The generalization of Mastroianni operators using the Durrmeyer’s method

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Abstract. In the present paper, we define a sequence of Durrmeyer’s type operators associated with Mastroianni operators and introduce a new operator.

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1. Introduction

In [5], [6] G. Mastroianni defined and studied a general class of linear positive approximation operators, which was generalized by O. Agratini, B. Della Vecchia [1]. In brief we recall this construction.

Taking $[0, \infty) := \mathbb{R}_+$, we consider the next spaces of functions:

$B(\mathbb{R}_+) = \{ f : \mathbb{R}_+ \longrightarrow \mathbb{R} | (\exists) M_f > 0 : |f(x)| \leq M_f \}$, a normed space with the uniform norm $\| f \|_B = \sup \{ |f(x)| : x \in \mathbb{R}_+ \}$;

$B_\rho(\mathbb{R}_+) = \{ f : \mathbb{R}_+ \longrightarrow \mathbb{R} | |f(x)| \leq N_f \rho(x), N_f > 0, \rho(x) = 1 + x^2 \}$, a normed space with the norm $\| f \|_\rho = \sup \{ \frac{|f(x)|}{\rho(x)} : x \geq 0 \} = \sup \{ \frac{|f(x)|}{1 + x^2} : x \geq 0 \}$;

$C_\rho(\mathbb{R}_+) = \{ f \in B_\rho(\mathbb{R}_+) | f \ \text{continuous function} \}$;

$C_\rho^*(\mathbb{R}_+) = \{ f \in C_\rho(\mathbb{R}_+) | (\exists) \lim_{x \to \infty} |f(x)| < \infty \}$.

The space $C_\rho^*(\mathbb{R}_+)$ endowed with the norm $\| f \|_\rho$ is a Banach space.

In our estimations we use the first modulus of continuity on a finite interval $[0, b]$, $b > 0$, $\omega_{[0,b]}(f; \delta) = \sup \{ |f(x + h) - f(x)| : 0 < h \leq \delta, x \in [0, b] \}$ and the Peetre’s K-functional defined as

$$K_2(f; \delta) = \inf \{ \| f - g \|_B + \delta \| g'' \|_B : g \in W^2_\infty \}$$,

where $W^2_\infty = \{ g \in C_B(\mathbb{R}_+) : g', g'' \in C_B(\mathbb{R}_+) \}$.

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It is known (see [9] p. 177, th. 2.4) that, there exists a positive constant $C$ such that $K_2(f; \delta) \leq C \omega_2(f; \sqrt{\delta})$, where

$$\omega_2(f; \sqrt{\delta}) = \sup_{0 < h \leq \sqrt{\delta}} \sup_{x \in [0, \infty)} \{|f(x + 2h) - 2f(x + h) + f(x)|\}.$$  

Let $(\Phi_n)_{n \geq 1}$ be a sequence of real functions defined on $[0, \infty) := \mathbb{R}_+$ which are infinitely differentiable on $\mathbb{R}_+$ and satisfy the conditions:

(i) $\Phi_n(0) = 1$, $n \in \mathbb{N}$;

(ii) for every $n \in \mathbb{N}$, $x \in \mathbb{R}_+$ and $k \in \mathbb{N} \cup \{0\} := \mathbb{N}_0$,

$$(−1)^k \Phi_n^{(k)}(x) \geq 0; \quad (1.1)$$

(iii) for each $(n, k) \in \mathbb{N} \times \mathbb{N}_0$ there exists a number $p(n, k) \in \mathbb{N}$ and a function $\alpha_{n,k} \in \mathbb{R}$ such that $\Phi_n^{(i+k)}(x) = (−1)^k \Phi_n^{(i)}(p(n, k)) \alpha_{n,k}(x)$, $i \in \mathbb{N}_0$, $x \in \mathbb{R}_+$ and

$$(iv) \lim_{n \to \infty} \frac{n}{p(n, k)} = \lim_{n \to \infty} \frac{\alpha_{n,k}(x)}{n^k} = 1.$$  

Remark 1.1. There is easy to see that

$$\lim_{n \to \infty} \frac{\Phi_n^{(k)}(0)}{n^k} = \lim_{n \to \infty} \frac{(−1)^k \alpha_{n,k}(0)}{n^k} = (−1)^k. \quad (1.2)$$

The Mastroianni operators $M_n : C_B(\mathbb{R}_+) \rightarrow C(\mathbb{R}_+)$ are defined by the following formula

$$M_n(f, x) = \sum_{k=0}^{\infty} m_{n,k}(x) f \left( \frac{k}{n} \right) \quad (1.3)$$

with the basis functions,

$$m_{n,k}(x) = \frac{(-x)^k \Phi_n^{(k)}(x)}{k!}. \quad (1.4)$$

For these operators and for the test functions $e_r(x) = x^r$, $r = 0, 1, 2$ the following results were obtained [5]:

$$M_n(e_0; x) = \Phi_n(0),$$

$$M_n(e_1; x) = -\frac{\Phi'_n(0)}{n} x,$$

$$M_n(e_2; x) = \frac{\Phi''_n(0)x^2 - \Phi'_n(0)x}{n^2}. \quad (1.5)$$

In terms of the hypergeometric and confluent hypergeometric functions, recent results, about Durrmeyer type operators [2], [3], [4], [8], have considered in the definition of the basis functions, the family’s functions:

$$\Phi_{n,c}(x) = \begin{cases} e^{-nx}, & c = 0, \; x \geq 0 \\ (1 + cx)^{-\frac{n}{c}}, & c \in \mathbb{N}, \; x \geq 0 \end{cases}$$

For these functions we have

$$\Phi_{n,c}^{(k+1)}(x) = -n \Phi_{n,c}^{(k)}(x), \; n > \max \{0, -c\}$$
respectively
\[ \Phi^{(i+k)}_{n,c}(x) = (-1)^k n_{[k,-c]} \Phi^{(i)}_{n+k,c,c}(x) \]
where \( n_{[k,-c]} = n(n+c)(n+2c) \cdots (n+k-1)c \) is the factorial power of order \( k \) of \( n \) with the increment \( -c \) and \( n_{[0,-c]} = 1 \).

So, the conditions (iii)-(iv) are true, for \( p(n,k) = n + kc \) and \( \alpha_{n,k}(x) = n_{[k,-c]} \).

In the next section we propose a Mastroianni–Durrmeyer operator, when the sequence of functions \( (\Phi_n)_{n \geq 1} \) satisfy the conditions (i)-(iv) and other supplementary conditions, is non-nominated.

2. Main results

Let \( (\Phi_n)_{n \geq 1} \) be the sequence of functions which satisfy the conditions (i)-(iv) and the next supplementary conditions, for any \( n \in \mathbb{N} \) and \( r, k \in \mathbb{N}_0 \):

(i) \( \lim_{x \to \infty} x^r \Phi^{(k)}_{n}(x) = 0 \)

(ii) \( \exists J_{n,k,r} := \int_0^\infty t^r \Phi^{(k)}_{n}(t)dt < \infty, (\exists) J_{n,0,r} := \int_0^\infty \Phi_{n}(t)dt \neq 0. \)

We define the operators of Durrmeyer type associated with Mastroianni operators (1.3)-(1.4) for each real value function \( f \in \mathbb{R}^\mathbb{R} \) for which the series exists:

\[ DM_n(f; x) = \sum_{k=0}^{\infty} m_{n,k}(x) \int_0^\infty m_{n,k}(t)f(t)dt = \int_0^\infty K_n(t,x)f(t)dt \quad (2.1) \]
with the kernel

\[ K_n(t, x) = \frac{1}{I_{n,0,0}} \sum_{k=0}^{\infty} m_{n,k}(x)m_{n,k}(t), I_{n,0,0} = J_{n,0,0} = \int_0^\infty \Phi_{n}(t)dt \neq 0. \quad (2.2) \]

Lemma 2.1. The next identity is true for any \( n \in \mathbb{N} \) and \( r, k \in \mathbb{N}_0 \)

\[ I_{n,k,r} = \frac{(r+1)_k}{k!} I_{n,0,r}, \]
where

\[ I_{n,k,r} := \int_0^\infty t^r m_{n,k}(t)dt = \frac{(-1)^k}{k!} J_{n,k,r+k} \]
and

\( (n)_k = n(n+1)(n+2) \cdots (n+k-1) = n_{[k,-1]}, \ (n)_0 = 1 \)
is the Pochhammer symbol or the factorial power of order \( k \) of \( n \) and the increment \(-1\). So, \( (1)_k = k!, \ (2)_k = (k+1)! \).

The proof suppose an easy computation, so we have omitted them. We remark that
\[ I_{n,0,r} = J_{n,0,r} = \int_0^\infty t^r \Phi_{n}(t)dt, r \geq 0, \] (the moments of the \( r \)-th order reported to \( \Phi_n \))
Using Lemma 2.1 we obtain

\[ I_{n,k,0} = \int_0^\infty m_{n,k}(t)dt = \frac{(-1)^k}{k!}J_{n,k,k}, \]

\[ J_{n,k,k} = (-1)^k k!J_{n,0,0}, \quad k \geq 0, \]

\[ I_{n,k,0} = I_{n,0,0} = \int_0^\infty \Phi_n(t)dt, \]

\[ I_{n,k,r} = \int_0^\infty t^r m_{n,k}(t)dt = \frac{(-1)^k}{k!} \int_0^\infty t^{r+k} \Phi_n^{(k)}(t)dt = \frac{(r+1)_k}{k!} \int_0^\infty t^r \Phi_n(t)dt = \frac{(r+1)_k}{k!} I_{n,0,r}, \quad \text{(the moments of the r-th order reported to } \Phi_n). \]

**Lemma 2.2.** The moments of the operators \( DM_n(f; x) \) are given for \( e_r(x) = x^r, \quad r \in \mathbb{N}_0 \) as

\[ DM_n \left( e_r; x \right) = \frac{I_{n,0,r}}{I_{n,0,0}} \sum_{k=0}^\infty \frac{(r+1)_k}{k!} m_{n,k}(x). \] (2.3)

Further, we have

\[ DM_n \left( e_0; x \right) = 1, \]

\[ DM_n \left( e_1; x \right) = \frac{I_{n,0,1}}{I_{n,0,0}} (1 - x \Phi_n'(0)), \]

\[ DM_n \left( e_2; x \right) = \frac{I_{n,0,2}}{2I_{n,0,0}} (x^2 \Phi_n''(0) - 4x \Phi_n'(0) + 2), \] (2.4)

\[ DM_n \left( (e_1 - xe_0)^2; x \right) = x^2 \left( \frac{I_{n,0,2}}{2I_{n,0,0}} \Phi_n''(0) + 2 \frac{I_{n,0,1}}{I_{n,0,0}} \Phi_n'(0) + 1 \right) - 2x \left( \frac{I_{n,0,2}}{I_{n,0,0}} \Phi_n'(0) + \frac{I_{n,0,1}}{I_{n,0,0}} \right) + \frac{I_{n,0,2}}{I_{n,0,0}}. \]

**Proof.** Using Lemma 2.1 we obtain

\[ DM_n \left( e_r; x \right) = \frac{1}{I_{n,0,0}} \sum_{k=0}^\infty m_{n,k}(x)I_{n,k,r} = \frac{I_{n,0,r}}{I_{n,0,0}} \sum_{k=0}^\infty m_{n,k}(x) \frac{(r+1)_k}{k!}. \]

\[ DM_n \left( e_0; x \right) = \frac{1}{I_{n,0,0}} \sum_{k=0}^\infty m_{n,k}(x)I_{n,k,0} = \frac{I_{n,0,0}}{I_{n,0,0}} \sum_{k=0}^\infty m_{n,k}(x) \frac{(1)_k}{k!} = 1, \]

\[ DM_n \left( e_1; x \right) = \frac{1}{I_{n,0,0}} \sum_{k=0}^\infty m_{n,k}(x)I_{n,k,1} = \frac{I_{n,0,1}}{I_{n,0,0}} \sum_{k=0}^\infty m_{n,k}(x) \frac{(2)_k}{k!} \]

\[ = \frac{I_{n,0,1}}{I_{n,0,0}} \sum_{k=0}^\infty m_{n,k}(x) \frac{(k+1)!}{k!} = \frac{I_{n,0,1}}{I_{n,0,0}} \sum_{k=0}^\infty m_{n,k}(x)(k+1) \]

\[ = \frac{nI_{n,0,1}}{I_{n,0,0}} \left( M_n \left( e_1; x \right) + \frac{1}{n} \right) = \frac{nI_{n,0,1}}{I_{n,0,0}} \left( \frac{\Phi_n'(0)}{n} x + \frac{1}{n} \right). \]
Because \( \text{DM}_n (e_2; x) = \frac{1}{I_{n,0,0}} \sum_{k=0}^{\infty} m_{n,k}(x) I_{n,k,2} = \frac{I_{n,0,2}}{I_{n,0,0}} \sum_{k=0}^{\infty} m_{n,k}(x) \frac{(3)_k}{k!} \)

\[
= \frac{I_{n,0,2}}{2I_{n,0,0}} \sum_{k=0}^{\infty} m_{n,k}(x) \frac{(k + 2)!}{k!}
\]

\[
= \frac{n^2 I_{n,0,2}}{2I_{n,0,0}} \left( M_n (e_2; x) + \frac{3}{n} M_n (e_1; x) + \frac{2}{n^2} \right)
\]

\[
= \frac{n^2 I_{n,0,2}}{2I_{n,0,0}} \left( \Phi''(0)x^2 - \Phi'(0)x \right) + \frac{3}{n^2} \frac{\Phi'(0)x}{x} + \frac{2}{n^2}.
\]

Because \( \text{DM}_n ((e_1 - xe_0)^2; x) = \text{DM}_n (e_2; x) - 2x \text{DM}_n (e_1; x) + x^2 \text{DM}_n (e_0; x) \) is easy to obtain the relation of enunciation. \( \square \)

**Lemma 2.3.** Let

\[
\text{DM}_n(f; x) = \text{DM}_n(f; x) - f \left( \frac{nI_{n,0,1}}{I_{n,0,0}} \left( 1 - \frac{n}{n} \Phi'(0) \right) \right) + f(x).
\]  \hspace{1cm} (2.5)

The following assertions hold:

\[
\text{DM}_n (e_0; x) = 1,
\]

\[
\text{DM}_n (e_1; x) = x,
\]

\[
\text{DM}_n (e_1 - xe_0; x) = 0.
\]

The proof suppose an easy computation, so we have omitted them.

Further, we consider the next conventions:

\[
|\text{DM}_n (e_1 - xe_0; x)| = \left| \frac{nI_{n,0,1}}{I_{n,0,0}} (1 - x\Phi'(0)) - x \right| := \lambda_n(x), \hspace{1cm} (2.6)
\]

\[
\text{DM}_n ((e_1 - xe_0)^2; x) = x^2 \left( \frac{nI_{n,0,2}}{2I_{n,0,0}} \Phi''(0) + 2 \frac{nI_{n,0,1}}{I_{n,0,0}} \Phi'(0) + 1 \right)
\]

\[
-2x \left( \frac{nI_{n,0,2}}{I_{n,0,0}} \Phi'(0) + \frac{nI_{n,0,1}}{I_{n,0,0}} \right) + \frac{nI_{n,0,2}}{I_{n,0,0}} = \beta_n(x).
\]  \hspace{1cm} (2.7)

From (1.1) we have \( \Phi_n(x) \geq 0, \Phi'_n(x) \leq 0, \Phi''_n(x) \geq 0, x \in \mathbb{R}_+ \) and so

\[
\beta_n(x) \leq x^2 \left( \frac{nI_{n,0,2}}{I_{n,0,0}} \Phi''(0) + 1 \right) - 2x \frac{nI_{n,0,2}}{I_{n,0,0}} \Phi'(0) + \frac{nI_{n,0,2}}{I_{n,0,0}} := \eta_n(x).
\]  \hspace{1cm} (2.8)

Because \( \text{DM}_n \) is a linear positive operator, using the Cauchy-Schwarz’s inequality we have

\[
\lambda_n(x) = |\text{DM}_n (e_1 - xe_0; x)| \leq \text{DM}_n ((e_1 - xe_0)^2; x) \leq \sqrt{\text{DM}_n ((e_1 - xe_0)^2; x)} = \sqrt{\beta_n(x)}.
\]

Let

\[
\gamma_n(x) = \beta_n(x) + \lambda_n^2(x) \leq 2\beta_n(x). \hspace{1cm} (2.9)
\]
Lemma 2.4. For every \( x \in \mathbb{R}_+ \) and \( f'' \in C_B(\mathbb{R}_+) \) we have

\[
|DM_n(f; x) - f(x)| \leq \frac{\|f''\|_B}{2} \gamma_n(x).
\]

Proof. Using Taylor's expansion

\[
f(t) = f(x) + (t - x)f'(x) + \int_x^t (t - u)f''(u)du
\]

we obtain with Lemma 2.3 that

\[
DM_n(f; x) - f(x) = DM_n\left(\int_x^t (t - u)f''(u)du; x\right).
\]

Because \( \left|\int_x^t (t - u)f''(u)du\right| \leq \|f''\|_B \frac{(t - x)^2}{2} \) using Lemma 2.2 we get

\[
|DM_n(f; x) - f(x)| \leq DM_n\left(\int_x^t (t - u)f''(u)du; x\right)
\]

\[
\leq \frac{\|f''\|_B}{2} DM_n((t - x)^2; x) + \frac{\|f''\|_B}{2} \left(\frac{I_{n,0,1}}{I_{n,0,0}}(1 - x\Phi_n(0)) - x\right)^2
\]

\[
\leq \frac{\|f''\|_B}{2} (\beta_n(x) + \lambda^2_n(x)).
\]

\( \square \)

Theorem 2.5. For every \( x \in \mathbb{R}_+ \) and \( f \in C_B(\mathbb{R}_+) \) the operators (2.1)-(2.2) satisfy the following relations

(i) If \( \lim_{n \to \infty} \frac{n^r I_{n,0,r}}{r! I_{n,0,0}} = 1 \), \( r = 0, 1, 2 \), then \( \lim_{n \to \infty} DM_n(f; x) = f(x) \),

(ii) \( |DM_n(f; x) - f(x)| \leq 2 \omega \left(f, \sqrt{\beta_n(x)}\right)\),

(iii) \( |DM_n(f; x) - f(x)| \leq 2C \omega_2 \left(f, \sqrt{\gamma_n(x)}\right) + \omega \left(f, \lambda_n(x)\right)\).

with \( \lambda_n(x), \beta_n(x), \eta_n(x), \gamma_n(x) \) defined as (2.6), (2.7), (2.8), (2.9).

Proof. (i) Because \( \lim_{n \to \infty} \frac{\Phi_n(k)(0)}{n^k} = \lim_{n \to \infty} \frac{(-1)^k \alpha_n(k)(0)}{n^k} = (-1)^k \) and \( \lim_{n \to \infty} \frac{n^r I_{n,0,r}}{r! I_{n,0,0}} = 1 \), \( r = 0, 1, 2 \) using Lemma 2.2 we have \( \lim_{n \to \infty} DM_n(e_r; x) = e_r(x), \) \( r = 0, 1, 2 \) and the Bohmann-Korovkin assure the conclusion (i) of the theorem.
(ii) Using a result of O. Shisha, B. Mond [7] with the modulus of continuity of \( f \) we obtain a quantitative estimation of the remainder of the approximation formula. Indeed,

\[
|DM_n(f; x) - f(x)| \leq \left(1 + \delta_n^{-1}(x)\right)\sqrt{DM_n\left((e_1 - xe_0)^2; x\right)} \omega(f, \delta_n(x)) .
\]

Taking

\[
\delta_n(x) = \sqrt{\beta_n(x)} = \sqrt{DM_n\left((e_1 - xe_0)^2; x\right)}
\]

\[
= \left\{ \frac{I_{n,0,2}}{2I_{n,0,0}} (x^2\Phi''_n(0) - 4x\Phi'_n(0) + 2) - 2x \frac{I_{n,0,1}}{I_{n,0,0}} (1 - x\Phi'_n(0)) + x^2 \right\}^{\frac{1}{2}}
\]

the proof of (ii) is completed.

(iii) From (2.5) we obtain for \( g \in W_\infty^2 \)

\[
|DM_n(f; x) - f(x)| \\
\leq \frac{|DM_n(f - g; x) - (f - g)(x) + DM_n(g; x) - g(x)|}{2} \\
+ \left| \frac{nI_{n,0,1}}{I_{n,0,0}} \left( \frac{1}{n} - \frac{\Phi'_n(0)}{n} \right) x \right| - f(x) \right| \\
\leq 2 \| f - g \|_B + \frac{\|g''\|_B}{2} \gamma_n(x) \right| \\
+ \left| \frac{nI_{n,0,1}}{I_{n,0,0}} \left( \frac{1}{n} - \frac{\Phi'_n(0)}{n} \right) x \right| - f(x) \right| \\
\leq 2 \| f - g \|_B + \frac{\|g''\|_B}{2} \gamma_n(x) + \omega(f, \lambda_n(x)) .
\]

Taking infimum over \( g \in W_\infty^2 \) on the right hand side, we get

\[
|DM_n(f; x) - f(x)| \leq 2K_2 (f, \gamma_n(x)) + \omega(f, \lambda_n(x)) \]

\[
\leq 2C \omega_2 \left(f, \sqrt{\gamma_n(x)}\right) + \omega(f, \lambda_n(x)) .
\]

Theorem 2.6. Let \( f \in C_\rho(\mathbb{R}_+) \) and \( \omega_{[0,b+1]}(f; \delta) \) be its modulus of continuity on the finite interval \([0, b + 1], b > 0\). Then

\[
\|DM_n(f) - f\|_{C[0,b]} \leq 3NF_n(b)(1 + b)^2 + 2\omega_{[0,b+1]} \left(f, \sqrt{\eta_n(b)}\right) ,
\]

with \( \eta_n(x) \) defined as (2.8).

Proof. Let \( x \in \mathbb{R}_+ \) and \( t > b + 1 \) because \( f \in C_\rho(\mathbb{R}_+) \) using the growth condition of \( f \) since \( t - x > 1 \) we have

\[
|f(t) - f(x)| \leq NF_2(2 + t^2 + x^2) \leq NF(2 + (t - x + x)^2 + x^2) \leq 3NF(t - x)^2(1 + b)^2 .
\]

For \( x \in \mathbb{R}_+, \delta > 0 \) and \( t < b + 1 \) we have

\[
|f(t) - f(x)| \leq \left(1 + \frac{|t - x|}{\delta}\right) \omega_{[0,b+1]} (f, \delta) .
\]
So,
\[ |f(t) - f(x)| \leq 3N_f(t - x)^2(1 + b)^2 + \left(1 + \frac{|t - x|}{\delta}\right) \omega_{[0,b+1]} (f, \delta). \]

and with (2.8) we obtain
\[
|DM_n(f; x) - f(x)| \leq 3N_f DM_n \left( (e_1 - xe_0)^2; x \right) (1 + b)^2 \\
+ \left(1 + \frac{DM_n \left( |e_1 - xe_0|; x \right)}{\delta}\right) \omega_{[0,b+1]} (f, \delta) \\
\leq 3N_f DM_n \left( (e_1 - xe_0)^2; x \right) (1 + b)^2 \\
+ \left(1 + \frac{\sqrt{DM_n \left( (e_1 - xe_0)^2; x \right)}}{\delta}\right) \omega_{[0,b+1]} (f, \delta) \\
\leq 3N_f \eta_n(b)(1 + b)^2 + \left(1 + \frac{\sqrt{\eta_n(b)}}{\delta}\right) \omega_{[0,b+1]} (f, \delta) \\
\leq 3N_f \eta_n(b)(1 + b)^2 + 2\omega_{[0,b+1]} \left(f, \sqrt{\eta_n(b)}\right). \quad \square
\]

References


