\mathcal{S}^{\ast}(\alpha, \beta)\) if and only if
\[
(1 + \beta)(1 - e^{-m}) + \frac{(\beta(1 - 2\alpha) - 1)}{m} (1 - e^{-m} - me^{-m}) \leq 2\beta(1 - \alpha).
\]

The proof of Theorem 4.2 is lines similar to the proof of Theorem 4.1, so we omitted the proof of Theorem 4.2.

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**References**


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