The weighted mean operator on $\ell^2$ with weight sequence $w_n = (n + 1)^p$ is hyponormal for $p = 2$

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Abstract. Posinormality is used to demonstrate that the weighted mean matrix whose weight sequence is the sequence of squares of positive integers is a hyponormal operator on $\ell^2$.

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1. Introduction

In this paper, attention will be focused on an example of a weighted mean matrix that does not satisfy the sufficient conditions for hyponormality given in [4]. Nor does it satisfy the key lemma used in [5], so a somewhat different approach will be required here. The computations here are much more complex than those in [5], and, because of that, the computer software package SAGE [6] has been used as an aid.

If $B(H)$ denotes the set of all bounded linear operators on a Hilbert space $H$, then $A \in B(H)$ is said to be is posinormal (see [1], [2]) if $AA^* = A^*PA$ for some positive operator $P \in B(H)$, called the interrupter, and $A$ is hyponormal if $A^*A - AA^* \geq 0$. Hyponormal operators are necessarily posinormal.

A lower triangular infinite matrix $M = [m_{ij}]$, acting through multiplication to give a bounded linear operator on $\ell^2$, is factorable if its entries are of the form

$$m_{ij} = \begin{cases} a_i c_j & \text{if } j \leq i \\ 0 & \text{if } j > i \end{cases}$$

where $a_i$ depends only on $i$ and $c_j$ depends only on $j$; the matrix $M$ is terraced if $c_j = 1$ for all $j$. A weighted mean matrix is a lower triangular matrix with entries $w_j/W_i$, where $\{w_j\}$ is a nonnegative sequence with $w_0 > 0$, and $W_i = \sum_{j=0}^{i} w_j$. A weighted mean matrix is factorable, with $a_i = 1/W_i$ and $c_j = w_j$ for all $i,j$. 
2. Main result

Under consideration here will be the weighted mean matrix $M$ associated with the weight sequence $w_n = (n+1)^2$. As was also the case for $w_n = n+1$, this example is not easily seen to be hyponormal directly from the definition and fails to satisfy the sufficient conditions for hyponormality given in [4, Corollary 1], but it survives the necessary condition given in [4, Corollary 2]. Encouraged by the latter, we set out to prove that $M$ is hyponormal. The next theorem will provide us our main tool – an expression for the interrupter $P$ associated with the matrix $M$.

**Theorem 2.1.** Suppose $M = [a_{ij}]$ is a lower triangular factorable matrix that acts as a bounded operator on $\ell^2$ and that the following conditions are satisfied:

(a) both $\{a_n\}$ and $\{a_n/c_n\}$ are positive decreasing sequences that converge to 0, and
(b) the matrix $B$ defined by $B = [b_{ij}]$ by

$$b_{ij} = \begin{cases} 
  c_{ij} - 1 \frac{a_{j+1}}{c_{j+1}a_j} & \text{if } i \leq j; \\
  0 & \text{if } i > j+1;
\end{cases}$$

is a bounded operator on $\ell^2$.

Then $M$ is posinormal with interrupter $P = B^* B$. The entries of $P = [p_{ij}]$ are given by

$$p_{ij} = \begin{cases} 
  c_{ij}^2a_{j+1}^2a_{j+1}^2 + (\sum_{k=0}^{j}c_{k}a_{j+1}c_{j}a_{j+1})^2 & \text{if } i = j; \\
  \frac{c_{ij}a_{j+1}^2}{c_{i}a_{j+1}^2} & \text{if } i > j; \\
  \frac{c_{ij}a_{j+1}^2}{c_{i}a_{j+1}^2} & \text{if } i < j.
\end{cases}$$

**Proof.** See [3]. \[\square\]

We are now ready for the main result. The induction step in the proof below was aided by explicit computations using the computer software package SAGE [6].

**Theorem 2.2.** The weighted mean matrix $M$ associated with the weight sequence $w_n = (n+1)^2$ is hyponormal.

**Proof.** One easily verifies that the weighed mean matrix $M$ associated with $w_n = (n+1)^2$ satisfies the hypotheses of Theorem 2.1. For $M$ to be hyponormal, we must have

$$\langle (M^*M - MM^*)f, f \rangle = \langle (M^*M - M^*PM)f, f \rangle = \langle (I - P)Mf, Mf \rangle \geq 0$$

for all $f$ in $\ell^2$. Consequently, we can conclude that $M$ will be hyponormal when $Q := I - P \geq 0$; we note that the range of $M$ contains all the $e_n$’s from the standard orthonormal basis for $\ell^2$. 

The weighted mean matrix with $w_n = (n + 1)^2$

Using the given weight sequence, we determine that the entries of $Q = [q_{mn}]$ are given by

$$q_{mn} = \begin{cases} 
\frac{60n^7 + 600n^6 + 2488n^5 + 5476n^4 + 6795n^3 + 4650n^2 + 1584n + 207}{30(n+1)^3(n+2)^2(n+3)(2n+5)} & \text{if } m = n; \\
-\frac{10m^3+52m^2+93m+57}{30(m+1)^2(m+2)^2(m+3)(2m+5)} & \text{if } m > n; \\
-\frac{10n^3+52n^2+93n+57}{30(n+1)^2(n+2)^2(n+3)(2n+5)} & \text{if } m < n.
\end{cases}$$

In order to show that $Q$ is positive, it suffices to show that $Q_N$, the $N^{th}$ finite section of $Q$ (involving rows $m = 0, 1, 2, ..., N$ and columns $n = 0, 1, 2, ..., N$), has positive determinant for each positive integer $N$. For columns $n = 0, 1, ..., N - 1$, we multiply the $(n + 1)^{st}$ column of $Q_N$ by

$$z_n := \frac{(n + 3)(2n + 3)(3n^2 + 7n + 3)}{(n + 1)(2n + 5)(3n^2 + 13n + 13)}$$

and subtract from the $n^{th}$ column. Call the new matrix $Q'_N$. Then we work with the rows of $Q'_N$. For $m = 0, 1, ..., N - 1$, we multiply the $(m + 1)^{st}$ row of $Q'_N$ by $z_m$ and subtract from the $m^{th}$ row. This leads to the tridiagonal form

$$Y_N := \begin{pmatrix}
d_0 & s_0 & 0 & \ldots & 0 & 0 \\
s_0 & d_1 & s_1 & \ldots & 0 & 0 \\
0 & s_1 & d_2 & \ldots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & d_{N-1} & s_{N-1} \\
0 & 0 & 0 & \ldots & s_{N-1} & d_N
\end{pmatrix},$$

where

$$d_n = q_{nn} - z_nq_{n,n+1} - z_n'(q_{n+1,n} - z_nq_{n+1,n+1}) = \frac{144n^{11} + 3192n^{10} + 31216n^9 + 177540n^8 + 651210n^7 + 1613062n^6}{(n+1)^2(n+2)^2(n+3)(2n+5)^2(2n+7)^2(3n^2+13n+13)^2} + \frac{2743061n^5 + 3168210n^4 + 2460693n^3 + 1192988n^2 + 323673n + 37086}{(n+1)^2(n+3)(n+4)(2n+5)^2(2n+7)(3n^2+13n+13)^2}$$

and $s_n = q_{n+1,n} - z_nq_{n+1,n+1} = -\frac{(n+3)(2n+3)(2n^2+10n+11)(3n^2+7n+3)}{(n+1)(n+2)(n+4)(2n+5)(2n+7)(3n^2+13n+13)}$ when $0 \leq n \leq N - 1$; and

$$d_N = \frac{60N^7 + 600N^6 + 2488N^5 + 5476N^4 + 6795N^3 + 4650N^2 + 1584N + 207}{30(N+1)^3(N+2)^2(N+3)(2N+5)}.$$

Note that $\det Y_N = \det Q'_N = \det Q_N$. Next we transform $Y_N$ into a triangular matrix with the same determinant, and we find that the new matrix has diagonal entries $\delta_n$, which are given by the recursion formula: $\delta_0 = d_0$, $\delta_n = d_n - s_{n-1}/\delta_{n-1} (1 \leq n \leq N)$. An induction argument shows that

$$\delta_n \geq \frac{30(n+3)^6(2n+3)^2(2n^2+10n+11)^2(3n^2+7n+3)^2}{(n+1)^2(n+4)(2n+5)^2(2n+7)(3n^2+13n+13)^2} g(n) > 0,$$

where $g(n) = 60n^7 + 1020n^6 + 7348n^5 + 29016n^4 + 67679n^3 + 93031n^2 + 69633n + 21860$ for $0 \leq n \leq N - 1$; note that $g(n)$ is the numerator obtained in $d_N$ when $N$ is replaced by $n + 1$. Since $d_N$ departs from the pattern set by the earlier $d_n$’s, $\delta_N$ must be handled separately:

$$\delta_N \geq \frac{60N^7 + 600N^6 + 2488N^5 + 5476N^4 + 6795N^3 + 4650N^2 + 1584N + 207}{30(N+1)^3(N+2)^2(N+3)(2N+5)} > 0.$$

Therefore $\det Q_N = \prod_{j=0}^{N} \delta_j > 0$, and the proof is complete. $\square$
For the induction step in the proof above, the initial estimate for \( \delta_n \) came from computing \( \frac{s^2}{d_N} \) and then replacing \( N \) by \( n + 1 \). From there, an adjustment was needed.

The verification of the induction step reduces to showing that a 19th degree polynomial is positive for all \( n \geq 1 \). Below is the command that was given to SAGE to execute.

\[
n = \text{var('n')}
\]
\[
\text{expand((30*(n+2)^4*(144*n^11 + 3192*n^10 + 31216*n^9 + 177540*n^8 + 651210*n^7 + 1613062*n^6 + 2743061*n^5 + 3186210*n^4 + 2460693*n^3 + 1192988*n^2 + 651210*n + 37086) - (n+1)*(n+4)*(2*n+5)*(2*n+7)*(3*n^2+13*n+13)^2*(60*(n-1)^7 + 1020*(n-1)^6 + 7348*(n-1)^5 + 29016*(n-1)^4 + 93031*(n-1)^3 + 69633*(n-1)^2 + 21860)))*}
\]
\[
(60*n^7 + 1020*n^6 + 7348*n^5 + 29016*n^4 + 67679*n^3 + 93031*n^2 + 69633*n + 21860) - 900*(n+1)*(n+2)^4*(n+3)^7*(2*n+10+n+11)^2*(3*n^2+7*n+3)^2)
\]

And this is the resulting SAGE worksheet output, which we denote by \( f(n) \).

\[
\begin{align*}
    f(0) &= -26586925200 \\
    f(1) &= 48058098267150 \\
    f(2) &= 29447930357308764 \\
    f(3) &= 2740303120043884194 \\
    f(4) &= 10061120108363615760
\end{align*}
\]

References


[5] Rhaly, H.C., Jr., Rhoades, B.E., *The weighted mean operator on \( \ell^2 \) with the weight sequence \( w_n = n + 1 \) is hyponormal*, New Zealand J. Math, in press.