

Adaptive algorithm for polyhedral approximation of 3D solids

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Abstract. In this paper we discuss theoretical foundations of developing general methods for volume-based approximation of three-dimensional solids. We construct an iterative method that can be used for approximation of regular subsets of \mathbb{R}^d ($d \in \mathbb{N}$) in particular \mathbb{R}^3 . We will define solid meshes and investigate the connection between solid meshes, regular sets and polyhedra. First the general description of the method will be given. The main idea of our algorithm is a kind of space partitioning with increasing atomic σ -algebra sequences. In every step one atom will be divided into two nonempty atoms. We define a volume-based distance metric and we give sufficient conditions for the convergence and monotonicity of the method. We show a possible application, a polyhedral approximation (or approximate convex decomposition) of triangular meshes.

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1. Introduction

Approximation of $2D$ or $3D$ subsets is useful many times. The main application is mesh simplification, i.e. we would like to approximate a mesh having n triangles with another mesh having $n' \ll n$ triangles. Mesh simplification algorithms sometimes use volume-based error metric [1], but the most common metric is the Hausdorff-distance, it can be found in many papers e.g. in [2]. A volume-based metric allows to approximate meshes which can not be realized to generate meshes which have a geometric realization. We investigated some topological properties of triangular meshes. There are some common topological error, e.g. holes, dangling faces, isolated faces, etc, a good summary of this topic can be read in [5]. Mesh repairing algorithms like [4] or [2] deal with changing the topological properties of meshes. Result of our method is

a common finite polyhedron, which always has an obvious geometric realization. The discussed algorithm can be applied to recover topological properties, if we define properly the measure of an object having topological errors. Another intensively studied related topic is the approximate convex decomposition, see e.g. [16]. It can be used in physical simulation e.g. collision detection, fracture simulation [12] and obviously finite element methods. Similar approaches exist for volume-like approximations, for example spatial decomposition and mesh generation. Common spatial decomposition algorithms like octree and k-d trees can be found in [9] and comparative analysis of some algorithms in [22]. There are some useful information about mesh generation in [19]. During our work we studied the above topics, and it became clear, that the methods are very similar to each other, but there is no general theory published yet that connects these subjects. This was our motivation to give a theoretical foundation, which can describe most of the volume-based approximation methods. Reader will notice, that our concept is similar to the construction of the measure and integral theory, where it is needed we refer to the literature.

2. Solids

The subject of our study is the solid mesh. The solids can be triangular meshes, polyhedra or solid geometries, e.g. a solid cube. The solid can be determined by its vertices and faces, as we discussed in [13]. We will define the solid as a special subset of \mathbb{R}^3 . General theory of solids can be found in [10], [11],[20], including the concept of regularity, which plays a central role, as we will see further. We draw up some requirements before defining the solid. Let us consider an $S \subset \mathbb{R}^d$ set. If S is a solid, it should meet the following expectations:

- S does not contain dangling faces, edges, isolated points or gaps
- S has volume and surface area
- S cannot be arbitrary large
- S cannot be decomposed into some parts satisfying the first 3 conditions.

As we will see, the requirements above can be translated as regularity, measurability, boundedness and connectedness. The last two criterions are obvious. In the followings, some properties of regularity will be detailed.

Let S be an arbitrary set. Denote $S^c, \bar{S}, \text{int}S, \text{ext}S, \partial S$ the complement, closure, interior, exterior and boundary of S , respectively. Let μ be the common Lebesgue-measure in \mathbb{R}^d , and denote \mathbb{B}^d the d dimensional unit ball with respect to the Euclidean norm denoted by $\|\cdot\|$, moreover let $B_r(x)$ be the ball with radius r centered at x . Topological subjects concerned with the notions of this paper can be found in [15].

Definition 2.1. *Let S be an arbitrary set. We define the regularized of S as*

$$S^* := \overline{\text{int}S}.$$

Definition 2.2. *S is said to be regular if $S = S^*$.*

Corollary 2.3. *If $S \subset \mathbb{R}^d$ and $S = S^*$ then*

$$\partial S = \overline{\partial \text{int}S} = \partial \text{int}S.$$

Therefore

$$S = S^* = \overline{\text{int}S} = \text{int}S \cup \partial \text{int}S = \text{int}S \cup \partial S.$$

Moreover $\mathbb{R}^d = \text{int}S \cup \partial S \cup \text{ext}S = S \cup \text{ext}S$, consequently

$$S^c = \text{ext}S.$$

Lemma 2.4. *If $S = S^*$ then $x \in \partial S$ if and only if*

$$\forall r > 0 : (B_r(x) \setminus \{x\}) \cap \text{int}S \neq \emptyset \quad \text{and} \quad (B_r(x) \setminus \{x\}) \cap \text{ext}S \neq \emptyset.$$

Proof. Since x is a boundary point, then each $B_r(x)$ satisfies $B_r(x) \cap S \neq \emptyset$ and $B_r(x) \cap S^c \neq \emptyset$. Notice, that $\text{int}S$ does not contain isolated points and S^* contains all of its limit points. Therefore $x \in \partial S$ if and only if

$$\forall r > 0 : (B_r(x) \setminus \{x\}) \cap S \neq \emptyset \quad \text{and} \quad (B_r(x) \setminus \{x\}) \cap S^c \neq \emptyset.$$

As a consequence of Corollary 2.3 $S^c = \text{ext}S$, i.e.

$$(B_r(x) \setminus \{x\}) \cap S^c \neq \emptyset \Leftrightarrow (B_r(x) \setminus \{x\}) \cap \text{ext}S \neq \emptyset. \quad (2.1)$$

On the other hand $x \in \partial S = \partial \text{int}S$, thus $B_r(x) \cap \text{int}S \neq \emptyset$ for all $r > 0$, and $x \in \partial S$ implies $(B_r(x) \setminus \{x\}) \cap \text{int}S \neq \emptyset$, hence

$$x \in \partial S \Rightarrow (B_r(x) \setminus \{x\}) \cap \text{int}S \neq \emptyset. \quad (2.2)$$

In the opposite direction $(B_r(x) \setminus \{x\}) \cap \text{int}S \neq \emptyset$ is sufficient to fulfill $(B_r(x) \setminus \{x\}) \cap S \neq \emptyset$, thus

$$(B_r(x) \setminus \{x\}) \cap \text{int}S \neq \emptyset \Rightarrow (B_r(x) \setminus \{x\}) \cap S \neq \emptyset. \quad (2.3)$$

(2.1),(2.2),(2.3) are sufficient to satisfy the statement. \square

We need a measure such that the concept of volume can be interpreted. Most of our theorems can be proved supposing that μ is an outer measure, for simplicity we suppose that μ is the Lebesgue-measure.

Corollary 2.5. *The closed (and opened) sets of \mathbb{R}^d are Lebesgue-measurable.*

It is known, that any open ball in \mathbb{R}^d has a positive Lebesgue-measure, because for all $x \in \mathbb{R}^d$

$$\mu(B_r(x)) = \frac{\pi^{d/2}}{\Gamma(d/2 + 1)} r^d > 0 \quad (2.4)$$

is the volume of the d -dimensional ball with radius r .

Corollary 2.6. *An open set $S \subset \mathbb{R}^d$ is empty if and only if $\mu(S) = 0$.*

If S is regular and $S \neq \emptyset$ then $\text{int}S \neq \emptyset$, since $\text{int}S = \emptyset$ implies $S = S^* = \overline{\text{int}S} = \overline{\emptyset} = \emptyset$.

Corollary 2.7. *A regular set $S \subset \mathbb{R}^d$ is empty if and only if $\mu(S) = 0$.*

We deal with regular, connected and bounded sets. It is easy to see, that there exists an isomorphism between every bounded subset of \mathbb{R}^d and the subsets of the unit ball, so it is enough to consider the subsets of \mathbb{B}^d without loss of generality.

Definition 2.8. *Let us define the following set:*

$$\Omega^d := \{S \subset \mathbb{B}^d \mid S \text{ connected, regular, and } \mu(\partial S) = 0\}.$$

If $S \in \Omega^d$ then we say S is a d -dimensional solid.

A commonly convex polyhedron is defined to be the finite intersection of half-spaces. Then a polyhedron can be defined as the finite union of convex polyhedra. By definition we have two important corollary with respect to polyhedra, see e.g. [14], [6], [3].

Corollary 2.9. *If B is a polyhedron, then B is a solid.*

Corollary 2.10. *Let B be a convex polyhedron, and H be a half-space, such that $B' := B \cap H$, $B'' := B \cap H^c$ with $\text{int}B' \neq \emptyset$, $\text{int}B'' \neq \emptyset$. Then $\overline{B'}$ and $\overline{B''}$ are convex polyhedra.*

3. Distance, approximation problem

In this section we use some notes from integral theory. The connection between measure and integral is studied in [8], [23]. The following concept of distance can be found in [8].

Definition 3.1. *Let us define the following function $\rho : 2^{\mathbb{R}^d} \times 2^{\mathbb{R}^d} \rightarrow \mathbb{R}_0^+$ which defines the distance between subsets A and B of \mathbb{R}^d*

$$\rho(A, B) := \mu(A\Delta B) \quad (A, B \subset \mathbb{R}^d)$$

where $A\Delta B := (A \setminus B) \cup (B \setminus A)$ denotes the symmetric difference.

We can see that if $\mu(A\Delta B) = 0$ then A and B may differ only on a set of measure zero. Denote χ_S the indicator function of some set S . If A, B are measurable and bounded then χ_A, χ_B are integrable. Therefore we can express the measure of set operations by integrals.

$$\mu(A \cap B) = \int_{\mathbb{R}^d} \chi_A \chi_B \, d\mu \tag{3.1}$$

$$\mu(A^c) = \mu(\mathbb{R}^d \setminus A) = \int_{\mathbb{R}^d} 1 - \chi_A \, d\mu \tag{3.2}$$

$$\mu(A \cup B) = \int_{\mathbb{R}^d} \chi_A + \chi_B - \chi_A \chi_B \, d\mu. \tag{3.3}$$

The distance can be reformulated as

$$\rho(A, B) = \int_{\mathbb{R}^d} \chi_A + \chi_B - 2\chi_A \chi_B \, d\mu.$$

There are connections between measure and integral. The range of χ is $\{0, 1\}$ then $\chi \equiv \chi^p$ for all $1 \leq p < +\infty$, consequently

$$\begin{aligned} \rho(A, B) &= \int_{\mathbb{R}^d} \chi_A^2 + \chi_B^2 - 2\chi_A \chi_B \, d\mu = \int_{\mathbb{R}^d} (\chi_A - \chi_B)^2 \, d\mu \\ &= \int_{\mathbb{R}^d} |\chi_A - \chi_B| \, d\mu = \int_{\mathbb{R}^d} |\chi_A - \chi_B|^p \, d\mu =: \|\chi_A - \chi_B\|_{L_\mu^p}^p. \end{aligned} \tag{3.4}$$

As we see, $\rho(A, B)$ equals to the p -th power of the common L_μ^p -norm of the measurable function $\chi_A - \chi_B$. This implies, that ρ is a pseudo-metric, since ρ is non-negative, triangle inequality holds and $\rho(A, A) = 0$, but $\rho(A, B) = 0$ does not imply $A = B$,

because A and B may differ on a set of measure zero. Now we show, that A and B can not differ in the case $\rho(A, B) = 0$ if they are regular sets.

Theorem 3.2. *Let $A, B \subset \mathbb{R}^d$ be regular sets. Then $\rho(A, B) = 0$ if and only if $A = B$.*

Proof. If $A = B$ then obviously $A \Delta B = \emptyset$, thus $\rho(A, B) = \mu(A \Delta B) = 0$. To verify the opposite direction, let us suppose $\rho(A, B) = 0$. Since A, B are regular sets, we can express the symmetric difference this way:

$$\begin{aligned} A \Delta B &= (A \setminus B) \cup (B \setminus A) = (A \cap B^c) \cup (B \cap A^c) \\ &= ((\text{int}A \cup \partial A) \cap \text{ext}B) \cup ((\text{int}B \cup \partial B) \cap \text{ext}A) \\ &(\text{int}A \cap \text{ext}B) \cup (\partial A \cap \text{ext}B) \cup (\text{int}B \cap \text{ext}A) \cup (\partial B \cap \text{ext}A). \end{aligned}$$

All four members of the last term are disjoint. Using properties of the measure we get

$$\begin{aligned} \rho(A, B) &= \mu(\text{int}A \cap \text{ext}B) + \mu(\text{int}B \cap \text{ext}A) \\ &+ \mu(\partial A \cap \text{ext}B) + \mu(\partial B \cap \text{ext}A) = 0. \end{aligned}$$

It can be seen $\rho(A, B) = 0$ if and only if every member has zero Lebesgue-measure. Notice that $\text{int}A$ and $\text{ext}B$ are open sets, therefore $\text{int}A \cap \text{ext}B$ is open, so they have zero measure if and only if they are empty. Since $\text{ext}A \cap \text{int}B$ is open, the followings are necessarily true:

$$\begin{aligned} \text{int}A \cap \text{ext}B &= \emptyset \\ \text{ext}A \cap \text{int}B &= \emptyset. \end{aligned}$$

Let us suppose, that $\partial A \cap \text{ext}B \neq \emptyset$. Then exists $x \in \mathbb{R}^d$ such for some $r > 0$ $B_r(x) \subset \text{ext}B$, moreover as a consequence of Lemma 2.4

$$(B_r(x) \setminus \{x\}) \cap \text{int}A \neq \emptyset.$$

Thus

$$(B_r(x) \setminus \{x\}) \cap \text{int}A \cap \text{ext}B \neq \emptyset,$$

i.e. $\text{int}A \cap \text{ext}B \neq \emptyset$, which is a contradiction. Consequently $\partial A \cap \text{ext}B$ is empty, similarly $\partial B \cap \text{ext}A$ is empty, as well. To sum up, $\rho(A, B) = 0$ implies $A \Delta B = \emptyset$, which proves the statement. □

As a consequence of the preceding theorem ρ is a metric on the regular subsets of \mathbb{R}^d . As Ω^d contains only regular sets, we get the following result.

Corollary 3.3. *ρ is a metric on Ω^d .*

The abstract approximation problem is the following:

Given $S \subset \mathbb{R}^d$ a solid and $\varepsilon > 0$ a positive real. We are looking for a solid $\tilde{S} \subset \mathbb{R}^d$ which has „better properties” than S in some sense, and $\mu(S, \tilde{S}) < \varepsilon$.

In the next sections we will discuss our construction to give a possible solution for the problem above. Here „better properties” refers to e.g. topological correctness, simplicity, convexity, etc. depending on the objective.

4. Construction

Consider $\mathcal{B} \subset 2^{\mathbb{B}^d}$ to be a finite atomic σ -algebra on \mathbb{B}^d , i.e.

$$\begin{aligned} \mathcal{B} &:= \sigma(B_i \subset \mathbb{B}^d : i = 0, 1, \dots, n) \quad (n \in \mathbb{N}) \\ B_i \cap B_j &= \emptyset \quad (i \neq j), \quad \cup_{i=0}^n B_j = \mathbb{B}^d. \end{aligned} \tag{4.1}$$

Then the functions contained in $L_\mu(\mathcal{B})$ are integrable, and by definition they are constant on an arbitrary B_i atom. Thus $L_\mu(\mathcal{B}) \subset L_\mu(\mathbb{B}^d)$, moreover $L_\mu(\mathcal{B})$ is a finite dimensional subspace in $L_\mu(\mathbb{B}^d)$. Denote by $\chi(X)$ the set of indicator functions defined on subsets of some set system X . Obviously $\chi(\mathcal{B}) \subset L_\mu(\mathcal{B})$. Moreover, if $S \subset \mathbb{B}^d$ then $\chi_S \in \chi(\mathbb{B}^d)$, i.e. a bijection can be defined between the subsets of \mathbb{B}^d and the functions from $\chi(\mathbb{B}^d)$.

Let us define the following functions:

$$\phi_i := \frac{1}{\sqrt{\mu(B_i)}} \chi_{B_i}. \tag{4.2}$$

It is easy to see, that $\{\phi_i\}_{i=0}^n$ forms an orthonormed system under the common inner product:

$$\langle \phi_i, \phi_j \rangle := \int_{\mathbb{B}^d} \phi_i \phi_j d\mu = \frac{1}{\sqrt{\mu(B_i)\mu(B_j)}} \int_{\mathbb{B}^d} \chi_{B_i} \chi_{B_j} d\mu.$$

If $i \neq j$ then $B_i \cap B_j = \emptyset$ implies that the inner product is zero, and obviously

$$\langle \phi_i, \phi_i \rangle := \frac{1}{\sqrt{\mu(B_i)^2}} \int_{\mathbb{B}^d} \chi_{B_i}^2 d\mu = \frac{1}{\mu(B_i)} \int_{B_i} d\mu = 1.$$

Consequently $\{\phi_i\}_{i=0}^n$ is an orthonormed system on a finite dimensional subspace of the Hilbert-space $L_\mu(\mathbb{B}^d)$. By Riesz projection theorem the best approximation in $L_\mu(\mathcal{B})$ of an arbitrary function $f \in L_\mu(\mathbb{B}^d)$ can be expressed by the Fourier-series with respect to \mathcal{B} , see e.g. [7],[17]:

$$\mathcal{F}^{\mathcal{B}} f := \sum_{i=0}^n \langle f, \phi_i \rangle \phi_i. \tag{4.3}$$

Let be given $S \subset \mathbb{B}^d$. Then we get the following formula for the i -th Fourier-coefficient of χ_S :

$$\begin{aligned} \langle \chi_S, \phi_i \rangle &= \frac{1}{\sqrt{\mu(B_i)}} \langle \chi_S, \chi_{B_i} \rangle = \frac{1}{\sqrt{\mu(B_i)}} \int_{\mathbb{B}^d} \chi_S \chi_{B_i} d\mu \\ &= \frac{1}{\sqrt{\mu(B_i)}} \int_{B_i \cap S} d\mu = \frac{1}{\sqrt{\mu(B_i)}} \mu(B_i \cap S). \end{aligned}$$

Thus the Fourier-series of χ_S is

$$\mathcal{F}^{\mathcal{B}} \chi_S := \sum_{i=0}^n \frac{1}{\sqrt{\mu(B_i)}} \mu(B_i \cap S) \phi_i = \sum_{i=0}^n \frac{\mu(B_i \cap S)}{\mu(B_i)} \chi_{B_i} = \sum_{i=0}^n b_i \chi_{B_i} \tag{4.4}$$

where

$$b_i := \frac{\mu(B_i \cap S)}{\mu(B_i)}. \tag{4.5}$$

$\mathcal{F}^{\mathcal{B}}\chi_S$ is also the best approximation of χ_S on the subspace spanned by \mathcal{B} . The main problem is, that $\mathcal{F}^{\mathcal{B}}\chi_S$ is generally not an element of $\chi(\mathbb{B}^d)$, therefore we can not assign a set from \mathbb{B}^d to the resulting function. To solve this problem, we will introduce the following operator.

Definition 4.1. *Let $f \in L_\mu(\mathbb{B}^d)$ be an arbitrary function and $\alpha \in (0, 1)$. Define the operator $\mathcal{X}_\alpha : L_\mu(\mathbb{B}^d) \rightarrow \chi(\mathbb{B}^d)$ as follows*

$$\mathcal{X}_\alpha f(x) := \begin{cases} 0, & f(x) \leq \alpha \\ 1, & f(x) > \alpha. \end{cases}$$

$\mathcal{X}_\alpha f$ is an approximation of f by an indicator function, and the resulting function depends on a real parameter $\alpha \in (0, 1)$. Later the importance of the value of α will be explained.

Corollary 4.2. *Let $S \subset \mathbb{B}^d$ be an arbitrary set, $\alpha \in (0, 1)$.*

$$\mathcal{X}_\alpha \mathcal{F}^{\mathcal{B}}\chi_S \in \chi(\mathcal{B})$$

in other words

$$\exists \tilde{S} \subset \mathbb{B}^d : \chi_{\tilde{S}} = \mathcal{X}_\alpha \mathcal{F}^{\mathcal{B}}\chi_S$$

namely

$$\tilde{S} := \{\mathcal{X}_\alpha \mathcal{F}^{\mathcal{B}}\chi_S = 1\}.$$

In our approach the set \tilde{S} is the approximation of S with respect to the system \mathcal{B} and α .

Our strategy is similar to Schipp’s construction in [21]. We would like to construct a sequence of increasing σ -algebras to refine the approximation. Let us suppose, we have a σ -algebra \mathcal{B}_n and a set S to be approximated. Now we can draw up the approximation with respect to \mathcal{B}_n as $S_n := \{\mathcal{X}_\alpha \mathcal{F}^{\mathcal{B}_n}\chi_S = 1\}$. We need to check if $\rho(S_n, S) < \varepsilon$. If not, then we have to construct a larger σ -algebra \mathcal{B}_{n+1} , compute the $(n + 1)$ -th approximation and its distance. Repeat this process while $\rho(S_n, S) \geq \varepsilon$. It seems to be easy, but we have to work out some conditions for refinement to ensure the convergence. Denote $[a..b]$ the interval of natural numbers between a and b , i.e. $[a..b] := [a, b] \cap \mathbb{N}$.

Let us consider the following atomic decomposition sequence of the unit ball

$$\begin{aligned} \mathcal{B}_n &= \sigma(B_0, B_1, \dots, B_n) \\ B_i \cap B_j &= \emptyset \quad (i \neq j), \quad \bigcup_{i=0}^n B_i = \mathbb{B}^d. \end{aligned} \tag{4.6}$$

By definition, the n -th σ -algebra is generated by exactly $n + 1$ subsets, then \mathcal{B}_{n+1} can be obtained only by splitting a generator element of \mathcal{B}_n , for details see [18]:

$$\exists k \in [0..n] : \mathcal{B}_{n+1} = \sigma(B_0, \dots, B_{k-1}, B'_k, B''_k, B_{k+1}, \dots, B_n)$$

such that

$$B'_k \neq \emptyset, \quad B''_k \neq \emptyset, \quad B'_k \cup B''_k = B_k, \quad B'_k \cap B''_k = \emptyset.$$

Then the indicator function of the n -th approximation can be defined by

$$\chi_{S_n} := \mathcal{X}_\alpha \mathcal{F}^{\mathcal{B}_n} \chi_S. \tag{4.7}$$

Denote $\text{diam}(A)$ the diameter of some set $A \subset \mathbb{B}^d$, i.e.

$$\text{diam}(A) := \sup_{x,y \in A} \|x - y\|. \tag{4.8}$$

We give a sufficient condition for the convergence of S_n .

Theorem 4.3. *Let (\mathcal{B}_n) be an increasing sequence of σ -algebra, where*

$$\mathcal{B}_n = \sigma(B_0^{(n)}, B_1^{(n)}, \dots, B_n^{(n)})$$

and consider $S \in \Omega^d$ to be a solid, an arbitrary $\alpha \in (0, 1)$ and let

$$S_n = \{\mathcal{X}_\alpha \mathcal{F}^{\mathcal{B}_n} \chi_S = 1\}.$$

If the diameter of all the generator atoms converge to zero, i.e.

$$\lim_{n \rightarrow \infty} \max_{k \in [0..n]} \text{diam}(B_k^{(n)}) = 0$$

then

$$\lim_{n \rightarrow \infty} S_n = S.$$

Proof. Since S is solid, $\lim_{n \rightarrow \infty} S_n = S$ if and only if $\lim_{n \rightarrow \infty} \rho(S_n, S) = 0$ as a consequence of Theorem 3.2, and

$$\begin{aligned} \rho(S_n, S) &= \int_{\mathbb{B}^d} |\chi_{S_n} - \chi_S| \, d\mu \\ &= \int_{\text{int}S} |\chi_{S_n} - \chi_S| \, d\mu + \int_{\text{ext}S} |\chi_{S_n} - \chi_S| \, d\mu + \int_{\partial S} |\chi_{S_n} - \chi_S| \, d\mu. \end{aligned} \tag{4.9}$$

Notice that any function in $\chi(\mathbb{B}^d)$ can be dominated by $\chi_{\mathbb{B}^d}$, which is integrable. Using this fact the third integral equals to zero, because $\mu(\partial S) = 0$.

We know by Lebesgue’s density theorem, that for almost every $x \in S$

$$\lim_{r \rightarrow 0} \frac{\mu(B_r(x) \cap S)}{\mu(B_r(x))} = 1. \tag{4.10}$$

This implies that

$$\begin{aligned} \forall x \in \text{int}S : \lim_{r \rightarrow 0} \frac{\mu(B_r(x) \cap S)}{\mu(B_r(x))} &= 1 \\ \forall x \in \text{ext}S : \lim_{r \rightarrow 0} \frac{\mu(B_r(x) \cap S)}{\mu(B_r(x))} &= 0. \end{aligned}$$

If $x \in \text{int}S$ then exists $r > 0$ such $B_r(x) \subset \text{int}S$. On the other hand since \mathcal{B}_n is atomic, then for all $n \in \mathbb{N}$ exists a unique k such $x \in B_k^{(n)}$. Since $\text{diam}(B_k^{(n)})$ tends to zero, we get $B_k^{(n)} \subset B_r(x)$ for sufficiently large n , i.e. pointwise convergence is true for any $x \in \text{int}S$. Similarly, pointwise convergence can be proved in the case $x \in \text{ext}S$. Now using dominant convergence theorem we have

$$\lim \int_{\text{int}S} |\chi_{S_n} - \chi_S| \, d\mu = \int_{\text{int}S} \lim |\chi_{S_n} - \chi_S| \, d\mu = 0$$

$$\lim \int_{\text{ext}S} |\chi_{S_n} - \chi_S| d\mu = \int_{\text{ext}S} \lim |\chi_{S_n} - \chi_S| d\mu = 0$$

to sum up, all the three integrals in (4.9) equals to 0 as $n \rightarrow \infty$, therefore $\rho(S_n, S) = 0$, which proves the theorem. \square

We should not require that the maximal diameter of the decomposition tends to zero, it would be too expensive in applications. To avoid this problem and to simplify the procedure, we have the following idea. If a Fourier-coefficient of an atom is exactly 0 or 1, it is unnecessary to split. Accurately, if every Fourier-coefficients tends to 0 or 1, then the preceding theorem is automatically satisfied.

Lemma 4.4. *Let be*

$$I_n := \{i \in [0..n] \mid b_i^{(n)} > \alpha\}$$

and

$$J_n := [0..n] \setminus I_n.$$

Then

$$S_n = \bigcup_{i \in I_n} B_i^{(n)}$$

moreover

$$\rho(S, S_n) = \sum_{i \in J_n} \mu(B_i^{(n)})b_i^{(n)} + \sum_{i \in I_n} \mu(B_i^{(n)})(1 - b_i^{(n)}).$$

Proof. By definition

$$\begin{aligned} S_n &= \{\mathcal{X}_\alpha \mathcal{F}^{\mathcal{B}^n} \chi_S = 1\} = \left\{ \mathcal{X}_\alpha \sum_{i=0}^n b_i^{(n)} \chi_{B_i} = 1 \right\} \\ &= \left\{ \sum_{i \in I_n} \chi_{B_i} = 1 \right\} = \bigcup_{i \in I_n} B_i^{(n)}. \end{aligned}$$

Using the formula above, we can write

$$\begin{aligned} \rho(S, S_n) &= \mu(S \Delta S_n) = \mu(S \cup S_n) - \mu(S \cap S_n) \\ &= \mu(S \cup \bigcup_{i \in I_n} B_i^{(n)}) - \mu(S \cap \bigcup_{i \in I_n} B_i^{(n)}) \\ &= \mu(S \cup \bigcup_{i \in I_n} (B_i^{(n)} \cap S^c)) - \mu(\bigcup_{i \in I_n} (B_i^{(n)} \cap S)) \\ &= \mu(S) - \mu(\bigcup_{i \in I_n} (B_i^{(n)} \cap S)) + \mu(\bigcup_{i \in I_n} (B_i^{(n)} \cap S^c)) \\ &= \mu(\bigcup_{i \in J_n} (B_i^{(n)} \cap S)) + \mu(\bigcup_{i \in I_n} (B_i^{(n)} \cap S^c)) \\ &= \sum_{i \in J_n} \mu(B_i^{(n)} \cap S) + \sum_{i \in I_n} \mu(B_i^{(n)}) - \mu(B_i^{(n)} \cap S) \\ &= \sum_{i \in J_n} \mu(B_i^{(n)})b_i^{(n)} + \sum_{i \in I_n} \mu(B_i^{(n)})(1 - b_i^{(n)}). \end{aligned} \quad \square$$

We can estimate the distance, if we define the relevant indices.

Definition 4.5. *Let be*

$$\Delta_n := \{i \in [0..n] \mid b_i^{(n)} \in (0, 1)\}.$$

An i index and the $B_i^{(n)}$ generator atom are said to be relevant if $i \in \Delta_n$.

Δ_n contains all the indices of atoms which have Fourier-coefficients not exactly equal to 0 or 1. Notice, that if $b_j^{(n)} = 0$ then $j \in J_n$ for all $\alpha \in (0, 1)$. Consequently in the first sum there is a multiplication with $b_j^{(n)} = 0$, moreover $I_n \cap J_n = \emptyset$ implies $j \notin I_n$ therefore j can be left from both sums. Similarly, if $b_i^{(n)} = 1$ then we multiply with $(1 - b_i^{(n)}) = 0$ in the second sum and $i \notin J_n$, therefore i can be left, as well. Because of this we have the following corollary.

Corollary 4.6.

$$\begin{aligned} \rho(S, S_n) &= \sum_{i \in J_n \cap \Delta_n} \mu(B_i^{(n)})b_i^{(n)} + \sum_{i \in I_n \cap \Delta_n} \mu(B_i^{(n)})(1 - b_i^{(n)}) \\ &\leq \sum_{i \in J_n \cap \Delta_n} \mu(B_i^{(n)}) + \sum_{i \in I_n \cap \Delta_n} \mu(B_i^{(n)}) = \sum_{i \in \Delta_n} \mu(B_i^{(n)}). \end{aligned}$$

We are ready to state a stronger convergence theorem.

Theorem 4.7. *Let (\mathcal{B}_n) be an increasing sequence of σ -algebra where*

$$\mathcal{B}_n = \sigma(B_0^{(n)}, B_1^{(n)}, \dots, B_n^{(n)})$$

and consider a solid $S \in \Omega^d$, an arbitrary $\alpha \in (0, 1)$ and let

$$S_n = \{\mathcal{X}_\alpha \mathcal{F}^{\mathcal{B}_n} \chi_S = 1\}.$$

If the diameter of all the relevant generator atoms converge to zero, i.e.

$$\lim_{n \rightarrow \infty} \max_{k \in \Delta_n} \text{diam}(B_k^{(n)}) = 0$$

then

$$\lim_{n \rightarrow \infty} S_n = S.$$

Proof. If $b_i^{(n)} = 0$ then $\mu(B_i^{(n)} \cap S) = \mu(B_i^{(n)})$. Since atoms and S are regular sets, we have $B_i^{(n)} \cap S = B_i^{(n)}$, therefore $\chi_{B_i^{(n)}}(x) = \chi_S(x)$ for all $x \in B_i^{(n)}$. The same result is true if $b_i^{(n)} = 1$. Consequently,

$$\begin{aligned} \rho(S_n, S) &= \int_{\mathbb{B}^d} |\chi_{S_n} - \chi_S| \, d\mu = \sum_{i=0}^n \int_{B_i^{(n)}} |\chi_{S_n} - \chi_S| \, d\mu \\ &= \sum_{i \in \Delta_n} \int_{B_i^{(n)}} |\chi_{S_n} - \chi_S| \, d\mu. \end{aligned}$$

Using the same idea as in Theorem 4.3 we can prove that the remaining integrals tend to zero as $n \rightarrow \infty$. □

We have convergence theorems, finally we would like to investigate the monotonicity of the convergence. The following theorem gives a monotonicity condition of the approximation method. We will show, that it is closely related to the value of α .

Theorem 4.8. (S_n) is an improving approximation sequence, i.e.

$$\rho(S_{n+1}, S) \leq \rho(S_n, S)$$

for all $n \in \mathbb{N}$ if and only if $\alpha = \frac{1}{2}$.

Proof. Let us suppose, that $B_k^{(n)} =: B \in \mathcal{B}_n$ is divided into two disjoint non-empty $B', B'' \in \mathcal{B}_{n+1}$ sets. We have to show, that the sequence $\delta_n := \rho(S, S_n) - \rho(S, S_{n+1})$ is non-negative, i.e. $\delta_n \geq 0$ ($n \in \mathbb{N}$). We have 6 possible outcomes with respect to the relationship of the Fourier-coefficients b, b', b'' and α , viz. it depends on the Fourier-coefficients if B is a subset of S_n and if B' or B'' or both B', B'' are subsets of S_{n+1} .

1. $b > \alpha, b' > \alpha, b'' > \alpha$

Then

$$\delta_n = \mu(B \cap S) - (\mu(B' \cap S) + \mu(B'' \cap S)) = 0.$$

2. $b \leq \alpha, b' \leq \alpha, b'' \leq \alpha$

Similarly,

$$\delta_n = \mu(B \cap S^c) - (\mu(B' \cap S^c) + \mu(B'' \cap S^c)) = 0.$$

3. $b > \alpha, b' \leq \alpha, b'' \leq \alpha$

This implies $\mu(B' \cap S) \leq \alpha \mu(B')$ and $\mu(B'' \cap S) \leq \alpha \mu(B'')$, therefore

$$\mu(B \cap S) = \mu(B' \cap S) + \mu(B'' \cap S) \leq \alpha(\mu(B') + \mu(B'')) = \alpha \mu(B)$$

consequently $b \leq \alpha$, and it is a contradiction. This case cannot be realized.

4. $b \leq \alpha, b' > \alpha, b'' > \alpha$

Similarly to the preceding case, this case is also impossible.

5. $b > \alpha, b' > \alpha, b'' \leq \alpha$

Under the assumptions

$$\begin{aligned} \delta_n &= \mu(B \cap S^c) - (\mu(B' \cap S^c) + \mu(B'' \cap S)) \\ &= (\mu(B \cap S^c) - \mu(B' \cap S^c)) - \mu(B'' \cap S) \\ &= \mu(B'' \cap S^c) - \mu(B'' \cap S) \\ &= \mu(B'') - 2\mu(B'' \cap S) \geq 0, \end{aligned}$$

if and only if

$$\mu(B'') \geq 2\mu(B'' \cap S)$$

$$\frac{1}{2} \geq b''.$$

This holds for arbitrary B'' only if $\alpha \leq \frac{1}{2}$.

6. $b \leq \alpha, b' \leq \alpha, b'' > \alpha$

In this case, similarly to the preceding we get

$$\delta_n = \mu(B'') - 2\mu(B'' \cap S^c) \geq 0,$$

if and only if

$$\mu(B'') \geq 2\mu(B'' \cap S^c)$$

$$\frac{1}{2} \geq \frac{\mu(B'' \cap S^c)}{\mu(B'')} = \frac{\mu(B'') - \mu(B'' \cap S)}{\mu(B'')} = 1 - b''$$

$$b'' \geq \frac{1}{2}.$$

It is true in general case only if $\alpha \geq \frac{1}{2}$.

In summary, 3. and 4. are impossible, 1. and 2. implies $\delta_n = 0$, in the cases 5. and 6. $\delta_n \geq 0$ is satisfied only if $\alpha \leq \frac{1}{2}$ and $\alpha \geq \frac{1}{2}$, respectively. Consequently $\delta_n \geq 0$ for all possible outcomes if and only if $\alpha = \frac{1}{2}$. \square

To guarantee the monotonicity of the convergence let us redefine the n -th approximation for $\alpha = \frac{1}{2}$

$$S_n := \{\mathcal{X}_{\frac{1}{2}} \mathcal{F}^{\mathcal{B}_n} \chi_S = 1\}. \tag{4.11}$$

Applying the above idea, we can give a general schema to develop volume-based approximation methods:

1. Let be given a solid $S \in \Omega^d$, moreover let $B_0^{(0)} \supseteq \mathbb{B}^d$ be an arbitrary superset of the unit ball, which is the unique generator set of the \mathcal{B}_0 σ -algebra.
2. **Choose** an index k , where $0 < b_k^{(n)} < 1$, and **divide** the set $B_k^{(n)}$ into two non-empty disjoint sets. In this way we obtain the algebra \mathcal{B}_{n+1} .
3. Compute the new Fourier-coefficients b'_k, b''_k as well as $\rho(S_n, S)$. While $\rho(S_n, S) \geq \varepsilon$ for some given $\varepsilon > 0$ tolerance, go back to step 2.
4. If $\rho(S_n, S) < \varepsilon$ we are done, we could define ∂S if it is needed.

It can be seen, there are two important questions unanswered, namely: how could we choose the k index and how could we divide the $B_k^{(n)}$ atom such that the assumptions of our convergence theorems are satisfied. Let us define the following functions.

Definition 4.9. The $\mathcal{C} : 2^{2^{\mathbb{B}^d}} \rightarrow \mathbb{N}$ type function is said to be a choosing function if

$$\forall n \in \mathbb{N} \exists! k \in [0..n] : \mathcal{C}(\mathcal{B}_n) = k.$$

Moreover $\mathcal{D} : 2^{\mathbb{B}^d} \rightarrow 2^{\mathbb{B}^d} \times 2^{\mathbb{B}^d}$ is said to be a dividing function if

$$\forall B \in 2^{\mathbb{B}^d} \exists! B', B'' \in 2^{\mathbb{B}^d} : B' \neq \emptyset, B'' \neq \emptyset$$

$$\mathcal{D}(B) = (B', B''), B' \cap B'' = \emptyset, B' \cup B'' = B.$$

With our new notations we can draw up the approximation schema more precisely.

VolumeBasedApproximation($S, B_0^{(0)}, \varepsilon, \mathcal{C}, \mathcal{D}$)

1. $n := 0$
2. $\mathcal{B}_n := \sigma(B_0^{(n)}, \dots, B_n^{(n)})$
3. $k := \mathcal{C}(\mathcal{B}_n)$
4. $S_n := \{\mathcal{X}_{\frac{1}{2}} \mathcal{F}^{\mathcal{B}_n} \mathcal{X}_S = 1\}$
5. **if** $\rho(S_n, S) \geq \varepsilon$ **then**
6. $\mathcal{B}_{n+1} := \sigma(B_0^{(n)}, \dots, B_{k-1}^{(n)}, \mathcal{D}(B_k^{(n)}), B_{k+1}^{(n)}, \dots, B_n^{(n)})$
7. $n := n + 1$
8. **goto** 3.
9. **else stop**

(4.12)

To develop a volume-based approximation algorithm we need only to define exactly the choosing function and the dividing function. As we can see $\mathcal{C}, \mathcal{D}, B_0^{(0)}, \varepsilon$ are free parameters, the result can be affected by all of them. In the next section we give a simple example for an approximation algorithm.

5. Application

We applied our method successfully to give approximation of three-dimensional triangular meshes. Let $d := 3, B_0^{(0)} \supset \mathbb{B}^3$ is the cube of side 2 centered at the origin, $\varepsilon > 0$ be an arbitrary real. Denote

$$\mathcal{C}_0(\mathcal{B}_n) := \left\{ k \in \Delta_n \mid \mu(B_k^{(n)}) = \max_{j \in \Delta_n} \mu(B_j^{(n)}) \right\}. \tag{5.1}$$

Let be $a, b \in \mathbb{B}^3$ and

$$H_{a,b} := \left\{ x \in \mathbb{R}^3 \mid \langle x - \frac{a+b}{2}, b - a \rangle \leq 0 \right\} \tag{5.2}$$

is a half-space, the points below the plane determined by the midpoint of a and b and the normal vector in the direction of $b - a$. Then we can define the following function

$$\mathcal{D}_0(B) := \{(B', B'') \mid B' = B \cap H_{a,b}, B'' = B \setminus B' \mid \|a - b\| = \text{diam}(B)\}. \tag{5.3}$$

It is obvious, that $|\mathcal{C}_0| \geq 1$ and $|\mathcal{D}_0| \geq 1$, therefore \mathcal{C}_0 and \mathcal{D}_0 are not functions in generally. It can be thought, that we can give some extra conditions to obtain functions, e.g. let k be the minimal index which satisfies (5.1), and let a have the smallest x, y, z coordinate value satisfying (5.3). Let us denote the functions describing these additional conditions with \mathcal{C}_1 and \mathcal{D}_1 respectively. For instance, we chose the maximal indices for k and for a using that a finite convex polyhedron's diameter can be spanned by only two of its vertices, and vertices can be indexed. Here we

used Corollary 2.10 i.e. a convex polyhedron divided a by plane results two convex polyhedra. As a consequence of the definition of \mathcal{C}_1 and \mathcal{D}_1 , it is easy to see that

$$\mathcal{C} := \mathcal{C}_1 \circ \mathcal{C}_0 \tag{5.4}$$

is a choosing function,

$$\mathcal{D} := \mathcal{D}_1 \circ \mathcal{D}_0 \tag{5.5}$$

is a dividing function.

Theorem 5.1. *Let $B_0^{(0)} \supset \mathbb{B}^3$, $S \subset \Omega^3$ and $\varepsilon > 0$, and let \mathcal{C}, \mathcal{D} be defined by (5.4), (5.5), (5.1), (5.3). Then the algorithm defined by (4.12) is convergent.*

Proof. In terms of Theorem 4.7 it is enough to prove, that the definitions of \mathcal{C} and \mathcal{D} ensure that

$$\lim_{n \rightarrow \infty} \max_{k \in \Delta_n} \text{diam}(B_k^{(n)}) = 0.$$

Let us suppose, that in the n -th step of the iteration the k -th atom has maximal diameter, i.e. $r := \max_{k \in \Delta_n} \text{diam}(B_k^{(n)})$, furthermore let us suppose $\mathcal{C}(B_n) = k$. Corollary 2.10 implies that $B_k^{(n)}$ is a convex polyhedron for all $n \in \mathbb{N}, k \in \Delta_n$ determined by a finite number of vertices. Because of this there are finite number of vertex pairs $(a_0, b_0), \dots, (a_{n_1}, b_{n_1})$ which satisfy $\|a_i - b_i\| \geq \frac{r}{2}$. Let us suppose, that we apply a \mathcal{D} function to $B_k^{(n)}$. By definition we choose a vertex pair (a_i, b_i) , and after the dividing operation there are no atoms with diameter determined by (a_i, b_i) , since the a_i and b_i vertices are assigned to different ones. Moreover, it is impossible, that after the operation new vertex pairs were formed, whose distance is greater than $\frac{r}{2}$. Therefore the new atoms generated from $B_k^{(n)}$ can have at least one less vertex pairs, which satisfy $\|a_i - b_i\| \geq \frac{r}{2}$. Consequently, after a finite number of iterations all the atoms obtained from $B_k^{(n)}$ will have the diameter strictly less than $\frac{r}{2}$, i.e.

$$\exists N_1 \in \mathbb{N} \forall k_1 \in \Delta_{N_1} : B_{k_1}^{(n)} \cap B_{k_1}^{(N_1)} \neq \emptyset \Rightarrow \text{diam}(B_{k_1}^{(N_1)}) < \frac{r}{2}.$$

On the other hand, we have finite number of atoms, $n + 1$ in the n -th iteration, so there are only a finite number of atoms B_0, \dots, B_{n_2} , which satisfy $\text{diam}(B_i) \geq \frac{r}{2}$. By definition of \mathcal{C} , if $n_2 > 0$ and $n > N_1$ we need to choose an atom for which $B_i \cap B_k^{(n)} = \emptyset$, since $\text{diam}(B_i) \geq \frac{r}{2} > \text{diam}(B_{k_1}^{(N_1)})$ for all $k_1 \in \Delta_{N_1}$. Due to the above explanation, after finitely many iterations all the atoms obtained from B_0, \dots, B_{n_2} will have the diameter strictly less than $\frac{r}{2}$, therefore *all* the atoms will have this property, as well. In other words

$$\exists N_2 \in \mathbb{N} : \max_{k_2 \in \Delta_{N_2}} \text{diam}(B_{k_2}^{(N_2)}) < \frac{r}{2}.$$

The above idea can be applied arbitrary many times. Using the fact that $r \leq 1$, we find $(\frac{r}{2})^n \rightarrow 0$ as $n \rightarrow \infty$. □

6. Results, Future Work

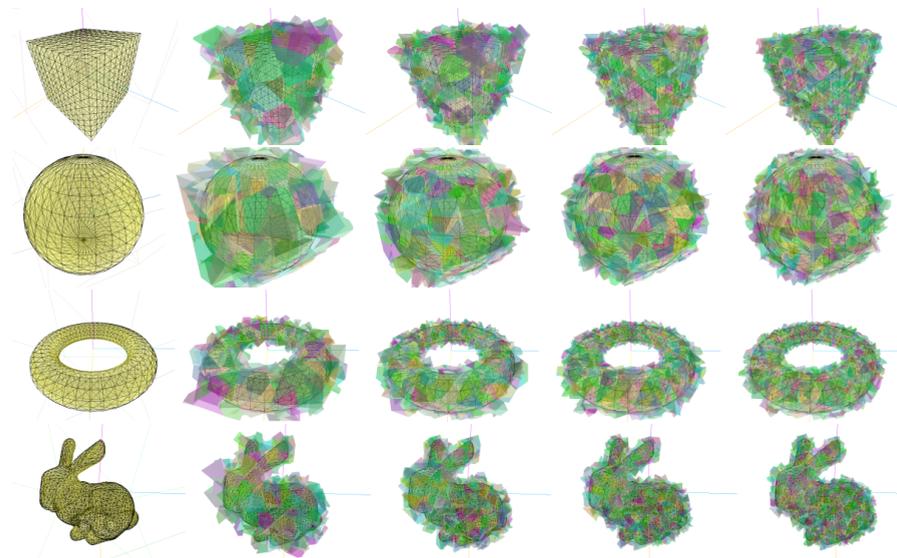


Figure 1. Test models were Cube, Sphere, Torus and Bunny. From left to right we can see the test model, and the resulting approximation, S_n after $n = 1000, 2000, 3000, 4000$ iterations, respectively.

TABLE 1. Results for our 4 test models. After the model name the number of vertices and faces are shown. The table contains the $d^*(S_n)$ maximal diameter and the $\rho^*(S_n, S)$ error estimation for iteration number n .

Cube (602, 1200)			Sphere (482, 960)		
n	$d^*(S_n)$	$\rho^*(S_n, S)$	n	$d^*(S_n)$	$\rho^*(S_n, S)$
1000	0,39	1,98	1000	0,43	3,16
2000	0,27	1,35	2000	0,31	2,31
3000	0,22	1,07	3000	0,26	1,90
4000	0,19	0,90	4000	0,22	1,63
Torus (576, 1152)			Bunny (2503, 4968)		
n	$d^*(S_n)$	$\rho^*(S_n, S)$	n	$d^*(S_n)$	$\rho^*(S_n, S)$
1000	0,31	1,14	1000	0,30	0,97
2000	0,22	0,84	2000	0,27	0,71
3000	0,18	0,69	3000	0,17	0,58
4000	0,16	0,60	4000	0,15	0,50

The algorithm obtained from our approximation schema was implemented by choosing and dividing functions defined in the last section. Some tests were performed on triangular meshes as we can see on Figure 1. We show 4 models: cube, sphere, torus, and the Stanford Bunny (see [24]) from top to bottom, respectively. From left

to right we can see 5 level of approximation, the test model and the resulting S_n after $n = 0, 1000, 2000, 3000, 4000$ iterations, respectively. The most important properties of the approximations were indicated in Table 6. These tables contain the maximal diameters

$$d^*(S_n) := \max_{k \in \Delta_n} \text{diam}(B_k^{(n)}) \quad (6.1)$$

and the estimation of the error, defined as

$$\rho^*(S_n, S) := \sum_{i \in \Delta_n} \mu(B_i^{(n)}) \quad (6.2)$$

where $\rho^*(S_n, S) \geq \rho(S_n, S)$ according to Corollary 4.6.

Our future plans are to work out some new choosing and dividing functions, to show that our method contains in particular most of the space partitioning methods as Octree, spatial decomposition, approximate or exact convex decomposition, etc. We are working on some strategies for choosing and dividing, that can have a dramatic effect on rate of convergence. We are designing an effective data structure for the decomposition, in addition we try to increase the efficiency of the implementation. The source code of the preparing software package will be publicly available in the future.

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