Spectral characterization of new classes of multicone graphs

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Abstract. This paper deals with graphs that are known as multicone graphs. A multicone graph is a graph obtained from the join of a clique and a regular graph. Let \( w, l, m \) be natural numbers and \( k \) is a natural number. It is proved that any connected graph cospectral with multicone graph \( K_w \triangleleft mECP_l^k \) is determined by its adjacency spectra as well as its Laplacian spectra, where \( ECP_l^k = K_3^k, K_3^k, \ldots, K_3^k \) \( l \) times. Also, we show that complements of some of these multicone graphs are determined by their adjacency spectra. Moreover, we prove that any connected graph cospectral with these multicone graphs must be perfect. Finally, we pose two problems for further researches.

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1. Introduction

All graphs considered here are simple and undirected. All notions on graphs that are not defined here can be found in [4, 5, 10, 12, 19]. Let \( \Gamma \) be a graph with \( n \) vertices, \( V(\Gamma) \) and \( E(\Gamma) \) be the sets of vertices and edges of \( \Gamma \), respectively. The complement of a graph \( \Gamma \), denoted by \( \overline{\Gamma} \), is the graph on the vertices set of \( \Gamma \) such that two vertices of \( \overline{\Gamma} \), are adjacent if and only if they are not adjacent in \( \Gamma \). The union of (disjoint) graphs \( \Gamma_1 \) and \( \Gamma_2 \) is denoted by \( \Gamma_1 \cup \Gamma_2 \), is the graph whose vertices (respectively, edges) set is the union of vertices (respectively, edges) set of \( \Gamma_1 \) and \( \Gamma_2 \). A graph consisting of \( k \) disjoint copies of an arbitrary graph \( \Gamma \) will be denoted by \( k\Gamma \). The join of two vertex disjoint graphs \( \Gamma_1 \) and \( \Gamma_2 \) is the graph obtained from \( \Gamma_1 \cup \Gamma_2 \) by joining each vertex in \( \Gamma_1 \) with every vertex in \( \Gamma_2 \). It is denoted by \( \Gamma_1 \nabla \Gamma_2 \). Let \( \Gamma \) be a graph with adjacency matrix \( A(\Gamma) \). The characteristic polynomial of \( \Gamma \) is \( \det(\lambda I - A(\Gamma)) \), and denoted by \( P_\Gamma(\lambda) \). The roots of \( P_\Gamma(\lambda) \) are called the adjacency eigenvalues of \( A(\Gamma) \). The eigenvalues and the spectrum of \( A(\Gamma) \) are also called the eigenvalues and the
spectrum of $\Gamma$, respectively. If we consider a matrix $L = D - A$ instead of $A$, where $D$ is the diagonal matrix of degree of vertices (in $\Gamma$), we get the Laplacian eigenvalues and the Laplacian spectrum, while in the case of matrix $SL(G) = D(\Gamma) + A(\Gamma)$, we get the signless Laplacian eigenvalues and the signless Laplacian spectrum, respectively. Since both matrices $A(\Gamma)$ and $L(\Gamma)$ are real symmetric matrices, their eigenvalues are all real numbers. Let $\lambda_1, \lambda_2, \ldots, \lambda_s$ be the distinct eigenvalues of $\Gamma$ with multiplicities $m_1, m_2, \ldots, m_s$, respectively. We denote the adjacency spectrum of $\Gamma$ by $\text{Spec}(\Gamma) = \{[\lambda_1]^{m_1}, [\lambda_2]^{m_2}, \ldots, [\lambda_s]^{m_s}\}$. Two graphs $\Gamma$ and $\Lambda$ are called cospectral, if $\text{Spec}(\Gamma) = \text{Spec}(\Lambda)$. A graph $\Gamma$ is said to be determined by its spectrum or DS for short, if $\text{Spec}(\Gamma) = \text{Spec}(\Lambda)$, follows that $\Gamma \cong \Lambda$. About the background of the question "which graphs are determined by their spectrums?", we refer to [15]. The friendship graph $F_n$ consists of $n$ edge-disjoint triangles that all of them meeting in one vertex, where $n$ is a natural number (see Figure 1). The friendship (or Dutch windmill or n-fan) graph $F_n$ is the graph that can be constructed by coalescing $n$ copies of the cycle graph $C_3$ of length 3 with a common vertex. By construction, the friendship graph $F_n$ is isomorphic to the windmill graph $Wd(3,n)$ [11]. The friendship theorem of Paul Erdős, Alfred Réyni and Vera T. Sós [12], states that graphs with the property that every two vertices have exactly one neighbour in common are exactly the friendship graphs. In [17, 18], it has been proposed that the friendship graph is DS with respect to its adjacency spectrum. This conjecture studied in [2, 8]. It is claimed in [8] that conjecture is valid. In [7], it is proved that if $\Gamma$ is any graph cospectral with $F_n$ ($n \neq 16$), then $\Gamma \cong F_n$. Abdollahi and Janbaz [3] presented a proof in special case of this topic. They proved that any connected graph cospectral with $F_n$ is isomorphic to $F_n$. Abdian and Mirafzal [1] characterized new classes of multicone graphs. In this paper, we present new classes of multicone graphs that friendship graphs are special classes of them and we show these graphs are DS with respect to their spectra. The plan of the present paper is as follows. In Section 2, we review some basic information and preliminaries. In Subsection 3.1, we show that any connected graph cospectral with multicone graph $K_w \heartsuit mECP^k_l$ (see Figures 1 and 2, for example) must be regular or bidegreed (Lemma 3.2). In Subsection 3.2, we prove that any connected graphs cospectral with $K_w \heartsuit mECP^k_l$ is determined by its adjacency spectra (Theorem 3.4). In Subsection 3.3, we prove that complement of $K_w \heartsuit mECP^k_l$ is DS with respect to their adjacency spectra (Theorem 3.7). In Subsection 3.4, we show that graphs $K_w \heartsuit mECP^k_l$ are DS with respect to their Laplacian spectra (Theorem 3.8). In Subsection 3.5, we show that any connected graph cospectral with multicone graph $K_w \heartsuit mECP^k_l$ must be perfect. We conclude with final remarks and open problems in Section 4.

2. Preliminaries

In this section, we give some facts that will be used in the proof of the main results.

A walk of length $m$ in a graph $\Gamma(V,E)$ is an alternating sequence:

$$v_1l_1v_2l_2v_3l_3\ldots v_ml_{m+1}$$
of vertices and edges that begins and ends with a vertex and has the added property that \(l_j\) is incident with both \(v_i\) and \(v_{i+1}\), where \(1 \leq i \leq m + 1\) and \(1 \leq j \leq m\). In graph \(\Gamma(V,E)\) a walk of length \(m\) is closed, if \(v_1 = v_{m+1}\).

**Lemma 2.1.** ([2, 14]) Let \(\Gamma\) be a graph. For the adjacency matrix and Laplacian matrix, the following can be obtained from the spectrum:

(i) The number of vertices,
(ii) The number of edges.

For the adjacency matrix, the following follows from the spectrum:

(iii) The number of closed walks of any length.
(iv) Being regular or not and the degree of regularity.
(v) Being bipartite or not.

For the Laplacian matrix, the following follows from the spectrum:

(vi) The number of spanning trees.
(vii) The number of components.
(viii) The sum of squares of degrees of vertices.

**Theorem 2.2.** ([5]) If \(\Gamma_1\) is \(r_1\)-regular with \(n_1\) vertices, and \(\Gamma_2\) is \(r_2\)-regular with \(n_2\) vertices, then the characteristic polynomial of the join \(\Gamma_1 \vee \Gamma_2\) is given by:

\[
P_{\Gamma_1 \vee \Gamma_2}(\lambda) = \frac{P_{\Gamma_1}(\lambda)P_{\Gamma_2}(\lambda)}{(\lambda - r_1)(\lambda - r_2)}((\lambda - r_1)(\lambda - r_2) - n_1n_2).
\]

**Proposition 2.3.** ([5]) Let \(\Gamma - j\) be the graph obtained from \(\Gamma\) by deleting the vertex \(j\) and all edges containing \(j\). Then \(P_{\Gamma - j}(\lambda) = P_{\Gamma}(\lambda) \sum_{i=1}^{m} \frac{\alpha_{ij}^2}{\lambda - \mu_i}\), where \(m\), \(\alpha_{ij}^2\) and \(P_{\Gamma}(\lambda)\) are the number of distinct eigenvalues of graph \(\Gamma\), the main angle of \(\Gamma\) and the characteristic polynomial of \(\Gamma\).

A graph is bidegreed if the set of degrees of its vertices consists of exactly two distinct elements. Also, the spectral radius \(\varrho(\Gamma)\) of \(\Gamma\) is the largest eigenvalue of its adjacency matrix \(A(\Gamma)\).

**Theorem 2.4.** ([3]) Let \(\Gamma\) be a simple graph with \(n\) vertices and \(m\) edges. Let \(\delta = \delta(\Gamma)\) be the minimum degree of vertices of \(\Gamma\) and \(\varrho(\Gamma)\) be the spectral radius of the adjacency matrix of \(\Gamma\). Then

\[
\varrho(\Gamma) \leq \frac{\delta - 1}{2} + \sqrt{2m - n\delta + \frac{(\delta + 1)^2}{4}}.
\]

Equality holds if and only if \(\Gamma\) is either a regular graph or a bidegreed graph in which each vertex is of degree either \(\delta\) or \(n - 1\).

A \(t\)-multipartite graph of order \(n\) is \(K_{b_1,\ldots,b_t}\), where \(b_1 + \ldots + b_t = n\). D. Cvetković, Doob and S. Simić [6] defined a generalized cocktail-party graph, denoted by \(GCP\), as a complete graph with some independent edges removed. A special case of this graph is the well-known cocktail-party graph \(CP(t)\) obtained from \(K_{2t}\) by removing \(t\) disjoint edges.

**Theorem 2.5.** ([1]) A graph has exactly one positive eigenvalue if and only if its non-isolated vertices form a complete multipartite graph.
Lemma 2.6. ([1]) Let $\Gamma$ be a connected non-regular graph with three distinct eigenvalues $\theta_0 > \theta_1 > \theta_2$. Then the following hold:

(i) $\Gamma$ has diameter two.
(ii) If $\theta_0$ is not an integer, then $\Gamma$ is complete bipartite.
(iii) $\theta_1 \geq 0$ with equality if and only if $\Gamma$ is complete bipartite.
(iv) $\theta_2 \leq -\sqrt{2}$ with equality if and only if $\Gamma$ is the path of length 2.

Proposition 2.7. ([12]) For a graph $\Gamma$, the following statements are equivalent:

(i) $\Gamma$ is $d$-regular.
(ii) $\varrho(\Gamma) = d_\Gamma$, the average vertex degree.
(iii) $G$ has $v = (1, 1, \ldots, 1)^t$ as an eigenvector for $\varrho(\Gamma)$.

Proposition 2.8. ([16]) Let $\Gamma$ be a disconnected graph that is determined by the Laplacian spectrum. Then the cone over $\Gamma$, the graph $\Lambda$; that is, obtained from $\Gamma$ by adding one vertex that is adjacent to all vertices of $\Gamma$, is also determined by its Laplacian spectrum.

Lemma 2.9. ([13]) Let $\Gamma$ be a graph on $n$ vertices. Then $n$ is Laplacian eigenvalue of $\Gamma$ if and only if $\Gamma$ is the join of two graphs.

Theorem 2.10. ([13]) Let $\Gamma$ and $\Lambda$ be two graphs with Laplacian spectrum $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ and $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_m$, respectively. Then the Laplacian spectra of $\overline{\Gamma}$ and $\Gamma \vartriangledown \Lambda$ are $n - \lambda_1, n - \lambda_2, \ldots, n - \lambda_{n-1}, 0$ and $n + m, m + \lambda_1, \ldots, m + \lambda_{n-1}, n + \mu_1, \ldots, n + \mu_{m-1}, 0$, respectively.

Lemma 2.11. ([12]) Let $G \neq K_1$ be connected with $P_1(\lambda) = \sum_{i=0}^{n} a_i \lambda^{n-i}$ and $\lambda = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n = \varrho(\Gamma)$, where $P_1(\lambda)$ is the characteristic polynomial of graph $\Gamma$ and $\lambda_i$ $(1 \leq i \leq n)$ is eigenvalue of $\Gamma$. The following are equivalent:

(i) $G$ is bipartite.
(ii) $a_{2i-1} = 0$ for all $1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil$.
(iii) $\lambda_i = -\lambda_{n+1-i}$ for $1 \leq i \leq n$.
(iv) $\varrho(\Gamma) = -\lambda$.

Moreover, $m(\lambda_i) = m(-\lambda_i)$, where $m(\lambda_i)$ denote the multiplicities of $\lambda_i$.

3. Main results

In the following, we show that any connected graph cospectral with multicone graphs $K_w \vartriangledown mECP^k_l$ are regular or bidegreed.

3.1. Connected bidegreed graph cospectral with multicone graphs $K_w \vartriangledown mECP^k_l$

Proposition 3.1. Let $G$ be a graph cospectral with multicone graphs $K_w \vartriangledown mECP^k_l$. Then

$$\text{Spec}(G) = \left\{ [0]^{(3^k l - 1)m}, [-1]^{w-1}, [3^k l - 3^k]^{m-1}, [-3^k]^{m-m}, [\frac{\chi + \sqrt{\chi^2 - 4\Theta}}{2}]^{1}, [\frac{\chi - \sqrt{\chi^2 - 4\Theta}}{2}]^{1} \right\},$$

where $\chi = w - 1 + 3^k l - 3^k$ and $\Theta = (w - 1)(3^k l - 3^k) - 3^k l w m$. 

Proposition 2.3 that
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Any connected graph cospectral with multicone graph
Lemma 3.3.
Proof. Let \( \Gamma \) be a graph cospectral with multicone graph \( K_w \vee mECP_i^k \). Then \( \Gamma \) is bidegreed in which any vertex of \( \Gamma \) is of degree \( w - 1 + 3^k l m \) or \( 3^k l - 3^k + w \).

Proof. It is obvious that \( \Gamma \) cannot be regular; since regularity of a graph can be determined by its spectrum. By contrary, we suppose that the sequence of degrees of vertices of graph \( \Gamma \) consists of at least three number. Hence the equality in Theorem 2.4 cannot happen for any \( \delta \). But, if we put \( \delta = 3^k l - 3^k + w \), then the equality in Theorem 2.4 holds. So, \( \Gamma \) must be bidegreed. Now, we show that \( \Delta = \Delta(\Gamma) = w - 1 + 3^k l m \). By contrary, we suppose that \( \Delta < w - 1 + 3^k l m \). Therefore, the equality in Theorem 2.4 cannot hold for any \( \delta \). But, if we put \( \delta = 3^k l - 3^k + w \), then this equality holds. This is a contradiction and so \( \Delta = 3^k l - 3^k + w \). Now, \( \delta = 3^k l - 3^k + w \), since \( \Gamma \) is bidegreed and \( \Gamma \) has \( w + 3^k l m \), \( \Delta = w - 1 + 3^k l m \) and
\[ w(w - 1 + 3^k l m) + 3^k l m(3^k l - 3^k + w) = w\Delta + 3^k l m(3^k l - 3^k + w) = \sum_{i=1}^{w+3^k l m} \deg v_i. \]
This completes the proof.

3.2. Spectral characterization of connected graphs cospectral with multicone graphs
\( K_1 \vee mECP_i^k \).

In this subsection, we show that multicone graphs \( K_1 \vee mECP_i^k \) are DS.

Lemma 3.3. Any connected graph cospectral with multicone graph \( K_1 \vee mECP_i^k \) is isomorphic to \( K_1 \vee mECP_i^k \).

Proof. Let \( \Gamma \) be a graph cospectral with multicone graph \( K_1 \vee mECP_i^k \). If \( m = 1 \) there is nothing to prove. Hence we suppose that \( m \neq 1 \). It is obvious that in this case \( \Gamma \) cannot be regular. First we show that \( \Gamma \) has one vertex of degree \( \Delta = 3^k l m \) and \( 3^k l m \) vertices of degree \( \delta = 3^k l - 3^k + 1 \). Let \( G \) has \( t \) vertex of degree \( \Delta = 3^k l m \). Hence
\[ t3^k l m + (3^k l m + 1 - t)(3^k l - 3^k + 1) = 3^k l m + 3^k l m(3^k l - 3^k + 1) = \sum_{i=1}^{1+3^k l m} \deg v_i \]
and so \( t = 1 \). Therefore, \( \Gamma \) has one vertex of degree \( \Delta = 3^k l m \), say \( j \). It follows from Proposition 2.3 that
\[ P_{r-j}(\lambda) = (\lambda - \mu_3)^{m-2}(\lambda - \mu_4)^{lm-m-1}(\lambda - \mu_5)^{3^k l m - lm - 1} \times [\alpha_{1j}^2 F + \alpha_{2j}^2 G + \alpha_{3j}^2 H + \alpha_{4j}^2 I + \alpha_{5j}^2 J], \]
where
\[ F = (\lambda - \mu_2)(\lambda - \mu_3)(\lambda - \mu_4)(\lambda - \mu_5), \]
\[ G = (\lambda - \mu_1)(\lambda - \mu_3)(\lambda - \mu_4)(\lambda - \mu_5), \]
where
\[ H = (\lambda - \mu_1)(\lambda - \mu_2)(\lambda - \mu_4)(\lambda - \mu_5), \]
\[ I = (\lambda - \mu_1)(\lambda - \mu_2)(\lambda - \mu_3)(\lambda - \mu_5), \]
\[ J = (\lambda - \mu_1)(\lambda - \mu_2)(\lambda - \mu_3)(\lambda - \mu_4), \]
\[
\begin{align*}
\mu_1 &= \frac{3^k l - 3^k + \sqrt{(3^k l - 3^k)^2 + 4(2^k l m)}}{2}, \\
\mu_2 &= \frac{3^k l - 3^k - \sqrt{(3^k l - 3^k)^2 + 4(2^k l m)}}{2}, \\
\mu_3 &= 3^k l - 3^k, \quad \mu_4 = -3^k \text{ and } \mu_5 = 0.
\end{align*}
\]

It is clear that \( P_{\Gamma-j}(\lambda) \) has \( 3^k l m \) roots. So, we have:
\[
\begin{align*}
\alpha + \beta + \gamma + 3^k l - 3^k &= -(m-2)\mu_3 + (lm - m - 1)\mu_4, \\
\alpha^2 + \beta^2 + \gamma^2 + (3^k l - 3^k)^2 &= 3^k l m (3^k l - 3^k) - [(m-2)\mu_3^2 + (lm - m - 1)\mu_4^2], \\
\alpha^3 + \beta^3 + \gamma^3 + (3^k l - 3^k)^3 &= 6m(3^k) (\frac{5}{3}) - [(m-2)\mu_3^3 + (lm - m - 1)\mu_4^3],
\end{align*}
\]

where \( \alpha, \beta \) and \( \gamma \) are the eigenvalues of graph \( \Gamma - j \). If we solve the above equations, then we will have: \( \alpha = -3^k, \beta = 0 \) and \( \gamma = 3^k l - 3^k \). Therefore,
\[
\text{spec}(\Gamma - j) = \left\{ [3^k l - 3^k]^m, [0]^{3^k l m - lm}, [-3^k]^{lm - m} \right\}.
\]

Graph \( \Gamma - j \) is regular and degree of its regularity is \( 3^k l - 3^k \). It follows from Theorem 2.4 that \( \Gamma - j = mK_{3^k, \ldots, 3^k} \) and so \( G - j = m\text{ECP}_l^k \). Hence \( \Gamma = K_1 \triangledown m\text{ECP}_l^k \).

This follows the result. \( \square \)

Up to now, we have shown that the multicone graphs \( K_1 \triangledown m\text{ECP}_l^k \) are DS. The natural question is; what happen for multicone graphs \( K_w \triangledown m\text{ECP}_l^k \)? we answer to this question in the following theorem.

**Theorem 3.4.** Any connected graph cospectral with multicone graph \( K_w \triangledown m\text{ECP}_l^k \) is isomorphic to \( K_w \triangledown m\text{ECP}_l^k \).

**Proof.** We solve the problem by induction on \( w \). If \( w = 1 \), there is nothing to prove. Let the claim be true for \( w \); that is, if \( \text{Spec}(\Gamma_1) = \text{Spec}(K_w \triangledown m\text{ECP}_l^k) \), then \( \Gamma_1 \cong K_w \triangledown m\text{ECP}_l^k \), where \( \Gamma_1 \) is a graph. We show that, if \( \text{Spec}(\Gamma) = \text{Spec}(K_{w+1} \triangledown m\text{ECP}_l^k) \), then \( \Gamma \cong K_{w+1} \triangledown m\text{ECP}_l^k \), where \( \Gamma \) is a graph. By Lemma 3.2, Theorem 2.4, Lemma 2.1 (iii) and in a similar manner of Lemma 3.3 for \( \Gamma - j \), where \( j \) is a vertex of degree \( w + 3^k l m \) belonging to \( \Gamma \), we obtain \( \text{Spec}(\Gamma - j) = \text{Spec}(K_w \triangledown m\text{ECP}_l^k) \). Therefore, the assertion holds. \( \square \)

In the following, we give another proof of the above theorem.
Proof. Let $\Gamma$ be a connected graph cospectral with multicone graph $K_w \uplus mECP^k$. By Lemma 3.2, $\Gamma$ has subgraph $L$ in which degree of any vertex of $L$ is $w - 1 + 3^k lm$. In other words, $\Gamma \cong K_w \uplus H$, where $H$ is a subgraph of $\Gamma$. Now, we remove the vertices of $K_w$ and we consider $3^k lm$ another vertices. Consider $H$ consisting of these $3^k lm$ vertices. $H$ is regular and degree of its regularity is $3^k l - 3^k$ and multiplicity of $3^k l - 3^k$ is $m$. By Theorem 2.2, $\text{Spec}(H) = \left\{ [3^k l - 3^k]^m, [0](3^k l - 1)^m, [-3^k]^m \right\}$. Now, it follows from Theorem 2.5 that $\text{Spec}(H) = \text{Spec}(mECP^k)$. This implies the result.

Corollary 3.5. Any connected graph cospectral with multicone graph

$$K_w \uplus mECP^k = K_w \uplus mK_3^k$$

is DS with respect to their adjacency spectrums.
3.3. Some complements of multicone graphs $K_w \triangledown mECP^k_l$ are DS with respect to their spectra.

In this subsection, we show that the complement of multicone graphs $K_w \triangledown mECP^k_l$ are DS with respect to their adjacency spectrum.

**Proposition 3.6.** Let $\Gamma$ be cospectral with complement of multicone graphs $K_w \triangledown mECP^k_l$. Then

$$\text{Spec}(\Gamma) = \left\{ \left[ 3^{k-1}lm - 3^k l + 3^k - 1 \right]^m, [-1]^{(3^k-1)l}, [3^k - 1]^{(l-1)m}, [0]^w \right\}.$$ 

**Proof.** Straightforward. \hfill $\square$

**Theorem 3.7.** The complement of multicone graph $K_w \triangledown ECP^k_l$ are DS with respect to their adjacency spectrum.

**Proof.** The proof of this theorem is the similar of Theorm 5.2 of [1]. Let

$$\text{Spec}(\Gamma) = \text{Spec}(K_w \triangledown ECP^k_l) = \left\{ [-1]^{(3^k-1)l}, [3^k - 1]^l, [0]^w \right\}.$$
If \( l = 1 \), by Lemma 2.1 ((i), (ii) and (iii)) the proof is clear (Also, by Theorem 2.5 the proof follows). Hence we suppose that \( l \neq 1 \). It is easy to see that \( \Gamma \) cannot be regular, since regularity of a graph can be determined by its spectrum. By contrary, we suppose that \( \Gamma \) is connected. So, we from Lemma 2.6 and Lemma 2.11 conclude that \( k = l = 1 \). This is a contradiction. Hence \( \Gamma \) cannot be regular, since regularity of a graph can be determined by its spectrum. By contrary, we suppose that \( \Gamma \) is connected. So, we from Lemma 2.6 and Lemma 2.11 conclude that \( k = l = 1 \). This is a contradiction. Hence \( \Gamma = \Gamma_1 \cup \Gamma_2 \cup \ldots \cup \Gamma_h \), where \( \Gamma_s \) is a connected component of \( \Gamma \) and \( 1 \leq s \leq h \). Now, we show that \( \Gamma_s \) cannot have three distinct eigenvalues. By contrary, we suppose that \( \Gamma_s \) has three distinct eigenvalues. In this case, if we also suppose \( \Gamma_s \) is non-regular, then it follows from Lemma 2.6 that \( \Gamma_s \) is a complete bipartite graph. Hence \( l = k = 1 \). This is a contradiction. Therefore, if \( \Gamma_s \) has three distinct eigenvalues, then it must be regular. Now, it follows from Theorem 2.5 that \( \Gamma_s \approx K_{1,1,\ldots,1} \). This is a contradiction. So, \( \Gamma_s \) cannot have three distinct eigenvalues. Therefore, it has one or two eigenvalue(s). Hence, any connected component of \( \Gamma \) is either isolated vertex or a complete graph. Hence \( \Gamma \approx \hat{w}K_1 \cup lK_3 \). This follows the result.

\[ \square \]

3.4. The multicone graphs \( K_w \triangledown mECP_l^k \) are determined by their Laplacian spectra

In this subsection, we show that any graph cospectral with multicone graph \( K_w \triangledown mECP_l^k \) is DS with respect to its Laplacian spectrum.

**Theorem 3.8.** Multicone graphs \( K_w \triangledown mECP_l^k \) are DS with respect to their Laplacian spectrum.

**Proof.** We solve the problem by induction on \( w \). If \( w = 1 \), there is nothing to prove. Let the claim be true for \( w \); that is,

\[
\text{Spec}(L(H)) = \text{Spec}(L(K_w \triangledown mECP_l^k))
\]

\[
= \left\{ [3^k l m + w]^w, [w]^{m-1}, [3^k l - 3^k + w]^{3^k l m - lm}, [3^k l + w]^{lm - m}, [0]^1 \right\}
\]

follows that \( H \cong K_w \triangledown mECP_l^k \). We show that the problem is true for \( w + 1 \); that is, we show that

\[
\text{Spec}(L(G)) = \text{Spec}(L(K_{w+1} \triangledown mECP_l^k))
\]

\[
= \left\{ [3^k l m + w + 1]^{w+1}, [w + 1]^{m-1}, [3^k l - 3^k + w + 1]^{3^k l m - lm}, [3^k l + w + 1]^{lm - m}, [0]^1 \right\}
\]

follows that \( G \cong K_{w+1} \triangledown mECP_l^k \). It follows from Lemma 2.9 that \( H \) and \( G \) are the join of two graphs. On the other hand,

\[
\text{Spec}(L(K_1 \triangledown H)) = \text{Spec}(L(G)) = \text{spec}(L(K_{w+1} \triangledown mECP_l^k)).
\]

Therefore, we must have \( G \cong K_1 \triangledown H \). Because, \( G \) is the join of two graphs and also according to spectrum of \( G \), must \( K_1 \) be joined to \( H \) and this is only available state. This completes the proof. \( \square \)

**Corollary 3.9.** Multicone graphs \( K_w \triangledown mECP_l^k = K_w \triangledown mK_3^k \) are DS with respect to their Laplacian spectra.
3.5. Some results about multicone graphs $K_w \triangle mECP^k_l$

In this subsection, we show that any graph cospectral with multicone graph $K_w \triangle mECP^k_l$ must be perfect. Also, we prove that any graph cospectral with multicone graph $K_w \triangle mECP^k_l$ with respect to Laplacian spectrum is perfect. In addition, we show that any graph cospectral with complement of multicone graph $K_w \triangle mECP^k_l$ is perfect.

Suppose $\chi(\Gamma)$ and $\omega(\Gamma)$ are chromatic number and clique number of graph $G$, respectively. A graph is perfect if $\chi(H) = \omega(H)$ for every induced subgraph $H$ of $\Gamma$. It is proved that a graph $G$ is perfect if and only if $\Gamma$ is Berge; that is, it contains no odd hole or antihole as induced subgraph, where odd hole and antihole are odd cycle, $C_m$ for $m \geq 5$, and its complement, respectively. Also, in 1972 Lovász proved that, a graph is perfect if and only if its complement is perfect (see [22] of [2]). Now, by Theorem 3.4, Theorem 3.7, Theorem 3.8 and by what was said in the previous sections we can conclude the following results.

**Theorem 3.10.** Let graph $\Gamma$ be cospectral with multicone graph $K_w \triangle mECP^k_l$. Then $\Gamma$ and $\overline{\Gamma}$ are perfect.

*Proof.* By what was said in the beginning of this section and Theorem 3.4 the proof is completed. \hfill $\square$

**Theorem 3.11.** Let $\Gamma$ be a graph and $\text{Spec}(L(\Gamma)) = \text{Spec}(L(K_w \triangle mECP^k_l))$. Then $\Gamma$ and $\overline{\Gamma}$ are perfect.

*Proof.* The proof is straightforward. \hfill $\square$

**Theorem 3.12.** Let $\Gamma$ be a graph and $\text{Spec}(\Gamma) = \text{Spec}(\overline{K_w \triangle mECP^k_l})$. Then $\Gamma$ and $\overline{\Gamma}$ are perfect.

*Proof.* It is obvious. \hfill $\square$

In the following, we pose two conjectures.

4. Final remarks and open problems

In this paper, we have shown any connected graph cospectral with multicone graph $K_w \triangle mECP^k_l$ is DS with respect to its spectra. Also, we have shown in special cases complement of these graphs are DS. In addition, we have proved any connected graph cospectral with these graph is perfect. On the other hand, It is obvious that, $F_n$ are special classes of multicone graphs $K_w \triangle mECP^k_l$ (one can also consider $k = 0$). In addition, $F_n$ are DS with respect to:

- (i) Their adjacency spectrum (if $n \neq 16$).
- (ii) Their Laplacian spectrum.
- (iii) Their signless Laplacian spectrum. Also, $\overline{F_n}$ are DS with respect to their adjacency spectrum, where $n \neq 2$.

Hence we pose the following conjectures.

**Conjecture 4.1.** Multicone graphs $K_w \triangle mECP^k_l$ are DS with respect to their signless Laplacian spectrum.
Conjecture 4.2. The complement of multicone graphs $K_w \nabla mECP_k^l$ are DS with respect to their adjacency spectrum.

References

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