On systems of semilinear hyperbolic functional equations

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Abstract. We consider a system of second order semilinear hyperbolic functional differential equations where the lower order terms contain functional dependence on the unknown function. Existence of solutions for \( t \in (0, T) \) and \( t \in (0, \infty) \), further, examples and some qualitative properties of the solutions in \((0, \infty)\) are shown.

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1. Introduction

In the present work we shall consider weak solutions of initial-boundary value problems of the form

\[
\begin{aligned}
&u''(t) + Q_j(u(t)) + \varphi(x) D_j h(u(t)) + H_j(t, x; u) + G_j(t, x; u, u') = F_j, \\
&t > 0, \ x \in \Omega, \quad j = 1, \ldots, N \\
&u(0) = u^{(0)}, \quad u'(0) = u^{(1)}
\end{aligned}
\]

where \( \Omega \subset \mathbb{R}^n \) is a bounded domain and we use the notations \( u(t) = (u_1(t), \ldots, u_N(t)) \), \( u(t) = (u_1(t, x), \ldots, u_N(t, x)) \), \( u' = (u'_1, \ldots, u'_N) = D_t u = (D_t u_1, \ldots, D_t u_N) \), \( u'' = D_t^2 u \), \( Q_j \) is a linear second order symmetric elliptic differential operator in the variable \( x \); \( h \) is a \( C^1 \) function having certain polynomial growth, \( H_j \) and \( G_j \) contain nonlinear functional (non-local) dependence on \( u \) and \( u' \), with some polynomial growth.

There are several papers on semilinear hyperbolic differential equations, see, e.g., [3], [4], [10], [14] and the references there. Semilinear hyperbolic functional equations were studied, e.g. in [5], [6], [7], with certain non-local terms, generally in the form of particular integral operators containing the unknown function. First order quasilinear evolution equations with non-local terms were considered, e.g., in [13] and [15], second
order quasilinear evolution equations with non-local terms were considered in [11], by using the theory of monotone type operators (see [2], [9], [16]).

This work was motivated by the classical book [9] of J.L. Lions on nonlinear PDEs where a single equation was considered in a particular case (semilinear hyperbolic differential equation). We shall use ideas of the above work.

Semilinear hyperbolic functional equations were considered in a previous work of the author (see [12]).

2. Existence in \((0, T)\)

Denote by \(\Omega \subset \mathbb{R}^n\) a bounded domain with sufficiently smooth boundary, and let \(Q_T = (0, T) \times \Omega\). Denote by \(W^{1,2}(\Omega)\) the Sobolev space with the norm

\[
\|u\| = \left[ \int_\Omega \left( \sum_{j=1}^n |D_j u|^2 + |u|^2 \right) \, dx \right]^{1/2}.
\]

Further, let \(V_j \subset W^{1,2}(\Omega)\) be closed linear subspaces of \(W^{1,2}(\Omega)\), \(V_j^*\) the dual space of \(V_j\), \(V = (V_1, ..., V_N)\), \(V^* = (V_1^*, ..., V_N^*)\), \(H = L^2(\Omega) \times \cdots \times L^2(\Omega)\), the duality between \(V_j^*\) and \(V_j\) (and between \(V^*\) and \(V\)) will be denoted by \(\langle \cdot, \cdot \rangle\), the scalar product in \(L^2(\Omega)\) and \(H\) will be denoted by \((\cdot, \cdot)\). Denote by \(L^2(0, T; V_j)\) and \(L^2(0, T; V)\) the Banach space of measurable functions \(u : (0, T) \to V_j\), \(u : (0, T) \to V\), respectively, with the norm

\[
\|u_j\|_{L^2(0, T; V_j)} = \left[ \int_0^T \|u_j(t)\|_{V_j}^2 \, dt \right]^{1/2}, \quad \|u\|_{L^2(0, T; V)} = \left[ \int_0^T \|u(t)\|_V^2 \, dt \right]^{1/2},
\]

respectively.

Similarly, \(L^\infty(0, T; V_j)\), \(L^\infty(0, T; V)\), \(L^\infty(0, T; L^2(\Omega))\), \(L^\infty(0, T; H)\) is the set of measurable functions \(u_j : (0, T) \to V_j\), \(u : (0, T) \to V\), \(u_j : (0, T) \to L^2(\Omega)\), \(u : (0, T) \to H\), respectively, with the \(L^\infty(0, T)\) norm of the functions \(t \mapsto \|u_j(t)\|_{V_j}\), \(t \mapsto \|u(t)\|_V\), \(t \mapsto \|u_j(t)\|_{L^2(\Omega)}\), \(t \mapsto \|u(t)\|_H\), respectively.

Now we formulate the assumptions on the functions in (1.1).

\((A_1)\). \(Q : V \to V^*\) is a linear continuous operator defined by

\[
\langle Q(u), v \rangle = \sum_{j=1}^N \langle Q_j(u), v_j \rangle = \sum_{j=1}^N \left[ \sum_{k=1}^N \langle Q_{jk}(u_k), v_j \rangle \right],
\]

\(u = (u_1, ..., u_N), \quad v = (v_1, ..., v_N),\)

where \(Q_{jk} : W^{1,2}(\Omega) \to [W^{1,2}(\Omega)]^*\) are continuous linear operators satisfying

\[
\langle Q_{jk}(u_k), v_j \rangle = \langle Q_{jk}(v_j), u_k \rangle, \quad Q_{jk} = Q_{kj}, \text{ thus } \langle Q(u), v \rangle = \langle Q(v), u \rangle
\]

for all \(u, v \in V\) and

\[
\langle Q(u), u \rangle \geq c_0 \|u\|_V^2 \text{ with some constant } c_0 > 0.
\]

\((A_2)\). \(\varphi : \Omega \to \mathbb{R}\) is a measurable function satisfying

\[c_1 \leq \varphi(x) \leq c_2 \text{ for a.a. } x \in \Omega\]
with some positive constants $c_1, c_2$.

(A₃). $h : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function satisfying

$$h(\eta) \geq 0, \quad |D_j h(\eta)| \leq \text{const} |\eta|^\lambda \text{ for } |\eta| > 1$$

where

$$1 < \lambda \leq \lambda_0 = \frac{n}{n-2} \text{ if } n \geq 3, \quad 1 < \lambda < \infty \text{ if } n = 2.$$

(A₄). $h : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function satisfying with some positive constants $c_3, c_4$

$$h(\eta) \geq c_3 |\eta|^{\lambda+1}, \quad |D_j h(\eta)| \leq c_4 |\eta|^\lambda \text{ for } |\eta| > 1, \quad n \geq 3 \text{ where } \lambda > \lambda_0 = \frac{n}{n-2},$$

$$|D_j h(\eta)| \leq c_4 |\eta|^\lambda \quad \text{for } |\eta| > 1, \quad n = 2 \text{ where } 1 < \lambda < \infty.$$

(A₅). $H_j : Q_T \times [L^2(Q_T)]^N \to \mathbb{R}$ are functions for which $(t, x) \mapsto H_j(t, x; u)$ is measurable for all fixed $u \in H$, $H_j$ has the Volterra property, i.e. for all $t \in [0, T]$, $H_j(t, x; u)$ depends only on the restriction of $u$ to $(0, t)$; the following inequality holds for all $t \in [0, T]$ and $u \in H$:

$$\int_{\Omega} |H_j(t, x; u)|^2 dx \leq c^* \left[ \int_{\Omega} \int_0^t h(u(\tau)) d\tau dx + \int_{\Omega} h(u) dx \right].$$

Finally, $(u^{(k)}) \to u$ in $[L^2(Q_T)]^N$ and $(u^{(k)}) \to u$ a.e. in $Q_T$ imply

$$H_j(t, x; u^{(k)}) \to H_j(t, x; u) \text{ for a.a. } (t, x) \in Q_T.$$

(A₅'). $G_j : Q_T \times [L^2(Q_T)]^N \times L^\infty(0, T; H) \to \mathbb{R}$ is a function satisfying: $(t, x) \mapsto G_j(t, x; u, w)$ is measurable for all fixed $u \in [L^2(Q_T)]^N$, $w \in L^\infty(0, T; H)$, $G_j$ has the Volterra property: for all $t \in [0, T]$, $G_j(t, x; u, w)$ depends only on the restriction of $u, w$ to $(0, t)$ and

$$G_j(t, x; u, w') = \varphi_j(t, x; u)u'_j(t) + \psi_j(t, x; u, u')$$

where

$$\varphi_j \geq 0, \quad |\varphi_j(t, x; u)| \leq \text{const} \quad (2.1)$$

if (A₃) is satisfied.

(A₅''). If (A₅') is satisfied, we assume instead of the second inequality in (2.1)

$$\int_{\Omega} |\varphi_j(t, x; u)|^2 dx \leq \text{const} \left[ \int_{Q_T} |u|^{2\mu} d\tau dx + \int_{\Omega} |u|^{2\mu} dx \right] \quad (2.2)$$

where $\mu \leq \frac{n+1}{n-1} \frac{\lambda-1}{\lambda+1}$.

Further, on $\psi_j$ we assume

$$\int_{\Omega} |\psi_j(t, x; u, w')|^2 dx \leq c_1 + c_2 \int_{Q_T} |u'|^2 d\tau$$

with some constants $c_1, c_2$.

Further, if $(u^{(\nu)}) \to u$ in $[L^2(Q_T)]^N$ then

$$\varphi_j(t, x; u^{(\nu)}) \to \varphi_j(t, x; u) \text{ for a.a. } (t, x) \in Q_T$$

and if

$$(u^{(\nu)}) \to u \text{ in } [L^2(Q_T)]^N \text{ and a.e. in } Q_T, \quad (w^{(\nu)}) \to w$$
weakening in $L^\infty(0,T;H)$ in the sense that for all fixed $g_1 \in L^1(0,T;H)$
\[
\int_0^T \langle g_1(t), w^{(\nu)}(t) \rangle dt \to \int_0^T \langle g_1(t), w(t) \rangle dt,
\]
then for a.a. $(t,x) \in Q_T$
\[
\psi_j(t,x;u^{(\nu)},w^{(\nu)}) \to \psi_j(t,x;u,w).
\]

**Theorem 2.1.** Assume $(A_1), (A_2), (A_3), (A_4), (A_5)$. Then for all $F \in L^2(0,T;H)$, $u^{(0)} \in V$, $u^{(1)} \in H$ there exists $u \in L^\infty(0,T;V)$ such that
\[
u' \in L^\infty(0,T;H), \quad u'' \in L^2(0,T;V^*)
\]
satisfies the system (1.1) in the sense: for a.a.t $\in [0,T]$, all $v \in V$
\[
\langle u''(t), v_j \rangle + \langle Q_j(u(t)), v_j \rangle + \int \phi(x)D_jh(u(t))v_jdx + \int H_j(t,x;u)v_jdx + \int G_j(t,x,u,u')v_jdx = (F_j(t), v_j) \quad j = 1, ..., N
\]
and the initial condition (1.2) is fulfilled.

If $(A_1), (A_2), (A_3), (A_4), (A_5)$ are satisfied then for all $F \in L^2(0,T;H)$, $u^{(0)} \in V \cap [L^{\lambda+1}(\Omega)]^N$, $u^{(1)} \in H$ there exists $u \in L^\infty(0,T;V \cap [L^{\lambda+1}(\Omega)]^N)$ such that
\[
u' \in L^\infty(0,T;H), \quad u'' \in L^2(0,T;[V \cap (L^{\lambda+1}(\Omega))^N]^*)
\]
and $u$ satisfies (1.1) in the sense: for a.a. $t \in [0,T]$, all $v_j \in V_j \cap L^{\lambda+1}(\Omega)$ (2.3) holds, further, the initial condition (1.2) is fulfilled.

**Proof.** We apply Galerkin’s method. Let $w_1^{(j)}, w_2^{(j)}, \ldots$ be a linearly independent system in $V_j$ if $(A_3)$ is satisfied and in $V_j \cap L^{\lambda+1}(\Omega)$ if $(A'_3)$ is satisfied such that the linear combinations are dense in $V_j$ and $V_j \cap L^{\lambda+1}(\Omega)$, respectively. We want to find the $m$-th approximation of $u$ in the form
\[
u_j^{(m)}(t) = \sum_{i=1}^m g_{lm}^{(j)}(t)w_i^{(j)} \quad (j = 1, 2, \ldots, N)
\]
where $g_{lm}^{(j)} \in W^{2,2}(0,T)$ if $(A_3)$ holds and $g_{lm}^{(j)} \in W^{2,2}(0,T) \cap L^\infty(0,T)$ if $(A'_3)$ holds such that
\[
\langle (u_j^{(m)})''(t), w_k^{(j)} \rangle + \langle Q(u^{(m)}(t)), w_j^{(j)} \rangle + \int \phi(x)D_jh(u^{(m)}(t))w_k^{(j)}dx
\]
\[
+ \int H_j(t,x;u^{(m)})w_k^{(j)}dx + \int G_j(t,x;u^{(m)},(u^{(m)})'w_k^{(j)}dx = (F_j(t), w_k^{(j)}),
\]
\[
k = 1, \ldots, m, \quad j = 1, \ldots, N
\]
\[
u_j^{(m)}(0) = u_j^{(m)}(0), \quad (u_j^{(m)})'(0) = u_j^{(m)}(0)
\]
where $u_j^{(m)}$, $u_j^{(m)}$ $(j = 1, 2, \ldots, N)$ are linear combinations of $w_1^{(j)}, w_2^{(j)}, \ldots, w_m^{(j)}$ satisfying
\[
(u_j^{(m)}) \to u_j^{(0)} \text{ in } V_j \text{ and } V_j \cap L^{\lambda+1}(\Omega), \text{ respectively, as } m \to \infty \quad (2.7)
\]
(u_{j1}^{(m)}) \to u_j^{(1)} in H as m \to \infty. \tag{2.8}

It is not difficult to show that all the conditions of the existence theorem for a system of functional differential equations with Carathéodory conditions are satisfied.

Thus, by using the Volterra property of \( G \) and \( H \), we obtain that there exists a solution of (2.5), (2.6) in a neighbourhood of 0 (see [8]). Further, the maximal solution of (2.5), (2.6) is defined in \([0, T]\). Indeed, multiplying (2.5) by \([g_{lm}]'(t)\) and taking the sum with respect to \( j \), and \( k \) we obtain

\[
\langle (u^{(m)})''(t), (u^{(m)})'(t) \rangle + \langle Q(u^{(m)}(t)), (u^{(m)})'(t) \rangle + \int_{\Omega} \varphi(x) \frac{d}{dt} [h(u^{(m)}(t))] dx
\]

\[
+ \int_{\Omega} (H(t, x; u^{(m)}), (u^{(m)})'(t)) dx + \int_{\Omega} (G(t, x; u^{(m)}, (u^{(m)})', (u^{(m)})'(t)) dx
\]

\[= \langle F(t), (u^{(m)})'(t) \rangle. \tag{2.9}\]

Integrating the above equality over \((0, t)\) we find (see, e.g., [16], [12])

\[
\frac{1}{2} \| (u^{(m)})'(t) \|_{H}^2 + \frac{1}{2} \langle Q(u^{(m)}(t)), u^{(m)}(t) \rangle + \int_{\Omega} \varphi(x) h(u^{(m)}(t)) dx
\]

\[+ \int_{0}^{t} \left[ \int_{\Omega} (H(\tau, x; u^{(m)}), (u^{(m)})') dx \right] d\tau + \int_{0}^{t} \left[ \int_{\Omega} (G(\tau, x; u^{(m)}, (u^{(m)})', (u^{(m)})') dx \right] d\tau
\]

\[= \int_{0}^{t} \left[ \langle F(\tau), (u^{(m)})'(\tau) \rangle \right] d\tau. \tag{2.10}\]

Hence, by using Young’s inequality, Sobolev’s imbedding theorem and the assumptions of our theorem, we obtain

\[
\| (u^{(m)})'(t) \|_{H}^2 + \int_{\Omega} h(u^{(m)}(t)) dx + \| u^{(m)}(t) \|_{V}^2
\]

\[\leq \text{const} \left\{ 1 + \int_{0}^{t} \left[ \| (u^{(m)})'(\tau) \|_{H}^2 + \int_{\Omega} h(u^{(m)}(\tau)) dx \right] d\tau \right\}
\]

where the constant is not depending on \( t \) and \( m \). Thus by Gronwall’s lemma

\[
\| (u^{(m)})'(t) \|_{H}^2 + \int_{\Omega} h(u^{(m)}(t)) dx \leq \text{const} \tag{2.11}\]

and thus

\[
\| u^{(m)}(t) \|_{V}^2 \leq \text{const} \tag{2.12}\]

Further, the estimates (2.11), (2.12) hold for all \( t \in [0, T] \) and all \( m \) and in the case \( \lambda > \lambda_0, n \geq 3 \)

\[
\| u^{(m)}(t) \|_{V \cap [L^{\lambda+1}(\Omega)]^n} \leq \text{const}. \tag{2.13}\]

By (2.11), (2.12), if \((A_3)\) is satisfied, there exist a subsequence of \((u^{(m)})\), again denoted by \((u^{(m)})\) and \( u \in L^\infty(0, T; V) \) such that

\[
(u^{(m)}) \to u \text{ weakly in } L^\infty(0, T; V), \tag{2.14}\]

\[(u^{(m)})' \to u' \text{ weakly in } L^\infty(0, T; H) \tag{2.15}\]
in the following sense: for any fixed \( g \in L^1(0, T; V^*) \) and \( g_1 \in L^1(0, T; H) \)
\[
\int_0^T \langle g(t), u^{(m)}(t) \rangle dt \to \int_0^T \langle g(t), u(t) \rangle dt,
\]
\[
\int_0^T (g_1(t), (u^{(m)}))'(t) dt \to \int_0^T (g_1(t), u'(t)) dt.
\]

Similarly, in the case \( \lambda > \lambda_0, n \geq 3, \) (when \( A_3' \) holds) there exist subsequence of \( (u^{(m)}) \) and \( u \in L^\infty(0, T; V \cap [L^{\lambda+1}(\Omega)]^N) \) such that
\[
(u^{(m)}) \to u \text{ weakly in } L^\infty(0, T; V \cap [L^{\lambda+1}(\Omega)]^N),
\]
which means: for any fixed \( g \in L^1(0, T; (V \cap L^{\lambda+1}(\Omega))^*) \)
\[
\int_0^T \langle g(t), u^{(m)}(t) \rangle dt \to \int_0^T \langle g(t), u(t) \rangle dt.
\]

Since the imbedding \( W^{1,2}(\Omega) \) into \( L^2(\Omega) \) is compact, by (2.14) – (2.16) we have for a subsequence
\[
(u^{(m)}) \to u \text{ in } L^2(0, T; H) = [L^2(Q_T)]^N \text{ and a.e. in } Q_T.
\]
(see, e.g., [9]). Finally, we show that the limit function \( u \) is a solution of problem (1.1), (1.2).

As \( Q : V \to V^* \) is a linear and continuous operator, by (2.14) for all \( v \in V \) and \( v \in V \cap [L^{\lambda+1}(\Omega)]^N \), respectively we have
\[
\langle (Q(u^{(m)}m)(t)), v \rangle \to \langle (Q(u(t)), v) \rangle \text{ weakly in } L^\infty(0, T)
\]
and by (2.15)
\[
\langle (u^{(m)})'(t), v \rangle = \frac{d}{dt} \langle (u^{(m)})(t), v \rangle \to \langle u'(t), v \rangle
\]
with respect to the weak convergence of the space of distributions \( D'(0, T) \).

Further, by (2.17) and the continuity of \( D_jh \)
\[
\varphi(x)D_jh(u_m(t)) \to \varphi(x)D_jh(u(t)) \text{ for a.e. } (t, x) \in Q_T.
\]

Now we show that for any fixed
\[
v \in L^2(0, T; V), \quad v \in L^2(0, T; V) \cap L^1(0, T; (L^{\lambda+1}(\Omega))^N),
\]
respectively, the sequence of functions
\[
\varphi(x)D_jh(u^{(m)}(t))v \quad j = 1, \ldots, N
\]
is equiintegrable in \( Q_T \). Indeed, if \( A_3 \) is satisfied then by Sobolev’s imbedding theorem and (2.12) for all \( t \in [0, T] \)
\[
\|\varphi(x)D_jh(u^{(m)}(t))\|_{L^2(\Omega)}^2 \leq \text{const} \|D_jh(u^{(m)}(t))\|_{L^2(\Omega)}^2
\]
\[
\leq \text{const} \left[ 1 + \int_{\Omega} |u^{(m)}(t)|^{2\lambda_0} dx \right] \leq \text{const} \left[ 1 + \|u_m(t)\|_{V^2}^{2\lambda_0} \right] \leq \text{const},
\]
because \( 2\lambda_0 = \frac{2n}{n-2} \) and \( W^{1,2}(\Omega) \) is continuously imbedded into \( L^{\frac{2n}{n-2}}(\Omega) \), thus
Cauchy–Schwarz inequality implies that the sequence of functions (2.21) is equiintegrable in \( Q_T \).
If \((A'_3)\) is satisfied then for all \(t \in [0, T]\)
\[
\int_{\Omega} |\varphi(x)D_j h(u^{(m)}(t))|^{\frac{\lambda + 1}{\lambda}} \, dx \leq \text{const} \int_{\Omega} [h(u^{(m)}(t)) + 1] \, dx \leq \text{const}
\]
thus Hölder’s inequality implies that the sequence (2.21) is equiintegrable in \(Q_T\). Consequently, by (2.20) and Vitali’s theorem we obtain that for any fixed
\[
v \in L^2(0, T; V), \quad v \in L^2(0, T; V) \cap L^1(0, T; L^{\lambda + 1}(\Omega)),
\]
respectively
\[
\lim_{m \to \infty} \int_{Q_T} \varphi(x)D_j h(u^{(m)}(t))v_j \, dt \, dx = \int_{Q_T} \varphi(x)D_j h(u(t))v_j \, dt \, dx \tag{2.22}
\]
and
\[
\varphi(x)D_j h(u(t)) \in L^2(0, T; V^*), \quad \varphi(x)D_j h(u(t)) \in L^\infty(0, T; L^{\lambda + 1}(\Omega)) \tag{2.23}
\]
if \((A_3), (A'_3)\) holds, respectively.

Further, by (2.17) and \((A_4)\)
\[
H_j(t, x; u^{(m)}) \to H_j(t, x; u) \text{ a.e. in } Q_T \tag{2.24}
\]
and by (2.11)
\[
\int_{Q_T} |H_j(t, x; u^m)|^2 \, dx \, dt \leq \text{const} \int_{Q_T} h(u^m(t)) \, dx \, dt \leq \text{const},
\]
hence, by Cauchy–Schwarz inequality, for any fixed \(v \in L^2(0, T; V)\), the sequence of functions \(H_j(t, x; u^{(m)})v_j\) is equiintegrable in \(Q_T\) \((j = 1, \ldots, N)\), thus by (2.24) and Vitali’s theorem
\[
\lim_{m \to \infty} \int_{Q_T} H_j(t, x; u^{(m)})v_j \, dt \, dx = \int_{Q_T} H_j(t, x; u)v_j \, dt \, dx \tag{2.25}
\]
and
\[
H(t, x; u) \in L^2(0, T; V^*).\]

Similarly, (2.15) – (2.17) and \((A_5)\) imply
\[
\psi_j(t, x; u^{(m)}, (u^{(m)})') \to \psi_j(t, x; u, u') \text{ a.e. in } Q_T \tag{2.26}
\]
and for arbitrary \(v \in L^2(0, T; V)\) the sequence of functions \(\psi_j(t, x; u^{(m)}, (u^{(m)})')v_j\) is equiintegrable in \(Q_T\) by Cauchy–Schwarz inequality, because by (2.11)
\[
\int_{Q_T} |\psi_j(t, x; u^{(m)}, (u^{(m)})')|^2 \, dt \, dx \leq \text{const} \left[ 1 + \int_{Q_T} |(u^{(m)})'|^2 \, dx \right] \, dt \leq \text{const}.
\]
Consequently, Vitali’s theorem implies that for \(j = 1, \ldots, N\)
\[
\lim_{m \to \infty} \int_{Q_T} \psi_j(t, x; u^{(m)}, (u^{(m)})')v_j \, dt \, dx = \int_{Q_T} \psi_j(t, x; u, u')v \, dt \, dx \tag{2.27}
\]
and
\[
\psi_j(t, x; u, u') \in L^2(0, T; V^*).\]

Further, by using Vitali’s theorem, we show that for arbitrary fixed \(v \in L^2(0, T; V)\)
\[
\varphi_j(t, x; u^{(m)})v_j \to \varphi_j(t, x; u)v_j \text{ in } L^2(Q_T), \quad j = 1, \ldots, N. \tag{2.28}
\]
Indeed, by \((A_5)\) and (2.17)

\[ \varphi_j(t, x; u^{(m)}) \rightarrow \varphi_j(t, x; u) \text{ for a.e. } (t, x) \in Q_T, \quad j = 1, \ldots, N. \tag{2.29} \]

Further, by \((A_5)\) \(|\varphi_j(t, x; u^{(m)})|^2\) is bounded and so for fixed \(v \in L^2(0, T; V)\) the sequence

\[ \int_{Q_T} |\varphi_j(t, x; u^{(m)})v_j - \varphi_j(t, x; u)v_j|^2 \, dt \, dx \leq \text{const} |v_j|^2 \]

is equiintegrable which implies with (2.29) by Vitali’s theorem (2.28). Consequently, by (2.15) we obtain

\[ \lim \int_{Q_T} \varphi_j(t, x; u^{(m)})'(t)v_j \, dt \, dx = \int_{Q_T} \varphi_j(t, x; u)'(t)v_j \, dt \, dx, \quad j = 1, \ldots, N \tag{2.30} \]

and \(\varphi(t, x; u')u' \in L^2(0, T; V^*)\).

If \((A'_5)\) (and \((A'_3)\)) is satisfied, then for a fixed \(v \in L^2(0, T; V) \cap [L^{\lambda+1}(Q_T)]^N\) we also have

\[ \varphi_j(t, x; u^{(m)})v_j \rightarrow \varphi_j(t, x; u)v_j \text{ in } L^2(Q_T), \quad j = 1, \ldots, N. \tag{2.31} \]

Indeed, by (2.11), (2.12) \((u^{(m)})\) is bounded in \(W^{1,2}(Q_T)\), hence it is bonded in \(L^{2(\frac{n+1}{n-1})}(Q_T)\). Thus Hölder’s inequality implies for any measurable \(M \subset Q_T\)

\[ \int_M |\varphi_j(t, x; u^{(m)})v_j - \varphi_j(t, x; u)v_j|^2 \, dt \, dx \leq \text{const} \left\{ \int_{Q_T} [u^{(m)}|^2 + |u^{(m)}|^2]^q \, dt \, dx \right\}^{1/q_1} \cdot \left\{ \int_M |v_j|^2 p_1 \right\}^{1/p_1} \]

where

\[ 2p_1 = \lambda + 1, \quad \frac{1}{p_1} + \frac{1}{q_1}, \]

thus

\[ 2\mu q_1 = 2\mu - \frac{p_1}{p_1 - 1} = 2\mu \frac{\lambda + 1}{\lambda - 1} \leq \frac{2(n + 1)}{n - 1} \]

since

\[ \mu \leq \frac{n + 1}{n - 1} \cdot \frac{\lambda - 1}{\lambda + 1}, \]

hence (2.29), (2.32) and Vitali’s theorem imply (2.31). Consequently, by (2.15) we obtain (2.30) (when \((A'_5)\) holds).

Now let

\[ v = (v_1, \ldots, v_N) \in V \text{ and } \chi_j \in C_0^\infty(0, T) \quad (j = 1, \ldots, N) \]

be arbitrary functions. Further, let \(z^M_j = \sum_{l=1}^M b_l u_l^{(j)}, \quad b_l \in \mathbb{R} \) be sequences of functions such that

\[ (z^M_j) \rightarrow v_j \text{ in } V_j \text{ and } V_j \cap L^{\lambda+1}(\Omega), \quad j = 1, \ldots, N. \tag{2.33} \]
respectively, as $M \to \infty$. Further, by (2.5) we have for all $m \geq M$
\begin{equation}
\int_0^T \langle -(u_j^{(m)})'(t), z_j^M \rangle \chi_j(t)dt + \int_0^T \langle Q(u^{(m)}(t)), z_j^M \rangle \chi_j(t)dt + \int_0^T \langle u^{(m)}(t), z_j^M \rangle \chi_j(t)dt
\end{equation}
\begin{equation}
+ \int_0^T \int_\Omega \varphi(x) D_j h(u^{(m)}(t)) z_j^M \chi_j(t)dtdx + \int_0^T \int_\Omega H_j(t, x; u^{(m)}) z_j^M \chi_j(t)dtdx
\end{equation}
\begin{equation}
+ \int_0^T \int_\Omega G_j(t, x; u^{(m)'}, u') z_j^M \chi_j(t)dtdx
\end{equation}
\begin{equation}
= \int_0^T \langle F_j(t), z_j^M \rangle \chi_j(t)dt.
\end{equation}
By (2.15), (2.18), (2.22), (2.25), (2.27), (2.30) we obtain from (2.34) as $m \to \infty$
\begin{equation}
- \int_0^T \langle u_j'(t), z_j^M \rangle \chi_j(t)dt + \int_0^T \langle Q_j(u(t)), z_j^M \rangle \chi_j(t)dt + \int_0^T \langle u^{(m)}(t), z_j^M \rangle \chi_j(t)dt
\end{equation}
\begin{equation}
+ \int_0^T \int_\Omega \varphi(x) D_j h(u(t)) z_j^M \chi_j(t)dtdx
\end{equation}
\begin{equation}
+ \int_0^T \int_\Omega H_j(t, x; u) z_j^M \chi_j(t)dtdx + \int_0^T \int_\Omega G_j(t, x; u, u') z_j^M \chi_j(t)dtdx
\end{equation}
\begin{equation}
= \int_0^T \langle F_j(t), z_j^M \rangle \chi(t)dt.
\end{equation}
From equality (2.35) and (2.33) we obtain as $M \to \infty$
\begin{equation}
- \int_0^T \langle u_j'(t), v_j \rangle \chi_j(t)dt + \int_0^T \langle Q_j(u(t)), v_j \rangle \chi_j(t)dt + \int_0^T \langle u^{(m)}(t), v_j \rangle \chi_j(t)dt
\end{equation}
\begin{equation}
+ \int_0^T \int_\Omega \varphi(x) D_j h(u(t)) v_j \chi_j(t)dtdx
\end{equation}
\begin{equation}
+ \int_0^T \int_\Omega H_j(t, x; u) v_j \chi_j(t)dtdx + \int_0^T \int_\Omega G_j(t, x; u, u') v_j \chi_j(t)dtdx
\end{equation}
\begin{equation}
= \int_0^T \langle F_j(t), v_j \rangle \chi_j(t)dt.
\end{equation}
Since $v_j \in V_j$ and $\chi_j \in C_0^\infty(0, T)$ are arbitrary functions, (2.36) means that
\begin{equation}
u_j'' \in L^2(0, T; V_j^*) and \ u_j'' \in L^2(0, T; (V \cap L^{\lambda+1}(\Omega))^*)
\end{equation}
respectively (see, e.g. [16]) and for a.a. $t \in [0, T]$
\begin{equation}
u_j'' + Q_j(u(t)) + \varphi(x) D_j h(u(t)) + H_j(t, x; u) + G_j(t, x; u, u') = F_j, \ j = 1, \ldots, N,
\end{equation}
i.e. we proved (1.1).
Now we show that the initial condition (1.2) holds. Since $u \in L^\infty(0, T; V)$, $u' \in L^\infty(0, T; H)$, we have $u \in C([0, T]; H)$ and for arbitrary $\chi_j \in C^\infty[0, T]$ with the properties $\chi_j(0) = 1$, $\chi_j(T) = 0$, all $j, k$
\begin{equation}
\int_0^T \langle u_j'(t), w_j^{(k)} \rangle \chi_j(t)dt = -(u_j(0), w_k^{(j)})_{L^2(\Omega)} - \int_0^T \langle u_j(t), w_k^{(j)} \rangle \chi_j(t)dt,
\end{equation}
\[ \int_0^T \langle (u_j^{(m)})'(t), w_k^{(j)} \rangle_{\chi_j(t)} dt = -(u_j^{(m)}(0), w_k^{(j)})_{L^2(\Omega)} - \int_0^T \langle u_j^{(m)}(t), w_k^{(j)} \rangle_{\chi_j(t)} dt. \]

Hence by (2.6), (2.7), (2.8), (2.14), (2.15), we obtain as \( m \to \infty \)

\[ (u(0), w_k^{(j)})_{L^2(\Omega)} = \lim_{m \to \infty} (u_j^{(m)}(0), w_k^{(j)})_{L^2(\Omega)} \]

\[ \lim_{m \to \infty} (u_j^{(m)}(0), w_k^{(j)})_{L^2(\Omega)} = (u(0), w_k^{(j)})_{L^2(\Omega)} \]

for all \( j \) and \( k \) which implies \( u(0) = u(0) \).

Similarly can be shown that \( u'(0) = u'(0) \).

3. Examples

Let the operator \( Q \) be defined by

\[ \langle Q_{jk}(u_k), v_j \rangle = \int_\Omega \left[ \sum_{l=1}^n a_{il}^{jk}(x)(D_l u_k)(D_i v_j) + d^{jk}(x) u_k v_j \right] dx \]

where \( a_{il}^{jk}, d^{jk} \in L^\infty(\Omega) \), \( a_{il}^{jk} = a_{il}^{jk} \sum_{i,l=1}^n a_{il}^{jj}(x) \xi_i \xi_l \geq c_1 |\xi|^2 \), \( d^{ii}(x) \geq c_0 \) with some positive constants \( c_0, c_1 \); further, \( a_{il}^{jk} = a_{il}^{kj} \) and for some \( \tilde{c}_0 < c_1 \)

\[ \|a_{il}^{jk}\|_{L^\infty(\Omega)} < \frac{\tilde{c}_0}{n-1}; \quad \|d^{jk}\|_{L^\infty(\Omega)} < \frac{\tilde{c}_0}{n-1} \text{ for } j \neq k. \]

Then assumption (A1) is satisfied.

If \( h \) is a \( C^1 \) function such that \( h(\eta) = |\eta|^\lambda+1 \) if \( |\eta| > 1 \) then (A3), (A3'), respectively, are satisfied.

Further, let \( \hat{h}_j : \mathbb{R}^N \to \mathbb{R} \) be continuous functions satisfying

\[ |\hat{h}_j(\eta)| \leq \text{const } |\eta|^\lambda \text{ for } |\eta| > 1, \quad j = 1, \ldots, N \]

with some positive constant. It is not difficult to show that operators \( H_j \) defined by one of the formulas

\[ H_j(t, x; u) = \chi_j(t, x) \hat{h}_j \left( \int_{Q_T} u_1(\tau, \xi) d\tau d\xi, \ldots, \int_{Q_T} u_N(\tau, \xi) d\tau d\xi \right), \]

\[ H_j(t, x; u) = \chi_j(t, x) \hat{h}_j \left( \int_0^t u_1(\tau, x) d\tau, \ldots, \int_0^t u_N(\tau, x) d\tau \right), \]

\[ H_j(t, x; u) = \chi_j(t, x) \hat{h}_j \left( \int_{\Omega} u_1(\tau, \xi) d\xi, \ldots, \int_{\Omega} u_N(\tau, \xi) d\xi \right), \]

\[ H_j(t, x; u) = \chi_j(t, x) \hat{h}_j(u_1(\tau_1(t), x), \ldots, u_N(\tau_k(t), x)) \text{ where } \]

\( \tau_k \in C^1, \quad 0 \leq \tau_k(t) \leq t, \quad \tau'_k(t) \geq c_1 > 0, \quad k = 1, \ldots, N \)

satisfy (A4) if \( \chi_j \in L^\infty(Q_T) \).

The operators \( \varphi_j, \psi_j \) may have forms, similar to the above forms of \( H_j \) with bounded continuous functions \( \hat{h}_j \). Then (A5) is fulfilled.
Remark. One can show uniqueness and continuous dependence of the solution of (1.1), (1.2) if the following additional conditions are satisfied:

\[ G_j(t,x;u,u') = \tilde{\varphi}_j(x)u'_j(t) \]

where \( \tilde{\varphi}_j \) is measurable and \( 0 \leq \tilde{\varphi}_j(x) \leq \text{const} \), \( h \) is twice continuously differentiable and

\[ |D_iD_kh(\eta)| \leq \text{const}|\eta|^{\lambda - 1} \text{ for } |\eta| > 1. \]

Further \( H_j(t,x;u) \) satisfy some Lipschitz condition with respect to \( u \).

4. Solutions in \((0, \infty)\)

Now we formulate and prove existence of solutions for \( t \in (0, \infty) \). Denote by \( L^p_{\text{loc}}(0, \infty; V) \) the set of functions \( u : (0, \infty) \to V \) such that for each fixed finite \( T > 0 \), their restrictions to \((0,T)\) satisfy \( u|_{(0,T)} \in L^p(0,T;V) \) and let \( Q_{\infty} = (0, \infty) \times \Omega \), \( L^\alpha_{\text{loc}}(Q_{\infty}) \) the set of functions \( u : Q_{\infty} \to \mathbb{R}^N \) such that \( u_j|_{Q_T} \in L^\alpha(Q_T) \) \((j = 1, \ldots, N)\) for any finite \( T \).

Now we formulate assumptions on \( H_j \) and \( G_j \).

\((B_1)\) The functions \( H_j : Q_{\infty} \times [L^2_{\text{loc}}(Q_{\infty})]^N \to \mathbb{R} \) are such that for all fixed \( u \in [L^2_{\text{loc}}(Q_{\infty})]^N \) the functions \( (t,x) \mapsto H_j(t,x;u) \) are measurable, \( H_j \) have the Volterra property (see \((A_4)\)) and for each fixed finite \( T > 0 \), the restrictions of \( H_j \) to \( Q_T \times [L^2(Q_T)]^N \) satisfy \((A_4)\).

Remark. Since \( H_j \) has the Volterra property, this restriction \( H^T_j \) is well defined by the formula

\[ H^T_j(t,x;\tilde{u}) = H_j(t,x;u), \quad (t,x) \in Q_T, \quad \tilde{u} \in [L^2(Q_T)]^N \]

where \( u \in [L^2_{\text{loc}}(Q_{\infty})]^N \) may be any function satisfying \( u(t,x) = \tilde{u}(t,x) \) for \((t,x) \in Q_T \).

\((B_5)\) The operators

\[ G_j : Q_{\infty} \times [L^2_{\text{loc}}(Q_{\infty})]^N \times L^\infty_{\text{loc}}(0, \infty; H) \to \mathbb{R} \]

are such that for all fixed \( u \in L^2_{\text{loc}}(0, \infty; V) \), \( w \in L^\infty_{\text{loc}}(0, \infty; H) \) the functions \((t,x) \mapsto G_j(t,x;u,w) \) are measurable, \( G_j \) have the Volterra property and for each fixed finite \( T > 0 \), the restrictions \( G^T_j \) of \( G_j \) to \( Q_T \times [L^2(Q_T)]^N \times L^\infty(0,T;H) \) satisfy \((A_5)\).

\((B'_5)\) It is the same as \((B_5)\) but \( G^T_j \) satisfy \((A'_5)\).

Theorem 4.1. Assume \((A_1) - (A_3), (B_4), (B_5)\). Then for all \( F \in L^2_{\text{loc}}(0, \infty; H) \), \( u^{(0)} \in V, u^{(1)} \in H \) there exists

\[ u \in L^\infty_{\text{loc}}(0, \infty; V) \text{ such that } u' \in L^\infty_{\text{loc}}(0, \infty; H), \quad u'' \in L^2_{\text{loc}}(0, \infty; V^*), \]

\( u \) satisfies \((1.1)\) for a.a. \( t \in (0, \infty) \) (in the sense, formulated in Theorem 2.1) and the initial condition \((1.2)\).

If \((A_1), (A_2), (A'_3), (B_4), (B_5)\) are fulfilled then for all \( F \in L^2_{\text{loc}}(0, \infty; H) \), \( u^{(0)} \in V \cap [L^{\lambda+1}(\Omega)]^N, u^{(1)} \in H \) there exists

\[ u \in L^\infty_{\text{loc}}(0, \infty; V \cap [L^{\lambda+1}(\Omega)]^N) \text{ such that } u' \in L^\infty_{\text{loc}}(0, \infty; H), \]

for all \( t \in (0, \infty) \).
\[ u'' \in L^2_{\text{loc}}(0, \infty; V^*) + L^\infty_{\text{loc}}(0, \infty; [L^{1+1}(\Omega)]^N) \subset L^2_{\text{loc}}(0, \infty; [V \cap (L^{1+1}(\Omega))^N]^*), \]

\[ u \text{ satisfies (1.1) for a.a. } t \in (0, \infty) \text{ (in the sense, formulated in Theorem 2.1) and the initial condition (1.2).} \]

Assume that the following additional conditions are satisfied: there exist \( T_0 \) and a function \( \gamma \in L^2(T_0, \infty) \) such that for \( t > T_0 \)
\[ |G(t, x; u, u')| \leq \gamma(t), |H(t, x; u)| \leq \gamma(t) \text{ and } \|F(t)\|_{V^*} \leq \gamma(t). \] (4.1)

Then for the above solution \( u \) we have
\[ u \in L^\infty(0, \infty; V), \quad u \in L^\infty(0, \infty; V \cap [L^{1+1}(\Omega)]^N), \text{ respectively and} \]
\[ u' \in L^\infty(0, \infty; H). \] (4.2)

Further, assume that there exists a positive constant \( \hat{c} \) such that
\[ \varphi_j(t, x; u) \geq \hat{c}, \quad (t, x) \in Q_\infty, \quad j = 1, \ldots, N \] (4.3)
and there exist \( F_\infty \in H, u_\infty \in V \) such that
\[ Q(u_\infty) = F_\infty, \quad F - F_\infty \in L^2(0, \infty; H), \] (4.4)
\[ |H_j(t, x; u)| \leq \beta(t, x), \quad |\psi_j(t, x; u, u')| \leq \beta(t, x), \quad |\varphi_j(t, x; u)| \leq \text{const} \] (4.5)
with some \( \beta \in L^2(0, \infty; L^2(\Omega)). \) Then for the above solution we have
\[ u \in L^\infty(0, \infty; V), \quad u \in L^\infty(0, \infty; V \cap [L^{1+1}(\Omega)]^N), \] (4.6)
\[ \|u'(t)\|_H \leq \text{const } e^{-\tilde{c}t}, \quad t \in (0, \infty) \] (4.7)
and there exists \( w^{(0)} \in H \) such that
\[ u(T) \to w^{(0)} \text{ in } H \text{ as } T \to \infty, \quad \|u(T) - w^{(0)}\|_H \leq \text{const } e^{-\tilde{c}T}. \] (4.8)

Finally, \( w^{(0)} \in V \) and
\[ Q(w^{(0)}) + \varphi D\gamma(w^{(0)}) = F_\infty. \] (4.9)

**Proof.** Similarly to the proof of Theorem 2.1, we apply Galerkin’s method and we want to find the \( m \)-th approximation of solution \( u = (u_1, \ldots, u_N) \) for \( t \in (0, \infty) \) in the form (see (2.4))
\[ u_j^{(m)}(t) = \sum_{l=1}^{m} g_{lm}^{(j)}(t) w_l^{(j)}, \quad j = 1, \ldots, N \]
where \( g_{lm}^{(j)} \in W^{2,2}_{\text{loc}}(0, \infty) \) if \( (A_3) \) is satisfied and \( g_{lm}^{(j)} \in W^{2,2}_{\text{loc}}(0, \infty) \cap L^\infty_{\text{loc}}(0, \infty) \) if \( (A'_3) \) is satisfied. Here \( W^{2,2}_{\text{loc}}(0, \infty) \) and \( L^\infty_{\text{loc}}(0, \infty) \) denote the set of functions \( g : (0, \infty) \to \mathbb{R} \) such that for all \( T \) the restriction of \( g \) to \( (0, T) \) belongs to \( W^{2,2}(0, T), L^\infty(0, T), \) respectively.

According to the arguments in the proof of Theorem 2.1, there exists a solution of (2.5), (2.6) in a neighbourhood of \( t = 0. \) Further, we obtain estimates (2.11), (2.12) and (2.13), respectively, for \( t \in [0, T] \) with sufficiently small \( T \) where on the right hand side are finite constants (depending on \( T \)). Consequently, the maximal solutions of (2.5), (2.6) are defined in \((0, \infty)\) and the estimates (2.11), (2.12), (2.13) hold for all
finite \( T > 0 \) (if \( t \in [0,T] \)), the constants on the right hand sides are depending only on \( T \).

Let \((T_k)_{k \in \mathbb{N}}\) be a monotone increasing sequence, converging to \(+\infty\). According to the arguments in the proof of Theorem 2.1, there is a subsequence \((u^{(m_1)})\) of \((u^{(m)})\) for which (2.14), (2.15) and (2.16) hold, respectively, with \( T = T_1 \). Further, there is a subsequence \((u^{(m_2)})\) of \((u^{(m_1)})\) for which (2.14), (2.15) and (2.16) hold, respectively, with \( T = T_2 \), etc. By a diagonal process we obtain a sequence \((u^{(m_m)})_{m \in \mathbb{N}}\) such that (2.14), (2.15), (2.16) hold for every fixed \( T > 0 \); further,

\[
\begin{align*}
  u &\in L^\infty_{loc}(0,\infty;V), \quad u' \in L^\infty_{loc}(0,\infty;H), \quad u'' \in L^2_{loc}(0,\infty;V^*) \text{ and} \\
  u &\in L^\infty_{loc}(0,\infty;V \cap [L^{\lambda+1}(\Omega)]^N), \quad u' \in L^\infty_{loc}(0,\infty;H), \\
  u'' &\in L^2_{loc}(0,\infty;V^*) + L^\infty_{loc}(0,\infty;[L^{\lambda+1}(\Omega)]^N),
\end{align*}
\]

respectively and (1.1) holds for \( t \in (0,\infty) \).

Now we consider the case when (4.1) holds. Then by (2.10) we obtain for all \( t \geq T_1 \geq T_0 \)

\[
\frac{1}{2} \| (u^{(m)})'(t) \|^2_H + \frac{1}{2} \| (Q(u^{(m)}),u^{(m)}(t)) + c_1 \int_\Omega h(u^{(m)}(t))dx
\]

\[
\leq \int_0^{T_1} \int_\Omega |(G(\tau,x;u^{(m)},(u^{(m)})',(u^{(m)})'((u^{(m)})'(\tau))|d\tau + \int_0^{T_1} \int_\Omega |(H(\tau,x;u^{(m)}),(u^{(m)})'(\tau))|d\tau
\]

\[
+ \int_0^{T_1} \int_\Omega |(F(\tau),(u^{(m)}),(u^{(m)}))|d\tau + 3\lambda(\Omega) \left[ \int_{T_1}^{\infty} |\gamma(\tau)|d\tau \right] \sup_{\tau \in [0,t]} \| (u^{(m)})'(\tau) \|_H.
\]

Choosing sufficiently large \( T_1 > 0 \), since \( \lim_{T_1 \to \infty} \int_{T_1}^{\infty} |\gamma(\tau)|d\tau = 0 \), we find

\[
\frac{1}{4} \| (u^{(m)})'(t) \|^2_H + \frac{1}{2} \| (Q(u^{(m)}),u^{(m)}(t)) + c_1 \int_\Omega h(u^{(m)}(t))dx \leq \text{const}
\]

for all \( t > 0 \), \( m \) which implies (4.2).

Finally, consider the case when (4.3) - (4.5) are satisfied, too. Denoting \((w^{(m)})\) of \((u^{(m)})\), for simplicity, by \( (w^{(m)})' = (u^{(m)})' \):

\[
\langle (w^{(m)})'(t), (w^{(m)})'(t) \rangle + \langle (Q(w^{(m)}),w^{(m)})(t),(w^{(m)}))' \rangle + \int_\Omega \varphi(x) \frac{d}{dt} [h(u^{(m)}(t))]dx + \int_\Omega (H(t,x;u^{(m)}),(u^{(m)})'(t)dx
\]

\[
+ \int_\Omega (G(t,x;u^{(m)}),(u^{(m)})'(t)dx + \int_\Omega (F(t) - F_\infty, (w^{(m)}))'(t)dx
\]

Integrating over \([0,t]\) we find (similarly to (2.10))

\[
\frac{1}{2} \| (w^{(m)})'(t) \|^2_H + \frac{1}{2} \| (Q(w^{(m)}),w^{(m)}(t)) + c_1 \int_\Omega h(u^{(m)}(t))dx
\]

\[
+ \varepsilon \int_0^t \left[ \int_\Omega |(w^{(m)})'(\tau)|^2 dx \right] d\tau
\]

\[
\leq \varepsilon \int_0^t \left[ \int_\Omega |(w^{(m)})'(\tau)|^2 dx \right] d\tau + C(\varepsilon) \int_0^t \| F(\tau) - F_\infty \|^2_H d\tau
\]
\[ + \frac{1}{2} \| (u^{(m)})' (0) \|_{H}^{2} + \frac{1}{2} \langle (Qu^{(m)})(0), u^{(m)}(0) \rangle + c_2 \int_{\Omega} h(u^{(m)}(0)) dx + \varepsilon \int_{0}^{t} \left( \int_{\Omega} |(w^{(m)})'(\tau)| dx \right) d\tau + C(\varepsilon) \| \beta \|_{L^{2}(0, \infty; H)}. \]

Choosing \( \varepsilon = \tilde{c}/4 \) we obtain
\[ \int_{0}^{t} \left[ \int_{\Omega} |(w^{(m)})'(\tau)|^{2} dx \right] d\tau \leq \text{const.} \quad (4.12) \]

Further, from (4.11), (4.12) we obtain
\[ \| (u^{(m)})'(t) \|_{H}^{2} + \tilde{c} \int_{0}^{t} \| (u^{(m)})'(\tau) \|_{H}^{2} d\tau \leq c^{*} \]

with some positive constant \( c^{*} \) not depending on \( m \) and \( t \). Thus by Gronwall’s lemma we find
\[ \| (u^{(m)})'(t) \|_{H}^{2} = \| (w^{(m)})'(t) \|_{H}^{2} \leq c^{*} e^{-\tilde{c}t}, \quad t > 0 \]

which implies (4.7) as \( m \to \infty \) (since \( (u^{(m)})' \to u' \) weakly in \( L^{\infty}(0, T; H) \)). Further, by (A1) one obtains from (4.11) that for all \( t > 0, m \)
\[ \| w^{(m)}(t) \|_{V} \leq \text{const}, \quad \| w^{(m)}(t) \|_{V \cap [L^{\lambda+1}(\Omega)]^{N}} \leq \text{const}, \]

respectively, which implies (4.6).

Further, for arbitrary \( T_1 < T_2 \)
\[ \| u(T_2) - u(T_1) \|_{H}^{2} = \langle u(T_2), u(T_2) - u(T_1) \rangle_{H} - \langle u(T_1), u(T_2) - u(T_1) \rangle_{H} \]
\[ = \int_{T_1}^{T_2} \langle u'(t), u(T_2) - u(T_1) \rangle dt = \int_{T_1}^{T_2} \langle u'(t), u(T_2) - u(T_1) \rangle_{H} dt \]
\[ \leq \| u(T_2) - u(T_1) \|_{H} \int_{T_1}^{T_2} \| u'(t) \|_{H} dt \]

which implies
\[ \| u(T_2) - u(T_1) \|_{H} \leq \int_{T_1}^{T_2} \| u'(t) \|_{H} dt. \quad (4.13) \]

Hence by (4.7)
\[ \| u(T_2) - u(T_1) \|_{H} \to 0 \text{ as } T_1, T_2 \to \infty \]

which implies (4.8) and by (4.10), (4.7) we obtain
\[ \| u(T) - w_0 \|_{H} \leq \int_{T}^{\infty} \| u'(t) \|_{H} dt \leq \text{const } e^{-\tilde{c}T}. \]

Now we show \( w_0 \in V \) and (4.9) holds. Since \( u \in L^{\infty}(0, \infty; V) \),
\[ (u(T_k)) \to w_0^{*} \text{ weakly in } V, \quad w_0^{*} \in V \quad (4.14) \]

for some sequence \( (T_k) \), \( \lim(T_k) = +\infty \). Clearly, (4.14) implies
\[ (u(T_k)) \to w_0^{*} \text{ weakly in } H, \]

thus by (4.8) \( w_0 = w_0^{*} \in V \) and (4.14) holds for arbitrary sequence \( (T_k) \) converging to \( +\infty \).
In order to prove (4.9), consider arbitrary fixed \( v \in V, v \in V \cap [L^{\lambda+1}(\Omega)]^N \), respectively and

\[ \chi_T(t) = \chi(t - T) \text{ where } \chi \in C_0^\infty(\mathbb{R}), \supp \chi \subset [0, 1], \int_0^1 \chi(t) dt = 1. \]

Multiply (2.3) by \( \chi_T(t) \) and integrate with respect to \( t \) on \( (0, \infty) \) and take the sum with respect to \( j \), then we obtain

\[
\int_0^\infty \langle u'', (t), v \rangle T(t) dt + \int_0^\infty \langle Q(u(t)), v \rangle T(t) dt + \int_0^\infty \left[ \int_0^1 (H(t, x; u), v) dx \right] T(t) dt
\]

\[ + \int_0^\infty \left[ \int_0^\infty (G(t, x; u, u'), v) dx \right] T(t) dt = \int_0^\infty (F(t), v) T(t) dt. \]

Let \( (T_k) \) be an arbitrary sequence converging to \(+\infty\) and consider (4.15) with \( T = T_k \).

For the first term on the left hand side of this equation we have by (4.7) (if \( T_k > 1 \))

\[
\int_0^\infty \langle u''(t), v \rangle T_k(t) dt = -\int_0^\infty \langle u'(t), v \rangle (\chi_{T_k})'(t) dt \to 0 \text{ as } k \to \infty. \quad (4.16)
\]

Further, by \((A_1), (4.14)\) and Lebesgue’s dominated convergence theorem

\[
\int_0^\infty \langle Q(u(t)), v \rangle T_k(t) dt = \int_0^\infty \langle Q(v), u(t) \rangle T_k(t) dt \quad (4.17)
\]

\[
= \int_0^1 \langle Q(v), u(T_k + \tau) \rangle \chi(\tau) d\tau \to \int_0^1 \langle Q(v), w_0 \rangle \chi(\tau) d\tau = \langle Q(v), w_0 \rangle
\]

\[
= \langle Q(w_0), v \rangle \text{ as } k \to \infty.
\]

For the third term on the left hand side of (4.15) we have

\[
\int_0^\infty \left[ \int_0^\infty \varphi(x)((Dh)(u(t)), v) dx \right] T_k(t) dt \quad (4.18)
\]

\[
= \int_0^1 \left[ \int_0^\infty \varphi(x)((Dh)(u(T_k + \tau)), v) dx \right] \chi(\tau) d\tau
\]

\[
\to \int_0^1 \left[ \int_0^\infty \varphi(x)((Dh)(w_0), v) dx \right] \chi(\tau) d\tau = \int_0^\infty \varphi(x)((Dh)(w_0), v) dx
\]

as \( k \to \infty \) since by (4.8)

\[
u(T_k + \tau) \to w_0 \text{ in } [L^2((0, 1) \times \Omega)]^N \text{ as } k \to \infty
\]

and thus for a.a. \((\tau, x) \in (0, 1) \times \Omega\) (for a subsequence), consequently

\[
(Dh)(u(T_k + \tau, x)) \to (Dh)(w_0(x)) \text{ for a.a. } (\tau, x) \in (0, 1) \times \Omega. \quad (4.19)
\]

By using Hölder’s inequality, \((A_3), (A_3)\), respectively and Vitali’s theorem, we obtain (4.18) from (4.19).
The fourth and fifth terms on the left hand side of (4.15) can be estimated by (4.5) and (4.7) as follows: for sufficiently large $k$

$$
\left| \int_0^\infty \left[ \int_\Omega (H(t,x;u),v)dx \right] \chi_{T_k}(t)dt \right| = \left| \int_0^\infty \left[ \int_\Omega (H(T_k + \tau,x;u),v)dx \right] \chi(\tau)d\tau \right|
$$

(4.20)

$$
\leq \int_0^\infty \left[ \int_\Omega \beta(T_k + \tau,x)|v|dx \right] |\chi(\tau)|d\tau \to 0 \text{ as } k \to \infty,
$$

(4.21)

$$
\leq \int_0^1 \left[ \int_\Omega \left| c_5 |u'(T_k + \tau)| + \beta(T_k + \tau,x) \right| |v|dx \right] |\chi(\tau)|d\tau \to 0.
$$

Finally, for the right hand side of (4.15) we obtain by using (4.4) and the Cauchy – Schwarz inequality

$$
\int_0^\infty (F(t),v)\chi_{T_k}(t)dt = \int_0^1 (F(T_k + \tau),v)\chi(\tau)d\tau \to \int_0^1 (F_\infty,v)\chi(\tau)d\tau = (F_\infty,v).
$$

(4.22)

From (4.15) – (4.18), (4.20) – (4.22) one obtains (4.9).

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References


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