Helicoidal surfaces with $\Delta^J r = Ar$ in 3-dimensional Euclidean space

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Abstract. In this paper we study the helicoidal surfaces in the 3-dimensional Euclidean space under the condition $\Delta^J r = Ar; J = I, II, III$, where $A = (a_{ij})$ is a constant $3 \times 3$ matrix and $\Delta^J$ denotes the Laplace operator with respect to the fundamental forms $I$, $II$ and $III$.

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1. Introduction

Let $r = r(u, v)$ be an isometric immersion of a surface $M^2$ in the Euclidean space $\mathbb{E}^3$.

The inner product on $\mathbb{E}^3$ is

$$g(X, Y) = x_1y_1 + x_2y_2 + x_3y_3,$$

where $X = (x_1, x_2, x_3), Y = (y_1, y_2, y_3) \in \mathbb{R}^3$. The Euclidean vector product $X \wedge Y$ of $X$ and $Y$ is defined as follows:

$$X \wedge Y = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1).$$

The notion of finite type immersion of submanifolds of a Euclidean space has been widely used in classifying and characterizing well known Riemannian submanifolds [6]. B.-Y. Chen posed the problem of classifying the finite type surfaces in the 3-dimensional Euclidean space $\mathbb{E}^3$. An Euclidean submanifold is said to be of Chen finite type if its coordinate functions are a finite sum of eigenfunctions of its Laplacian $\Delta$ [6]. Further, the notion of finite type can be extended to any smooth functions on a submanifold of a Euclidean space or a pseudo-Euclidean space. Since then the theory of submanifolds of finite type has been studied by many geometers.

A well known result due to Takahashi [18] states that minimal surfaces and spheres are the only surfaces in $\mathbb{E}^3$ satisfying the condition

$$\Delta r = \lambda r, \ \lambda \in \mathbb{R}.$$
In [10] Ferrandez, Garay and Lucas proved that the surfaces of $E^3$ satisfying
\[ \Delta H = AH, \ A \in \text{Mat}(3, 3) \]
are either minimal, or an open piece of sphere or of a right circulaire cylindre.

In [7] M. Choi and Y. H. Kim characterized the minimal helicoid in terms of pointwise 1-type Gauss map of the first kind. In [2] M. Bekkar and H. Zoubir classified the surfaces of revolution with non zero Gaussian curvature $K_G$ in the 3-dimensional Lorentz-Minkowski space $E^3_1$, whose component functions are eigenfunctions of their Laplace operator, i.e.
\[ \Delta^{II} r_i = \lambda_i r_i, \ \lambda_i \in \mathbb{R}. \]

In [9] F. Dillen, J. Pas and L. Verstraelen proved that the only surfaces in $E^3$ satisfying
\[ \Delta r = Ar + B, \ A \in \text{Mat}(3, 3), \ B \in \text{Mat}(3, 1), \]
are the minimal surfaces, the spheres and the circular cylinders.

In [1] Ch. Baba-Hamed and M. Bekkar studied the helicoidal surfaces without parabolic points in $E^3_1$, which satisfy the condition
\[ \Delta^{II} r_i = \lambda_i r_i, \]
where $\Delta^{II}$ is the Laplace operator with respect to the second fundamental form.

In [13] G. Kaimakamis and B.J. Papantoniou classified the first three types of surfaces of revolution without parabolic points in the 3-dimensional Lorentz–Minkowski space, which satisfy the condition
\[ \Delta^{II} r = Ar, \ A \in \text{Mat}(3, 3). \]

We study helicoidal surfaces $M^2$ in $E^3$ which are of finite type in the sense of B.-Y. Chen with respect to the fundamental forms $I$, $II$ and $III$, i.e., their position vector field $r(u, v)$ satisfies the condition
\[ \Delta^J r = Ar; \ J = I, II, III, \]
where $A = (a_{ij})$ is a constant $3 \times 3$ matrix and $\Delta^J$ denotes the Laplace operator with respect to the fundamental forms $I$, $II$ and $III$. Then we shall reduce the geometric problem to a simpler ordinary differential equation system.

In [14] G. Kaimakamis, B.J. Papantoniou and K. Petoumenos classified and proved that such surfaces of revolution in the 3-dimensional Lorentz-Minkowski space $E^3_1$ satisfying
\[ \Delta^{III} \overrightarrow{r} = A \overrightarrow{r} \]
are either minimal or Lorentz hyperbolic cylinders or pseudospheres of real or imaginary radius, where $\Delta^{III}$ is the Laplace operator with respect to the third fundamental form. S. Stamatakis and H. Al-Zoubi in [17] classified the surfaces of revolution with non zero Gaussian curvature in $E^3$ under the condition
\[ \Delta^{III} r = Ar, \ A \in \text{Mat}(3, \mathbb{R}). \]

On the other hand, a helicoidal surface is well known as a kind of generalization of some ruled surfaces and surfaces of revolution in a Euclidean space $E^3$ or a Minkowski space $E^3_1$ ([5], [8], [12]).
2. Preliminaries

Let $\gamma : I \subset \mathbb{R} \to P$ be a plane curve in $\mathbb{E}^3$ and let $l$ be a straight line in $P$ which does not intersect the curve $\gamma$ (axis). A helicoidal surface in $\mathbb{E}^3$ is a surface invariant by a uniparametric group $G_{l,c} = \{ g_v : \mathbb{E}^3 \to \mathbb{E}^3 ; v \in \mathbb{R} \}$ of helicoidal motions. The motion $g_v$ is called a helicoidal motion with axis $l$ and pitch $c$. If we take $c = 0$, then we obtain a rotations group about the axis $l$.

A helicoidal surface in $\mathbb{E}^3$ which is spanned by the vector $(0, 0, 1)$ and with pitch $c \in \mathbb{R}^\ast$ as follows:

$$r(u, v) = \begin{pmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ 0 \\ \varphi(u) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ cv \end{pmatrix}, c \in \mathbb{R}^\ast.$$ 

Next, we will use the parametrization of the profile curve $\gamma$ as follows:

$$\gamma(u) = (u, 0, \varphi(u)).$$

Therefore, the surface $M^2$ may be parameterized by

$$r(u, v) = (u \cos v, u \sin v, \varphi(u) + cv) \quad (2.1)$$

in $\mathbb{E}^3$, where $(u, v) \in I \times [0, 2\pi], c \in \mathbb{R}^\ast$.

A surface $M^2$ is said to be of finite type if each component of its position vector field $r$ can be written as a finite sum of eigenfunctions of the Laplacian $\Delta$ of $M^2$, that is, if

$$r = r_0 + \sum_{i=1}^{k} r_i,$$

where $r_i$ are $\mathbb{E}^3$-valued eigenfunctions of the Laplacian of $(M^2, r)$: $\Delta r_i = \lambda_i r_i, \lambda_i \in \mathbb{R}, i = 1, 2, ..., k$ [6]. If $\lambda_i$ are different, then $M^2$ is said to be of $k$-type.

The coefficients of the first fundamental form and the second fundamental form are

$$E = g_{11} = g(r_u, r_u), F = g_{12} = g(r_u, r_v), G = g_{22} = g(r_v, r_v);$$

$$L = h_{11} = g(r_{uu}, N), M = h_{12} = g(r_{uv}, N), N = h_{22} = g(r_{vv}, N),$$

where $N$ is the unit normal vector to $M^2$.

The Laplace-Beltrami operator of a smooth function

$$\varphi : M^2 \to \mathbb{R}, (u, v) \mapsto \varphi(u, v)$$

with respect to the first fundamental form of the surface $M^2$ is the operator $\Delta^I$, defined in [15] as follows:

$$\Delta^I \varphi = \frac{-1}{\sqrt{|EG - F^2|}} \left[ \frac{\partial}{\partial u} \left( \frac{G \varphi_u - F \varphi_v}{\sqrt{|EG - F^2|}} \right) - \frac{\partial}{\partial v} \left( \frac{F \varphi_u - E \varphi_v}{\sqrt{|EG - F^2|}} \right) \right]. \quad (2.2)$$

The second differential parameter of Beltrami of a function

$$\varphi : M^2 \to \mathbb{R}, (u, v) \mapsto \varphi(u, v)$$
with respect to the second fundamental form of $M^2$ is the operator $\Delta^{II}$ which is defined by [15]

$$\Delta^{II} \varphi = \frac{-1}{\sqrt{|LN - M^2|}} \left[ \frac{\partial}{\partial u} \left( \frac{N \varphi_u - M \varphi_v}{\sqrt{|LN - M^2|}} \right) + \frac{\partial}{\partial v} \left( \frac{L \varphi_v - M \varphi_u}{\sqrt{|LN - M^2|}} \right) \right],$$  \hspace{1cm} (2.3)

where $LN - M^2 \neq 0$ since the surface has no parabolic points.

In the classical literature, one write the third fundamental form as

$$III = e^{11} du^2 + 2e^{12} dudv + e^{22} dv^2.$$

The second Beltrami differential operator with respect to the third fundamental form $III$ is defined by

$$\Delta^{III} = -\frac{1}{\sqrt{|e|}} \left( \frac{\partial}{\partial x^i} (\sqrt{|e|} e^{ij} \frac{\partial}{\partial x^j}) \right),$$  \hspace{1cm} (2.4)

where $e = \det(e_{ij})$ and $e^{ij}$ denote the components of the inverse tensor of $e_{ij}$.

If $r = r(u,v) = (r_1 = r_1(u,v), r_2 = r_2(u,v), r_3 = r_3(u,v))$ is a function of class $C^2$ then we set

$$\Delta^J r = (\Delta^J r_1, \Delta^J r_2, \Delta^J r_3); \hspace{1cm} J = I, II, III.$$

The mean curvature $H$ and the Gauss curvature $K_G$ are, respectively, defined by

$$H = \frac{1}{2(EG - F^2)}(EN + GL - 2FM)$$

and

$$K_G = \frac{LN - M^2}{EG - F^2}.$$

Suppose that $M^2$ is given by (2.1).

**3. Helicoidal surfaces with $\Delta^I r = Ar$ in $E^3$**

The main result of this section states that the only helicoidal surfaces $M^2$ of $E^3$ satisfying the condition

$$\Delta^I r = Ar$$  \hspace{1cm} (3.1)

on the Laplacian are open pieces of helicoidal minimal surfaces.

The coefficients of the first and the second fundamental forms are:

$$E = 1 + \varphi'^2, \hspace{0.5cm} F = c \varphi', \hspace{0.5cm} G = c^2 + u^2;$$  \hspace{1cm} (3.2)

$$L = \frac{u \varphi''}{W}, \hspace{0.5cm} M = -\frac{c}{W}, \hspace{0.5cm} N = \frac{u^2 \varphi'}{W},$$  \hspace{1cm} (3.3)

where $W = \sqrt{EG - F^2} = \sqrt{u^2(1 + \varphi'^2) + c^2}$ and the prime denotes derivative with respect to $u$.

The unit normal vector of $M^2$ is given by

$$N = \frac{1}{W} (u \varphi' \cos v - c \sin v, c \cos v + u \varphi' \sin v, -u).$$
From these we find that the mean curvature $H$ and the curvature $K_G$ of (3.2) are given by

$$H = \frac{1}{2W^3} \left( u^2 \varphi' (1 + \varphi'^2) + 2c^2 \varphi' + u \varphi'' (c^2 + u^2) \right)$$

and

$$K_G = \frac{1}{W^4} \left( u^3 \varphi' \varphi'' - c^2 \right). \quad (3.4)$$

If a surface $M^2$ in $\mathbb{E}^3$ has no parabolic points, then we have $u^3 \varphi' \varphi'' - c^2 \neq 0$.

The Laplacian $\Delta^I$ of $M^2$ can be expressed as follows:

$$\Delta^I = -\frac{1}{W^2} ((c^2 + u^2) \frac{\partial^2}{\partial u^2} - 2c \varphi' \frac{\partial^2}{\partial u \partial v} + (1 + \varphi'^2) \frac{\partial^2}{\partial v^2})$$

$$-\frac{1}{W^4} (u^3 (1 + \varphi'^2) + c^2 u (1 - \varphi'^2) - u^2 \varphi' \varphi'' (c^2 + u^2)) \frac{\partial}{\partial u}$$

$$-\frac{1}{W^4} (-c \varphi'' (c^2 + u^2) + cu \varphi' (1 + \varphi'^2)) \frac{\partial}{\partial v}.$$

Accordingly, we get

$$\Delta^I r = -2H N. \quad (3.5)$$

The equation (3.1) by means of (3.2) and (3.5) gives rise to the following system of ordinary differential equations

$$(u \varphi' A(u) - a_{11} u) \cos v - (c A(u) + a_{12} u) \sin v = a_{13} (\varphi + cv) \quad (3.6)$$

$$(u \varphi' A(u) - a_{22} u) \sin v + (c A(u) - a_{21} u) \cos v = a_{23} (\varphi + cv) \quad (3.7)$$

$$-u A(u) = a_{31} u \cos v + a_{32} u \sin v + a_{33} (\varphi + cv), \quad (3.8)$$

where

$$A(u) = \frac{2H}{W}. \quad (3.9)$$

On differentiating (3.6), (3.7) and (3.8) twice with respect to $v$ we have

$$a_{13} = a_{23} = a_{33} = 0, \quad A(u) = 0. \quad (3.10)$$

From (3.10) we obtain

$$-a_{11} u \cos v - a_{12} u \sin v = 0$$

$$-a_{22} u \sin v - a_{21} u \cos v = 0$$

$$a_{31} u \cos v + a_{32} u \sin v = 0. \quad (3.11)$$

But $\cos$ and $\sin$ are linearly independent functions of $v$, so we finally obtain $a_{ij} = 0$.

From (3.9) we obtain $H = 0$. Consequently $M^2$, being a minimal surface.

**Theorem 3.1.** Let $r : M^2 \to \mathbb{E}^3$ be an isometric immersion given by (2.1). Then $\Delta^I r = Ar$ if and only if $M^2$ has zero mean curvature.
4. Helicoidal surfaces with $\Delta^I r = Ar$ in $\mathbb{E}^3$

In this section we are concerned with non-degenerate helicoidal surfaces $M^2$ without parabolic points satisfying the condition
\[ \Delta^I r = Ar. \] (4.1)

By a straightforward computation, the Laplacian $\Delta^I$ of the second fundamental form $II$ on $M^2$ with the help of (3.3) and (2.3) turns out to be
\[
\Delta^I = -\frac{W}{R} \left( u^2 \varphi' \frac{\partial^2}{\partial u^2} + u \varphi'' \frac{\partial^2}{\partial v^2} + 2c \frac{\partial^2}{\partial u \partial v} \right) \\
- \frac{W}{2R^2} u \left( -\varphi' (\varphi'' - \varphi^{n2}) u^4 + \varphi'^2 \varphi'' u^3 - 2c^2 \varphi'' u - 4c^2 \varphi' \right) \frac{\partial}{\partial u} \\
+ \frac{W}{2R^2} cu^2 \left( (\varphi' \varphi'' + \varphi^{n2}) u + 3 \varphi' \varphi'' \right) \frac{\partial}{\partial v},
\]
where $R = u^3 \varphi' \varphi'' - c^2$.

Accordingly, we get
\[
\Delta^I r(u, v) = \begin{pmatrix} (u \varphi' \cos v - c \sin v) P(u) \\ (u \varphi' \sin v + c \cos v) P(u) \\ u \varphi'^2 P(u) - u^2 Q(u) \end{pmatrix}, \tag{4.2}
\]
where
\[
P(u) = -\frac{W}{2R^2} \left( (\varphi'' - \varphi^{n2}) u^4 - \varphi' \varphi'' u^3 + 4c^2 \right) \tag{4.3}
\]
\[
Q(u) = -\frac{W}{2R^2} \left( 4 \varphi'^2 \varphi'' u^3 - c^2 (\varphi'' + \varphi' \varphi''') u - 7c^2 \varphi' \varphi'' \right). \tag{4.4}
\]

Therefore, the problem of classifying the helicoidal surfaces $M^2$ given by (2.1) and satisfying (4.1) is reduced to the integration of this system of ordinary differential equations
\[
(u \varphi' P(u) - a_{11} u) \cos v - (cP(u) + a_{12} u) \sin v = a_{13} (\varphi + cv) \\
(u \varphi' P(u) - a_{22} u) \sin v + (cP(u) - a_{21} u) \cos v = a_{23} (\varphi + cv) \\
u \varphi'^2 P(u) - u^2 Q(u) = a_{31} u \cos v + a_{32} u \sin v + a_{33} (\varphi + cv).
\]

**Remark 4.1.** We observe that
\[ c^2 P(u) + u^3 Q(u) = 2W. \] (4.5)

But $\cos v$ and $\sin v$ are linearly independent functions of $v$, so we finally obtain
\[ a_{32} = a_{31} = a_{33} = a_{13} = a_{23} = 0. \]

We put $a_{11} = a_{22} = \alpha$ and $a_{21} = -a_{12} = \beta$, $\alpha, \beta \in \mathbb{R}$. Therefore, this system of equations is equivalently reduced to
\[
\begin{cases}
\varphi' P(u) = \alpha \\
cP(u) = \beta u \\
\varphi'^2 P(u) - uQ(u) = 0.
\end{cases} \tag{4.6}
\]
Now, let us examine the system of equations (4.6) according to the values of the constants \( \alpha \) and \( \beta \).

**Case 1.** Let \( \alpha = 0 \) and \( \beta \neq 0 \).

In this case the system (4.6) is reduced equivalently to

\[
\begin{cases}
\varphi' = 0 \\
cP(u) = -\beta u \\
Q(u) = 0.
\end{cases}
\]  

From (4.7) we have \( P''(u) = 0 \). From (4.5) and the fact that \( c \neq 0 \) we have a contradiction. Hence there are no helicoidal surfaces of \( \mathbb{E}^3 \) in this case which satisfy (4.1).

**Case 2.** Let \( \alpha \neq 0 \) and \( \beta = 0 \).

In this case the system (4.6) is reduced equivalently to

\[
\begin{cases}
\varphi'P(u) = \alpha \\
\quad P(u) = 0.
\end{cases}
\]

But this is not possible. So, in this case there are no helicoidal surfaces of \( \mathbb{E}^3 \).

**Case 3.** Let \( \alpha = \beta = 0 \).

In this case the system (4.6) is reduced equivalently to

\[
\begin{cases}
P(u) = 0 \\
Q(u) = 0.
\end{cases}
\]

From (4.5) we have \( W = 0 \), which is a contradiction. Consequently, there are no helicoidal surfaces of \( \mathbb{E}^3 \) in this case.

**Case 4.** Let \( \alpha \neq 0 \) and \( \beta \neq 0 \).

In this case the system (4.6) is reduced equivalently to

\[
\varphi(u) = \frac{\alpha c}{\beta} \ln(u) + k, \quad k \in \mathbb{R}.
\]  

By using (4.6) and (4.8), we obtain

\[
\begin{cases}
P(u) = \frac{\beta c}{\alpha^2 c} u \\
Q(u) = \frac{\alpha^2 c}{\beta^2} u.
\end{cases}
\]  

Substituting (4.9) into (4.5), we get

\[
\frac{c^2 (\alpha^2 + \beta^2)^2}{\beta^2} u^2 = 4(u^2 + \frac{c^2 \alpha^2}{\beta^2} + c^2).
\]

Then

\[
\begin{cases}
c^2 (\alpha^2 + \beta^2) = 0 \\
c^2 (\alpha^2 + \beta^2)^2 = 4 \beta^2.
\end{cases}
\]

From the first equation we have \( \alpha = \beta = 0 \), which is a contradiction. Hence, there are no helicoidal surfaces of \( \mathbb{E}^3 \) in this case.

Consequently, we have:

**Theorem 4.2.** Let \( r : M^2 \rightarrow \mathbb{E}^3 \) be an isometric immersion given by (2.1). There are no helicoidal surfaces in \( \mathbb{E}^3 \) without parabolic points, satisfying the condition \( \Delta^J r = Ar \).
Theorem 4.3. If $K_G = a \in \mathbb{R}\{0\}$, then
\[ \Delta^{II} r(u,v) = -2N. \] (4.10)

Proof. If $K_G = a \in \mathbb{R}\{0\}$, then $\frac{\partial K_G}{\partial u} = 0$. From (3.4) we obtain
\[ -\varphi'\varphi''u^4 + \varphi'\varphi''u^5 + 7c^2\varphi'\varphi''u^2 + c^2\varphi'\varphi''u^2 - 3\varphi'\varphi''u^3 + 4c^2u - \varphi'^2\varphi''u^4 + 4c^2\varphi'^2u = -(\varphi'\varphi'' + \varphi'^2\varphi'')u^5 - c^2\varphi'\varphi''u^3 \] (4.11)
By using (4.3), (4.4) and (4.11) we get
\[ uP(u) - u^2Q(u) = \frac{W}{2R^2}(\varphi'^2\varphi''u^4 - 4c^2\varphi'^2u - \varphi'^2\varphi''u^5 - \varphi'^2\varphi''u^5) = -\varphi'^2uP(u). \] (4.12)
From (4.2) and (4.12) we deduce that
\[ \Delta^{II} r(u,v) = WP(u)N. \] (4.13)
From (4.5) and (4.12) we have that
\[ P(u) = \frac{2}{W}. \] (4.14)
By using (4.13) and (4.14) we get (4.10). \[ \Box \]

5. Helicoidal surfaces with $\Delta^{III} r = Ar$ in $\mathbb{E}^3$

In this section we are concerned with non-degenerate helicoidal surfaces $M^2$ without parabolic points satisfying the condition
\[ \Delta^{III} r = Ar. \] (5.1)
The components of the third fundamental form of the surface $M^2$ is given by
\[ e_{11} = \frac{1}{W^4}(c^2(\varphi' + u\varphi'')^2 + c^2 + u^4\varphi'^2), \] (5.2)
\[ e_{12} = -\frac{c}{W^2}(\varphi' + u\varphi''), \]
\[ e_{22} = \frac{1}{W^2}(c^2 + u^2\varphi'^2), \]
hence
\[ e = \frac{1}{W^6}(u^3\varphi'\varphi'' - c^2)^2. \]
The Laplacian of $M^2$ can be expressed as follows:
\[ \Delta^{III} = -\frac{1}{|e|}(W\frac{c^2 + u^2\varphi'^2}{c^2 - u^3\varphi'\varphi''} \partial^2 + 2cW\frac{\varphi' + u\varphi''}{c^2 - u^3\varphi'\varphi''} \partial^2 + \frac{1}{W}(\frac{c^2(\varphi' + u\varphi'')^2 + c^2 + u^4\varphi'^2}{c^2 - u^3\varphi'\varphi''} \partial^2 + \frac{d}{du}W\frac{c^2 + u^2\varphi'^2}{c^2 - u^3\varphi'\varphi''} \partial + c\frac{d}{du}W\frac{\varphi' + u\varphi''}{c^2 - u^3\varphi'\varphi''} \partial). \] (5.3)
By using (5.1) and (5.3) we get
\[
\begin{align*}
\Delta^{III}(u \cos v) &= -u \varphi' Q(u) \cos v - cQ(u) \sin v \\
\Delta^{III}(u \sin v) &= cQ(u) \cos v - u \varphi' Q(u) \sin v \\
\Delta^{III}(\varphi(u) + cv) &= P(u).
\end{align*}
\]

Hence
\[
\Delta^{III}_{r}(u,v) = \begin{pmatrix}
- u \varphi' Q(u) \cos v - cQ(u) \sin v \\
- cQ(u) \cos v - u \varphi' Q(u) \sin v \\
P(u)
\end{pmatrix}, \quad (5.4)
\]

where
\[
Q(u) = \frac{W^2}{(c^2 - u^3 \varphi' \varphi'')^3} \left( W^2 u^2 (c^2 + u^2 \varphi'^2) \varphi''' + 3c^2 u^2 \varphi' + 3c^2 u^2 \varphi'^3 \right), \quad (5.5)
\]
\[
P(u) = \frac{-W^2}{(c^2 - u^3 \varphi' \varphi'')^3} \left( W^2 u(c^2 + u^2 \varphi'^2)^2 \varphi'''' + 4c^4 u^2 \varphi'' + 
7c^2 u^4 \varphi'^2 \varphi''' - 2u^7 \varphi^{3,\varphi'^2} + 3c^6 \varphi'' + 15c^4 u^2 \varphi'^2 \varphi''
- 3c^2 u^5 \varphi'^3 \varphi'' + 9c^2 u^4 \varphi'^4 \varphi'' - 3u^7 \varphi^{5,\varphi'^2} + 2c^4 u \varphi'
+ 4c^4 u \varphi'^3 + 3c^2 u^3 \varphi'^3 + 3c^2 u^3 \varphi'^5
+ 3c^4 u^3 \varphi' \varphi'' + c^2 u^5 \varphi' \varphi'' + c^2 u^6 \varphi'^3 + c^4 u^4 \varphi'^3 \right), \quad (5.6)
\]

From (5.5) and (5.6) we have
\[
P(u) = \frac{-u}{W} L(u) - \left( \frac{c^2 + u^2 \varphi'^2}{u^3 \varphi' \varphi'' - c^2} \right) WL'(u), \quad (5.7)
\]
\[
Q(u) = \frac{-1}{W} L(u) + \left( \frac{u}{u^3 \varphi' \varphi'' - c^2} \right) WL'(u),
\]

where \(L(u) = h_{11} e^{11} + 2h_{12} e^{12} + h_{22} e^{22} = \frac{2H}{K_G} \).

**Remark 5.1.** We observe that
\[
uP(u) + (c^2 + u^2 \varphi'^2)Q(u) = -W \left( \frac{2H}{K_G} \right), \quad (5.8)
\]

The equation (5.1) by means of (2.1) and (5.4) gives rise to the following system of ordinary differential equations
\[
\begin{align*}
- u \varphi' Q(u) \cos v - cQ(u) \sin v &= a_{11} u \cos v + a_{12} u \sin v + a_{13} (\varphi + cv) \\
cQ(u) \cos v - u \varphi' Q(u) \sin v &= a_{21} u \cos v + a_{22} u \sin v + a_{23} (\varphi + cv) \\
P(u) &= a_{31} u \cos v + a_{32} u \sin v + a_{33} (\varphi + cv).
\end{align*}
\]

But \( \cos v \) and \( \sin v \) are linearly independent functions of \( v \), so we finally obtain
\[
a_{32} = a_{31} = a_{33} = a_{13} = a_{23} = 0.
\]
We put \(-a_{11} = a_{22} = \lambda_1\) and \(a_{21} = -a_{12} = \lambda_2\), \(\lambda_1, \lambda_2 \in \mathbb{R}\). Therefore, this system of equations is equivalently reduced to

\[
\begin{align*}
\varphi' Q(u) &= \lambda_1 \\
cQ(u) &= \lambda_2 u \\
P(u) &= 0.
\end{align*}
\]  

(5.9)

Therefore, the problem of classifying the surfaces \(M^2\) given by (2.1) and satisfying (5.1) is reduced to the integration of this system of ordinary differential equations.

**Case 1.** Let \(\lambda_1 = 0\) and \(\lambda_2 \neq 0\).

In this case the system (5.9) is reduced equivalently to

\[
\begin{align*}
\varphi' Q(u) &= 0 \\
cQ(u) &= \lambda_2 u \\
P(u) &= 0.
\end{align*}
\]  

(5.10)

Differentiating (5.10), we obtain \(P''(u) = 0\), which is a contradiction. Hence there are no helicoidal surfaces of \(E^3\) in this case which satisfy (5.1).

**Case 2.** Let \(\lambda_1 \neq 0\) and \(\lambda_2 = 0\).

In this case the system (5.9) is reduced equivalently to

\[
\begin{align*}
\varphi' Q(u) &= \lambda_1 \\
cQ(u) &= 0 \\
P(u) &= 0.
\end{align*}
\]

But this is not possible. So, in this case there are no helicoidal surfaces of \(E^3\).

**Case 3.** Let \(\lambda_1 = \lambda_2 = 0\).

In this case the system (5.9) is reduced equivalently to

\[
\begin{align*}
\varphi' Q(u) &= 0 \\
Q(u) &= 0.
\end{align*}
\]

From (5.8) we have \(H = 0\). Consequently \(M^2\), being a minimal surface.

**Case 4.** Let \(\lambda_1 \neq 0\) and \(\lambda_2 \neq 0\).

In this case the system (5.9) is reduced equivalently to

\[
\varphi(u) = \frac{\lambda_1 c}{\lambda_2} \ln(u) + a, \quad a \in \mathbb{R}.
\]  

(5.11)

If we substitute (5.11) in (5.5) we get \(Q(u) = 0\). So we have a contradiction and therefore, in this case there are no helicoidal surfaces of \(E^3\).

Consequently, we have:

**Theorem 5.2.** Let \(r : M^2 \to E^3\) be an isometric immersion given by (2.1). Then \(\Delta^{III} r = Ar\) if and only if \(M^2\) has zero mean curvature.

**Theorem 5.3.** If \(\frac{2H}{K_G} = \alpha \in \mathbb{R} \setminus \{0\}\), then

\[
\Delta^{III} r(x, y) = -\frac{2H}{K_G} N.
\]
Proof. From (5.7) we have

\[ P(u) = uQ(u). \]  

Finally, (5.12) and (5.4) give

\[ \Delta^{III} r(u,v) = Q(u)(-u\varphi' \cos v + c \sin v, c \cos v - u\varphi' \sin v, u) \]
\[ = -\frac{1}{W} \left( \frac{2H}{K_G} \right) (-u\varphi' \cos v + c \sin v, -c \cos v - u\varphi' \sin v, u) \]
\[ = -\frac{2H}{K_G} N. \]

□

References

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