Bilateral inequalities for harmonic, geometric and Hölder means

Mira-Cristiana Anisiu and Valeriu Anisiu

Abstract. For $0 < a < b$, the harmonic, geometric and Hölder means satisfy $H < G < Q$. They are special cases ($p = -1, 0, 2$) of power means $M_p$. We consider the following problem: find all $\alpha, \beta \in \mathbb{R}$ for which the bilateral inequalities

$$\alpha H(a, b) + (1 - \alpha) Q(a, b) < G(a, b) < \beta H(a, b) + (1 - \beta) Q(a, b)$$

hold $\forall 0 < a < b$. Then we replace in the bilateral inequalities the mean $Q$ by $M_p, p > 0$ and address the same problem.

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1. Introduction

We consider bivariate means $m : \mathbb{R}_+^2 \to \mathbb{R}$ which are symmetric ($m(b, a) = m(a, b)$ for all $a, b > 0$) and homogeneous ($m(\lambda a, \lambda b) = \lambda m(a, b)$ for all $a, b, \lambda > 0$).

For two means $m_1$ and $m_2$ we write $m_1 \leq m_2$ if and only if $m_1(a, b) \leq m_2(a, b)$ for every $a, b > 0$, and $m_1 < m_2$ if and only if $m_1(a, b) < m_2(a, b)$ for all $a, b > 0$ with $a \neq b$.

Since we are dealing with strict inequalities, we may and shall assume in the following that $0 < a < b$.

We consider the bivariate means

$$A(a, b) = \frac{a + b}{2}; \quad G(a, b) = \sqrt{ab}; \quad H(a, b) = \frac{2ab}{a + b}; \quad Q(a, b) = \left(\frac{a^2 + b^2}{2}\right)^{1/2}; \quad (1.1)$$

$$M_p(a, b) = \begin{cases} \left(\frac{a^p + b^p}{2}\right)^{1/p}, & \text{for } p \neq 0 \\ \sqrt{ab}, & \text{for } p = 0, \end{cases} \quad (1.2)$$

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which are known as the arithmetic, geometric, harmonic, Hölder and power means, respectively. Properties and comparison of standard means can be found in [3].

The means from (1.1) are comparable:

$$\min < H < G < A < Q < \max,$$

where min and max are the trivial means given by \((a, b) \mapsto \min(a, b)\) and \((a, b) \mapsto \max(a, b)\). The power means are monotonic in \(p\), and \(M_{-1} = H, M_0 = G, M_1 = A,\) and \(M_2 = Q\).

Recently, many bilateral inequalities between means have been proved ([1], [2], [4], [5], [6]). We mention one of them, which was the starting point for this paper, and refers to the means \(G, A\) and \(Q\).

**Theorem 1.1.** [2] The double inequality

$$\alpha G(a, b) + (1 - \alpha)Q(a, b) < A(a, b) < (1 - \beta)Q(a, b) + \beta G(a, b)$$

holds if and only if \(\alpha \geq 1/2\) and \(\beta \leq 1 - \sqrt{2}/2\).

In what follows we shall prove a similar result for the means \(H, G\) and \(Q\). Afterwards we consider the more general case of the means \(H, G\) and \(M_p, p > 0\). We show that for \(p = 5/2\) the auxiliary function \(f\) is still monotone and we formulate an open problem.

**2. Main result**

For \(0 < a < b\), the geometric, harmonic and Hölder means satisfy \(H < G < Q\). We shall find all the values of \(\alpha\) and \(\beta\) in order that the geometric mean to be strictly between the combination of \(H\) and \(Q\) with parameters \(\alpha\), respectively \(\beta\). Due to the homogeneity of all the means considered here, we may denote \(t = b/a, t > 1\), and write in the following \(m(t)\) instead of \(m(1, t) = (1/a)m(a, b)\). For any three means \(m_1 < m_2 < m_3\), the double inequality

$$\alpha m_1(t) + (1 - \alpha)m_3(t) < m_2(t) < \beta m_1(t) + (1 - \beta)m_3(t)$$

(2.1)

is equivalent to

$$\beta < f(t) < \alpha,$$

(2.2)

where

$$f(t) = \frac{m_3(t) - m_2(t)}{m_3(t) - m_1(t)}.$$  

(2.3)

We shall prove the following result.

**Theorem 2.1.** The double inequality

$$\alpha H(t) + (1 - \alpha)Q(t) < G(t) < \beta H(t) + (1 - \beta)Q(t), \ \forall t > 1$$

holds if and only if \(\alpha \geq 1\) and \(\beta \leq 2/3\). The function

$$f_1(t) = \frac{Q(t) - G(t)}{Q(t) - H(t)}$$

is strictly increasing on \((1, \infty)\).
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Proof. The function $f_1$ is given by

$$f_1(t) = \frac{((2t^2 + 2)^{1/2} - 2t^{1/2})(t + 1)}{(2t^2 + 2)^{1/2}t + (2t^2 + 2)^{1/2} - 4t}.$$  \hspace{1cm} (2.4)

We substitute $t = s^2$, $s > 1$ and get

$$f_1(s^2) = \frac{((2s^4 + 2)^{1/2} - 2s)(s^2 + 1)}{(2s^4 + 2)^{1/2}s^2 + (2s^4 + 2)^{1/2} - 4s^2}.$$  

The numerator of the derivative of this expression is

$$4 \left( s^8 - 4s^7 + 2s^6 + 2(2s^4 + 2)^{1/2}s^4 - 2(2s^4 + 2)^{1/2}s^2 - 2s^2 + 4s - 1 \right)$$

$$= 4(s^2 - 1)(s^6 - 4s^5 + 3s^4 - 4s^3 + 3s^2 - 4s + 1 + 2(2s^4 + 2)^{1/2}s^2)$$

and the denominator is obviously positive. We shall prove that

$$g_1(s) = s^6 - 4s^5 + 3s^4 - 4s^3 + 3s^2 - 4s + 1 + 2(2s^4 + 2)^{1/2}s^2$$

is positive for $s > 1$, hence $f_1$ is strictly increasing. We write $g_1(s) = 0$ as

$$s^6 - 4s^5 + 3s^4 - 4s^3 + 3s^2 - 4s + 1 = -2(2s^4 + 2)^{1/2}s^2,$$  \hspace{1cm} (2.5)

square both sides and get

$$(s^8 - 4s^7 - 4s^5 + 6s^4 - 4s^3 - 4s + 1)(s - 1)^4 = 0.$$  

Denoting by $h_1(s) = s^8 - 4s^7 - 4s^5 + 6s^4 - 4s^3 - 4s + 1$ we get

$$h_1(s + 4) = s^8 + 28s^7 + 336s^6 + 2236s^5 + 8886s^4 + 20956s^3 + 26640s^2 + 12604s - 2831,$$

which has only one change of sign. We apply Descartes’ rule of signs for $h_1(s + 4)$ and we obtain that the polynomial $h_1(s)$ has a single root greater than 4. We denote by $k_1(s)$ the 6th degree polynomial in the left hand side of (2.5) and get

$$k_1(s + 4) = s^6 + 20s^5 + 163s^4 + 684s^3 + 1523s^2 + 1620s + 545.$$  \hspace{1cm} (2.6)

Then the polynomial (2.6) is positive on $s > 4$, hence $g_1(s) = 0$ has no solutions on $s > 1$. It follows that $f_1$ is strictly increasing on $(1, \infty)$. Since $\lim_{t \to 1} f_1(t) = 2/3$ and $\lim_{t \to \infty} f_1(t) = 1$, the theorem is proved. \hfill \Box

We try to see if a similar result can be obtained by taking instead of $M_2 = Q$ another power mean. For $p = 5/2$ we can prove

**Theorem 2.2.** The double inequality

$$\alpha H(t) + (1 - \alpha)M_{5/2}(t) < G(t) < \beta H(t) + (1 - \beta)M_{5/2}(t), \ \forall t > 1$$

holds if and only if $\alpha \geq 1$ and $\beta \leq 5/7$. The function

$$f_2(t) = \frac{M_{5/2}(t) - G(t)}{M_{5/2}(t) - H(t)}$$

is strictly increasing on $(1, \infty)$. 

Proof. We have

\[ f_2(t) = \frac{\left(\frac{1}{2} t^{5/2} + \frac{1}{2}\right)^{2/5} - t^{1/2}}{\left(\frac{1}{2} t^{5/2} + \frac{1}{2}\right)^{2/5} - \frac{2t}{t+1}}. \]  

(2.7)

By substituting \( t = s^2 \), \( s > 1 \) we get

\[ f_2(s^2) = \frac{(16s^5 + 16)^{2/5} - 4s)(s^2 + 1)}{(s^2 + 1)(16s^5 + 16)^{2/5} - 8s^2}. \]

We differentiate the above function and obtain its numerator

\[ 32(s - 1)(2s^8 - 6s^7 - 2s^6 - 2s^5 - 2s^3 - 2s^2 - 6s + 2 + s^2 (s + 1)(16s^5 + 16)^{2/5}), \]

the denominator being positive. We denote \( g_2(s) = 2s^8 - 6s^7 - 2s^6 - 2s^5 - 2s^3 - 2s^2 - 6s + 2 + s^2 (s + 1)(16s^5 + 16)^{2/5} \)
and we write \( g_2(s) = 0 \) as

\[ \frac{2(s^8 - 3s^7 - s^6 - s^5 - s^3 - s^2 - 3s + 1)}{s^2(s + 1)} = -(16s^5 + 16)^{2/5}. \]  

(2.8)

We apply the 5th power to both sides of (2.8) and get \( h_2(s) = 0 \), where

\[ h_2(s) = s^{30} - 10s^{29} + 25s^{28} + 20s^{27} - 50s^{26} - 196s^{25} - 150s^{24} + 320s^{23} \]
\[ + 1305s^{22} + 2090s^{21} + 2439s^{20} + 2320s^{19} + 2550s^{18} + 3460s^{17} + 4760s^{16} \]
\[ + 5240s^{15} + 4760s^{14} + 3460s^{13} + 2550s^{12} + 2320s^{11} + 2439s^{10} + 2090s^{9} \]
\[ + 1305s^{8} + 320s^{7} - 150s^{6} - 196s^{5} - 50s^{4} + 20s^{3} + 25s^{2} - 10s + 1. \]

Using the Sturm sequence, we obtain that \( h_2(s) \) has no roots in \((1, \infty)\). It follows that \( h_2(s) > 0 \) on \((1, \infty)\), and the derivative of \( f_2(t) \) is positive on this interval, hence \( f_2(t) \) is strictly increasing. Since \( \lim_{t \to 1} f_2(t) = 5/7, \lim_{t \to \infty} f_2(t) = 1 \), the theorem is proved. \( \square \)

Remark 2.3. We can consider the function

\[ f_3(t) = \frac{M_p(t) - G(t)}{M_p(t) - H(t)} \]

for arbitrary \( p > 0 \). It is easy to check that \( \lim_{t \to 1} f_3(t) = p/(p+1) \) and \( \lim_{t \to \infty} f_3(t) = 1 \). It remains to study the monotonicity of \( f_3 \). In the following theorem we prove that, for \( p > 5/2 \), the function \( f_3 \) is not monotone on \((1, \infty)\).

Theorem 2.4. For \( p > 5/2 \), the infimum of the function \( f_3 \) on \((1, \infty)\) satisfies the inequality

\[ \inf_{t > 1} f_3(t) < \frac{p}{p+1}. \]

Proof. Let \( p > 5/2 \). The function \( f_3 \) is given by

\[ f_3(t) = \frac{\left(\frac{1}{2} t^{p} + \frac{1}{2}\right)^{1/p} - t^{1/2}}{\left(\frac{1}{2} t^{p} + \frac{1}{2}\right)^{1/p} - \frac{2t}{t+1}}, \]
and after the substitution \( t = s^2 \), \( s > 1 \) we get

\[
f_3(s^2) = \frac{(\frac{1}{2}s^{2p} + \frac{1}{2})^{1/p} - s}{(s^2 + 1)(\frac{1}{2}s^{2p} + \frac{1}{2})^{1/p} - 2s^2}.
\]

The Taylor series for \( s_0 = 1 \) reads

\[
\frac{p}{p+1} - \frac{p(2p - 5)}{12(p+1)}(s - 1)^2 + \frac{p(2p - 5)}{12(p+1)}(s - 1)^3 + O((s - 1)^4), \text{ for } s \to 1
\]

and its derivative will be

\[
-\frac{p(2p - 5)}{6(p+1)}(s - 1) + O((s - 1)^2).
\]

It follows that the derivative is negative at least for \( s > 1 \) close to 1, hence \( f_3 \) decreases and \( \inf_{t>1} f_3(t) < p/(p+1). \)

Based on the results in theorems 2.1 and 2.2, we formulate the following

**Open problem.** Prove that the function \( f_3 \) is strictly increasing on \((1, \infty)\) for each \( p \in (0, 5/2] \). Then, for each \( p \in (0, 5/2] \), the double inequality

\[
\alpha H(t) + (1 - \alpha)M_p(t) < G(t) < \beta H(t) + (1 - \beta)M_p(t), \forall t > 1
\]

will be true if and only if \( \alpha \geq 1 \) and \( \beta \leq p/(p+1). \)

**References**


Mira-Cristiana Anisiu
“Tiberiu Popoviciu” Institute of Numerical Analysis
Romanian Academy
P.O. Box 68, 400110 Cluj-Napoca, Romania
e-mail: mira@math.ubbcluj.ro
Valeriu Anisiu
“Babeș-Bolyai” University
Faculty of Mathematics and Computer Sciences
1, Kogălniceanu Street, 400084 Cluj-Napoca, Romania
e-mail: anisiu@math.ubbcluj.ro