On vector variational-like inequalities and vector optimization problems in Asplund spaces

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Abstract. In this paper, we consider different kinds of generalized invexity for vector valued functions and a vector optimization problem. Some relations between some vector variational-like inequalities and a vector optimization problem are established using the properties of Mordukhovich limiting subdifferentials under $C - \eta$–strong pseudomonotonicity.


Keywords: Vector variational-like inequalities, Vector optimization problem, Mordukhovich limiting subdifferentials, Asplund space, $C - \eta$–strong pseudomonotonicity.

1. Introduction

In 1998, Giannessi [9] first used, so called, Minty type vector variational inequality (in short, $MVVI$) to establish the necessary and sufficient conditions for a point to be an efficient solution of a vector optimization problem (in short, $(VOP)$) for differentiable and convex functions. Since then, several researchers have studied $(VOP)$ by using different kinds of $MVVI$ under different assumptions, see [1, 2, 10, 15, 19] and the references therein. Consequently, vector variational inequalities have been generalized in various directions, in particular, vector variational-like inequality problems, see [1, 13, 14, 20, 23, 28] and the references therein. The vector variational-like inequalities are closely related to the concept of the invex and preinvex functions which generalize the notion of the convexity of functions. The concept of the invexity was first introduced by Hanson [12]. More recently, the characterization and applications for generalized invexity were studied by many authors, see [11, 13, 19, 21, 24, 25, 27] and the references therein.

The relation between the vector variational inequality and the smooth vector optimization problem has been studied by many authors (see, for example, [9, 23, 26] and the references therein). Yang et al. [26] extended the result of Giannessi [9, 10] for differentiable but pseudoconvex functions. Yang and Yang [23] gave some relations...
between Minty variational-like inequalities and the vector optimization problems for differentiable but pseudo-invex vector-valued functions. Yang et al. [25, 26] and Garzon et al. [6, 7] studied the relations between generalized invexity of a differentiable function and generalized monotonicity of its gradient mapping. Very recently, Rezaie and Zafarani [20] showed some relations between the vector variational-like inequalities and vector optimization problems for nondifferential functions under generalized monotonicity. Al-Homidan and Ansari [1] studied the relation among the generalized Minty vector variational-like inequality, generalized Stampacchia vector variational-like inequality and vector optimization problems for nondifferential and nonconvex functions with Clarke’s generalized directional derivative and then, Ansari and Lee [2] showed that for pseudoconvex functions with upper Dini directional derivative, similar results holds. Ansari, Rezaie and Zafarani [3] considered generalized Minty vector variational-like inequality problems, Stampacchia vector variational-like inequality problems and nonsmooth vector optimization problems under nonsmooth pseudo-invexity assumptions. They also considered the weak formulations of generalized Minty vector variational-like inequality problems and generalized Stampacchia vector variational-like inequality problems in a very general setting and established the existence results for their solutions. The main results in [1] and [20] were obtained in the setting of Clarke subdifferential. Since the class of Clarke subdifferential is larger than the class of Mordukhovich subdifferential, it is necessary to study the vector variational-like inequalities and vector optimization problems in the setting of Mordukhovich subdifferential (see [5, 16, 17]). Oveisiha and Zafarani [18] established some properties of pseudo-invex functions and Mordukhovich limiting subdifferentials and relations between vector variational-like inequalities and vector optimization problems. Chen and Huang [4] considered the Minty vector variational-like inequality, Stampacchia vector variational-like inequality and the weak formulations of these inequalities, defined by means of Mordukhovich limiting subdifferentials in Asplund spaces. They established some relations between the vector variational-like inequalities and vector optimization problems using the properties of Mordukhovich limiting subdifferential. Farajzadeh et al. [8] considered generalized variational-like inequalities with set-valued mappings in topological spaces, which include as a special case the strong vector variational-like inequalities. Motivated and inspired by the work mentioned above, in this paper we consider the Minty vector variational-like inequality, Stampacchia vector variational-like inequality and the weak formulations of these inequalities, defined by means of Mordukhovich limiting subdifferentials in Asplund spaces. Some relations between vector variational-like inequalities and a vector optimization problem (respectively, between Minty vector variational-like inequality and Stampacchia vector variational-like inequality) are established using the properties of Mordukhovich limiting subdifferentials under different kinds of generalized invexity (respectively, $C - \eta$—strong pseudomonotonicity).
2. Preliminaries

Let $X$ be a Banach space endowed with a norm $\|\cdot\|$ and $X^*$ its dual space with a norm $\|\cdot\|^*$. Denote $\langle\cdot,\cdot\rangle, [x,y], |x,y|$ the dual pair between $X$ and $X^*$, the line segment for $x,y \in X$ and $[x,y]\setminus\{x,y\}$, respectively. Let $\Omega$ be a nonempty open subset of $X$.

When functions are not differentiable, we use the concept of subdifferential: Fréchet subdifferential, Limiting subdifferential and Clarke-Rockafellar subdifferential.

**Definition 2.1.** Let $X$ be a Banach space and $f : X \to \mathbb{R} \cup \{\infty\}$ a proper l.s.c. function. We say that $f$ is Fréchet-subdifferentiable and $\xi^*$ is Fréchet-subderivative of $f$ at $x$ $(\xi^* \in \partial_f f(x))$ if $x \in \text{dom } f$ and
\[
\liminf_{\|h\| \to 0} \frac{f(x+h) - f(x) - \langle \xi^*, h \rangle}{\|h\|} \geq 0.
\]

**Definition 2.2.** [16] Let $x \in \Omega$ and $\varepsilon \geq 0$. The set of $\varepsilon$ – normals to $\Omega$ at $x$ is defined by
\[
\hat{N}_\varepsilon(x, \Omega) = \{x^* \in X^* | \limsup_{u \in \Omega} \frac{\langle x^*, u-x \rangle}{\|u-x\|} \leq \varepsilon\}.
\]

If $x \notin \Omega$, we put $\hat{N}_\varepsilon(x, \Omega) = \emptyset$ for all $\varepsilon \geq 0$.

**Definition 2.3.** [16] Let $\bar{x} \in \Omega$. Then $x^* \in X^*$ is a limiting normal to $\Omega$ at $\bar{x}$ if there are sequences $\varepsilon_k \searrow 0$, $x_k \Omega \to \bar{x}$ and $x_k^* \w^* \to \bar{x}^*$ such that $x_k^* \in \hat{N}_\varepsilon(x_k, \Omega)$, for all $k \in \mathbb{N}$. The set of such normals
\[
N(\bar{x}, \Omega) = \limsup_{x_k \searrow \bar{x}} \hat{N}_\varepsilon(x, \Omega)
\]
is the limiting normal cone to $\Omega$ at $\bar{x}$. If $\bar{x} \notin \Omega$, we put $N(\bar{x}, \Omega) = \emptyset$.

**Remark 2.4.** Note that the symbol $u \Omega \to x$ means that $u \to x$ with $u \in \Omega$. The symbol $\w^*$ stands for convergence in weak* topology.

**Definition 2.5.** [16] Considering the extended-real-valued function $\varphi : X \to \bar{\mathbb{R}} = [-\infty, +\infty]$ we say that $\varphi$ is proper if $\varphi(x) > -\infty$ for all $x \in X$ and its domain, $\text{dom} \varphi = \{x \in X : \varphi(x) < \infty\}$, is nonempty. The epigraph of $\varphi$ is defined as
\[
\text{epi} \varphi = \{(x, a) \in X \times \mathbb{R} | \varphi(x) \leq a\}.
\]

**Definition 2.6.** [16] Considering a point $\bar{x} \in X$ with $| \varphi(\bar{x}) | < \infty$, the set
\[
\partial_L \varphi(\bar{x}) = \{x^* \in X^* | (x^*, -1) \in N((\bar{x}, \varphi(\bar{x})), \text{epi } \varphi)\}
\]
is the limiting subdifferential of $\varphi$ at $\bar{x}$ and its elements are limiting subdifferentials of $\varphi$ at this point. If $| \varphi(\bar{x}) | = \infty$, we put $\partial_L \varphi(\bar{x}) = \emptyset$.

**Remark 2.7.** [16] It is well known that
\[
\partial_F f(x) \subseteq \partial_L f(x) \subseteq \partial_C f(x),
\]
where $\partial_C f$ is the Clarke subdifferential.
Definition 2.8. A Banach space $X$ is Asplund, or it has the Asplund property, if every convex continuous function $\varphi : U \to \mathbb{R}$ defined on an open convex subset $U$ of $X$ is Fréchet differentiable on a dense subset of $U$.

Remark 2.9. One of the most popular Asplund spaces is any reflexive Banach space [16].

Theorem 2.10. [16] Let $X$ be an Asplund space and $\varphi : X \to \mathbb{R}$ be proper and l.s.c. around $x \in \text{dom} \varphi$, then
\[ \partial_L \varphi(x) = \limsup_{x \to x} \partial_F \varphi(x). \]

For more details and applications, see [16].

Definition 2.11. Let $\eta : X \times X \to X$. A subset $\Omega$ of $X$ is said to be invex with respect to $\eta$ if for any $x, y \in \Omega$ and $\lambda \in [0, 1]$, we have $y + \lambda \eta(x, y) \in \Omega$.

Hereafter, unless otherwise specified, we assume that $X$ is an Asplund space and $\Omega \subseteq X$ is a nonempty open invex set with respect to the mapping $\eta : \Omega \times \Omega \to X$.

Definition 2.12. A mapping $\eta : \Omega \times \Omega \to X$ is said to be skew if for any $x, y \in \Omega$,
\[ \eta(x, y) + \eta(y, x) = 0. \]

Definition 2.13. Let $x_0 \in \Omega$. A mapping $\eta : \Omega \times \Omega \to X$ is said to be skew at $x_0$ if for any $x \in \Omega$, $x \neq x_0$,
\[ \eta(x, x_0) + \eta(x_0, x) = 0. \]

Definition 2.14. [21] Let $f : \Omega \to \mathbb{R}$ be a function. $f$ is said to be

1. weakly quasi invex with respect to $\eta$ on $\Omega$ if for any $x, y \in \Omega$,
\[ f(x) \leq f(y) \Rightarrow \exists \xi^* \in \partial_L f(y) \langle \xi^*, \eta(x, y) \rangle \leq 0; \]

2. quasi invex with respect to $\eta$ on $\Omega$ if for any $x, y \in \Omega$,
\[ f(x) \leq f(y) \Rightarrow \forall \xi^* \in \partial_L f(y) \langle \xi^*, \eta(x, y) \rangle \leq 0; \]

3. pseudo invex with respect to $\eta$ on $\Omega$ if for any $x, y \in \Omega$,
\[ \langle \xi^*, \eta(x, y) \rangle \geq 0, \exists \xi^* \in \partial_L f(y) \Rightarrow f(x) \geq f(y). \]

In some results of the paper we need to consider some further assumptions on $\eta$. These assumptions are known in invexity literature (Jabarootian and Zafarani (2006) [13]).

Condition C. Let $\eta : \Omega \times \Omega \to X$. Then for any $x, y \in \Omega$, $\lambda \in [0, 1]$,
\[ \begin{cases} C_1 : \eta(x, y + \lambda \eta(x, y)) = (1 - \lambda) \eta(x, y); \\ C_2 : \eta(y, y + \lambda \eta(x, y)) = -\lambda \eta(x, y). \end{cases} \]

Remark 2.15. Yang et al. [27] have shown that if $\eta : \Omega \times \Omega \to X$ satisfies condition C, then for all $x, y \in \Omega$, $\lambda \in [0, 1]$,
\[ \eta(y + \lambda \eta(x, y), y) = \lambda \eta(x, y). \]
Definition 2.16. Let \( \eta : \Omega \times \Omega \rightarrow X \), \( x_0 \in \Omega \). We say that \( \eta : \Omega \times \Omega \rightarrow X \) satisfies condition \( C \) at \( x_0 \) if for all \( x \in \Omega \), \( \lambda \in [0,1] \),
\[ \eta(x_0 + \lambda \eta(x,x_0),x_0) = \lambda \eta(x,x_0). \]

Definition 2.17. Let \( f = (f_1, \ldots, f_n) : \Omega \rightarrow \mathbb{R}^n \) be a vector-valued function and \( x_0 \in \Omega \). \( f \) is said to be

1. pseudo – invex with respect to \( \eta \) on \( \Omega \) if for any \( x, y \in \Omega \),
   \[ f(x) - f(y) \in -\mathbb{R}_+^n \setminus \{0\} \implies \langle \partial_L f(y), \eta(x,y) \rangle \subseteq -\mathbb{R}_+^n \setminus \{0\}; \]
2. quasi – invex with respect to \( \eta \) on \( \Omega \) if for any \( x, y \in \Omega \),
   \[ \langle \xi^*, \eta(x,y) \rangle \in \mathbb{R}_+^n \setminus \{0\}, \exists \xi^* \in \partial_L f(y) \implies f(x) - f(y) \in \mathbb{R}_+^n \setminus \{0\}; \]
3. weakly – quasi – invex with respect to \( \eta \) on \( \Omega \) if for any \( x, y \in \Omega \),
   \[ \langle \partial_L f(y), \eta(x,y) \rangle \subseteq \mathbb{R}_+^n \setminus \{0\} \implies f(x) - f(y) \in \mathbb{R}_+^n \setminus \{0\}; \]
4. weakly – quasi – invex at \( x_0 \) with respect to \( \eta \) if for any \( x \in \Omega \),
   \[ \langle \partial_L f(x_0), \eta(x,x_0) \rangle \subseteq \mathbb{R}_+^n \setminus \{0\} \implies f(x) - f(x_0) \in \mathbb{R}_+^n \setminus \{0\}. \]

Remark 2.18. Next, we provide an example which shows that a function \( f = (f_1, \ldots, f_n) \) it can be pseudo-invex with respect to \( \eta \) on \( \Omega \) and there exists \( k, 1 \leq k \leq n \), such that \( f_k \) is not pseudo-invex with respect to \( \eta \) on \( \Omega \).

Example 2.19. Let us consider \( X = \mathbb{R} \), \( \Omega = [-1,1] \), \( f = (f_1, f_2) : \Omega \rightarrow \mathbb{R}^2 \) defined as
\[ f_1(x) = \begin{cases} \sqrt{x}, & x \geq 0, \\ x, & x < 0, \end{cases} \quad f_2(x) = x \]
and \( \eta : \Omega \times \Omega \rightarrow \mathbb{R} \) defined as
\[ \eta(x,y) = x - y. \]

We have
\[ \partial_L f(x) = \begin{cases} (\frac{1}{\sqrt{x}}, 1), & x > 0, \\ [0, \infty) \times \{1\}, & x = 0, \\ (1,1), & x < 0. \end{cases} \]

It is not difficult to see that \( f \) is pseudo-invex with respect to \( \eta \). Function \( f_1 \) is not pseudo-invex with respect to \( \eta \) on \( \Omega \) because for \( x = -1, y = 0 \) there exists \( \xi^* = 0 \in \partial_L f(y) \) such that \( \langle \xi^*, \eta(x,y) \rangle = 0 \) and \( f(x) < f(y) \).

Definition 2.20. \[8\] A set valued mapping \( F : \Omega \rightarrow 2^X \) is said to be \( C - \eta - \text{strong pseudomonotone} \) if for any \( x, y \in \Omega \),
\[ \langle Fx, \eta(x,y) \rangle \not\in -C \setminus \{0\} \implies \langle Fy, \eta(y,x) \rangle \subseteq -C(y). \]

Definition 2.21. A set valued mapping \( F : \Omega \rightarrow 2^X \) is said to be \((C,K) - \eta - \text{strong pseudomonotone} \) if for any \( x, y \in \Omega \),
\[ \langle Fx, \eta(x,y) \rangle \not\in C \implies \langle Fy, \eta(y,x) \rangle \subseteq K. \]
Definition 2.22. A set valued mapping \( F : \Omega \to 2^{X^*} \) is said to be strictly \((C,K)\)–\(\eta\)–strong pseudomonotone if for any \( x, y \in \Omega, x \neq y \),

\[
\langle Fx, \eta(x,y) \rangle \nsubseteq C \implies \langle Fy, \eta(y,x) \rangle \subseteq K.
\]

Let \( f = (f_1, ..., f_n) : \Omega \to \mathbb{R}^n \) be a vector-valued function, where \( f_i : \Omega \to \mathbb{R} \) \((i = 1, ..., n)\) is non-differentiable locally Lipschitz function.

In this paper, we consider the following vector optimization problem:

\[
\text{(VOP)} \quad \text{Minimize } f(x) = (f_1(x), ..., f_n(x))
\]

subject to \( x \in \Omega \).

Definition 2.23. A point \( x_0 \in \Omega \) is said to be an efficient (or Pareto) solution (respectively, weak efficient solution) of \((\text{VOP})\) if for all \( x \in \Omega \),

\[
f(x) - f(x_0) = (f_1(x) - f_1(x_0), ..., f_n(x) - f_n(x_0)) \notin -\mathbb{R}^n_+ \setminus \{0\},
\]

(respectively, \( f(x) - f(x_0) = (f_1(x) - f_1(x_0), ..., f_n(x) - f_n(x_0)) \notin -\text{int}\mathbb{R}^n_+ \)),

where \( \mathbb{R}^n_+ \) is the nonnegative orthant of \( \mathbb{R}^n \) and \( 0 \) is the origin of the nonnegative orthant.

3. Characterization

We consider the following Minty vector variational-like inequality problems and Stampacchia vector variational-like inequality problems.

\( (\text{GGMVVVLIP}) \) Find \( x_0 \in \Omega \) such that, for all \( x \in \Omega \) and all \( \xi_i \in \partial_L f_i(x) \) \((i = 1, ..., n)\),

\[
\langle \xi_i^*, \eta(x,x_0) \rangle = (\langle \xi_1^*, \eta(x,x_0) \rangle, ..., \langle \xi_n^*, \eta(x,x_0) \rangle) \notin -\mathbb{R}^n_+ \setminus \{0\}.
\]

\( (\text{GMVVVLIP}) \) Find \( x_0 \in \Omega \) such that, for all \( x \in \Omega \) there exists \( \xi_i \in \partial_L f_i(x) \) \((i = 1, ..., n)\),

\[
\langle \xi_i^*, \eta(x,x_0) \rangle = (\langle \xi_1^*, \eta(x,x_0) \rangle, ..., \langle \xi_n^*, \eta(x,x_0) \rangle) \notin -\mathbb{R}^n_+ \setminus \{0\}.
\]

\( (\text{WGGMVVLIP}) \) Find \( x_0 \in \Omega \) such that, for all \( x \in \Omega \) and all \( \xi_i \in \partial_L f_i(x) \) \((i = 1, ..., n)\),

\[
\langle \xi_i^*, \eta(x,x_0) \rangle = (\langle \xi_1^*, \eta(x,x_0) \rangle, ..., \langle \xi_n^*, \eta(x,x_0) \rangle) \notin -\text{int}\mathbb{R}^n_+.
\]

\( (\text{WGMVVVLIP}) \) Find \( x_0 \in \Omega \) such that, for all \( x \in \Omega \) there exists \( \xi_i \in \partial_L f_i(x) \) \((i = 1, ..., n)\),

\[
\langle \xi_i^*, \eta(x,x_0) \rangle = (\langle \xi_1^*, \eta(x,x_0) \rangle, ..., \langle \xi_n^*, \eta(x,x_0) \rangle) \notin -\text{int}\mathbb{R}^n_+.
\]

\( (\text{SVVLIP}) \) Find \( x_0 \in \Omega \) such that, for all \( x \in \Omega \) there exists \( \xi_i \in \partial_L f_i(x_0) \) \((i = 1, ..., n)\),

\[
\langle \xi_i^*, \eta(x,x_0) \rangle = (\langle \xi_1^*, \eta(x,x_0) \rangle, ..., \langle \xi_n^*, \eta(x,x_0) \rangle) \notin -\mathbb{R}^n_+ \setminus \{0\}.
\]

\( (\text{WSVVVLIP}) \) Find \( x_0 \in \Omega \) such that, for all \( x \in \Omega \) there exists \( \xi_i \in \partial_L f_i(x_0) \) \((i = 1, ..., n)\),

\[
\langle \xi_i^*, \eta(x,x_0) \rangle = (\langle \xi_1^*, \eta(x,x_0) \rangle, ..., \langle \xi_n^*, \eta(x,x_0) \rangle) \notin -\text{int}\mathbb{R}^n_+.
\]
Theorem 3.1. If \( x_0 \) is a solution of \((SVVLIP)\), \( \partial_L f \) is strictly \((\text{int} \mathbb{R}_+^n, \text{int} \mathbb{R}_+^n) - \eta\)–strong pseudomonotone, \( \eta \) is skew at \( x_0 \) and \( \langle \partial_L f(x_0), \eta(x_0, x_0) \rangle \subseteq \mathbb{R}_+^n \), then \( x_0 \) is a solution of \((GGMVVLIP)\).

Proof. Suppose that \( x_0 \) is not a solution of \((GGMVVLIP)\).
Since \( \langle \partial_L f(x_0), \eta(x_0, x_0) \rangle \subseteq \mathbb{R}_+^n \) it follows that there exist \( \bar{x} \in \Omega, \bar{x} \neq x_0, \bar{\zeta} \in \partial_L f(\bar{x}) \) such that
\[
\langle \bar{\zeta}, \eta(\bar{x}, x_0) \rangle \in -\mathbb{R}_+^n \setminus \{0\}.
\]
Therefore,
\[
\langle \bar{\zeta}, \eta(\bar{x}, x_0) \rangle \notin \text{int} \mathbb{R}_+^n. \tag{3.1}
\]
Since \( \partial_L f \) is strictly \((\text{int} \mathbb{R}_+^n, \text{int} \mathbb{R}_+^n) - \eta\)–strong pseudomonotone, by (3.1) we obtain
\[
\langle \partial_L f(x_0), \eta(x_0, \bar{x}) \rangle \subseteq \text{int} \mathbb{R}_+^n. \tag{3.2}
\]
Since \( \eta \) is skew at \( x_0 \), by (3.2) it follows that
\[
\langle \partial_L f(x_0), \eta(\bar{x}, x_0) \rangle \subseteq -\text{int} \mathbb{R}_+^n,
\]
which contradicts the fact that \( x_0 \) is a solution of \((SVVLIP)\). Therefore, it follows that \( x_0 \) is a solution of \((GGMVVLIP)\). \( \square \)

Example 3.2. Let us consider \( X = \mathbb{R}, \ \Omega = [-1, 1], \ f : \Omega \to \mathbb{R} \) defined as
\[
f(x) = \begin{cases} \sqrt{x}, & x \geq 0, \\ -x, & x < 0. \end{cases}
\]
and \( \eta : \Omega \times \Omega \to \mathbb{R} \) defined as
\[
\eta(x, y) = x - y.
\]
We have
\[
\partial_L f(x) = \begin{cases} \frac{1}{2\sqrt{x}}, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}
\]
and \( \partial_L f \) is strictly \((\text{int} \mathbb{R}_+, \text{int} \mathbb{R}_+) - \eta\)–strong pseudomonotone. It is not difficult to see that \( x_0 = 0 \) is a solution of \((SVVLIP)\) and \( \eta \) is skew at \( x_0 \). Therefore, \( x_0 \) is a solution of \((GGMVVLIP)\).

Corollary 3.3. If \( x_0 \) is a solution of \((SVVLIP)\), \( \partial_L f \) is strictly \((\text{int} \mathbb{R}_+, \text{int} \mathbb{R}_+) - \eta\)–strong pseudomonotone and \( \eta \) is skew at \( x_0 \), then \( x_0 \) is a solution of \((GGMVVLIP)\).

Corollary 3.4. If \( x_0 \) is a solution of \((WSVVLIP)\), \( \partial_L f \) is strictly \((\text{int} \mathbb{R}_+, \text{int} \mathbb{R}_+) - \eta\)–strong pseudomonotone, \( \eta \) is skew at \( x_0 \) and \( \langle \partial_L f(x_0), \eta(x_0, x_0) \rangle \subseteq \mathbb{R}_+^n \setminus \{0\} \), then \( x_0 \) is a solution of \((GGMVVLIP)\).

Corollary 3.5. If \( x_0 \) is a solution of \((WSVVLIP)\), \( \partial_L f \) is strictly \((\text{int} \mathbb{R}_+, \text{int} \mathbb{R}_+) - \eta\)–strong pseudomonotone and \( \eta \) is skew at \( x_0 \), then \( x_0 \) is a solution of \((WGMVVLIP)\).

Theorem 3.6. If \( x_0 \) is a solution of \((VOP)\), \( f \) is quasi-invex with respect to \( \eta \) on \( \Omega \) and \( \eta \) is skew, then \( x_0 \) is a solution of \((GGMVVLIP)\).
Proof. Suppose that \( x_0 \) is not a solution of \((GGMVVLIP)\). It follows that there exist \( x \in \Omega, \zeta \in \partial_L f(x) \) such that we have
\[
\nonumber \langle \zeta, \eta(x, x_0) \rangle \in -\mathbb{R}_+^n \setminus \{0\}. \tag{3.3}
\]
Since \( \eta \) is skew, by (3.3) we obtain
\[
\nonumber \langle \zeta, \eta(x_0, x) \rangle \in \mathbb{R}_+^n \setminus \{0\}.
\]
Since \( f \) is quasi-invex, it follows that
\[
\nonumber f(x_0) - f(x) \in \mathbb{R}_+^n \setminus \{0\},
\]
which contradicts the fact that \( x_0 \) is a solution of \((VOP)\). Therefore, \( x_0 \) is a solution of \((GGMVVLIP)\).

Remark 3.7. In [4] (Theorem 3.1) the authors obtained this result by assuming that \( f_i (i = 1, \ldots, n) \) are invex with respect to \( \eta \) on \( \Omega \). Next, we provide an example which shows that a function \( f = (f_1, \ldots, f_n) \) it can be quasi-invex with respect to \( \eta \) on \( \Omega \) and there exists \( k, 1 \leq k \leq n \), such that \( f_k \) is not invex with respect to \( \eta \) on \( \Omega \).

Example 3.8. Let us consider \( X = \mathbb{R}, \Omega = [-\frac{1}{5}, \frac{1}{5}] \), \( f = (f_1, f_2) : \Omega \to \mathbb{R}^2 \) defined as
\[
\nonumber f_1(x) = \begin{cases} 
  x^2 + 2x, & x > 0, \\
  -x, & x \leq 0,
\end{cases}
\]
\[
\nonumber f_2(x) = \begin{cases} 
  x^3 - 2x^2 + x, & x \geq 0, \\
  -x, & x < 0,
\end{cases}
\]
and \( \eta : \Omega \times \Omega \to \mathbb{R} \) defined as
\[
\nonumber \eta(x, y) = x - y.
\]
We have
\[
\nonumber \partial_L f(x) = \begin{cases} 
  (2x + 2, 3x^2 - 4x + 1), & x > 0, \\
  (k, t), & k \in \{2, -1\}, t \in \{1, -1\}, x = 0, \\
  (-1, -1), & x < 0.
\end{cases}
\]
It is easy to observe that \( x_0 = 0 \) is a solution of \((VOP)\), \( \eta \) is skew and function \( f \) is quasi-invex with respect to \( \eta \) on \( \Omega \). Function \( f_2 \) is not invex with respect to \( \eta \) on \( \Omega \) because for \( x = 1, y = 0 \) we obtain
\[
\nonumber f_2(1) - f_2(0) < \langle \xi^*, \eta(1, 0) \rangle,
\]
for \( \xi^* = 1 \).

Corollary 3.9. If \( x_0 \) is a solution of \((VOP)\), \( f \) is quasi-invex with respect to \( \eta \) on \( \Omega \) and \( \eta \) is skew, then \( x_0 \) is a solution of \((GMVVLIP)\).

Theorem 3.10. If \( x_0 \) is a solution of \((VOP)\), \( f \) is weakly quasi-invex at \( x_0 \) with respect to \( \eta \) on \( \Omega \) and \( \eta \) is skew at \( x_0 \), then \( x_0 \) is a solution of \((GMVVLIP)\).

Proof. Suppose that \( x_0 \) is not a solution of \((GMVVLIP)\). Therefore, there exists \( x \in \Omega \) such that for all \( \xi^* \in \partial_L f(x) \) we have
\[
\nonumber \langle \xi^*, \eta(x, x_0) \rangle \in -\mathbb{R}_+^n \setminus \{0\}. \tag{3.4}
\]
Hence,
\[ \langle \partial_L f(\bar{x}), \eta(\bar{x}, x_0) \rangle \subseteq -\mathbb{R}^n_+ \setminus \{0\}. \]  \hfill (3.5)

Since \( \eta \) is skew at \( x_0 \) we obtain
\[ \langle \partial_L f(x), \eta(x_0, x) \rangle \subseteq \mathbb{R}^n_+ \setminus \{0\}. \]

Since \( f \) is weakly quasi-invex at \( x_0 \) with respect to \( \eta \) on \( \Omega \) it follows that
\[ f(x_0) - f(\bar{x}) \in \mathbb{R}^n_+ \setminus \{0\}, \]
which contradicts the fact that \( x_0 \) is a solution of \((VOP)\). Therefore, \( x_0 \) is a solution of \((GMVVVLIP)\).

**Remark 3.11.** In [18] (Theorem 13) the authors obtained this result by assuming that \( f_i(i = 1, ..., n) \) are pseudo-invex with respect to \( \eta \) on \( \Omega \). Next, we provide an example which shows that a function \( f = (f_1, ..., f_n) \) it can be weakly quasi-invex with respect to \( \eta \) on \( \Omega \) and there exists \( k, 1 \leq k \leq n \), such that \( f_k \) is not pseudo-invex with respect to \( \eta \) on \( \Omega \).

**Example 3.12.** Let us consider \( X = \mathbb{R}, \Omega = [-1, 1], f = (f_1, f_2) : \Omega \to \mathbb{R}^2 \) defined as
\[
\begin{align*}
f_1(x) &= \begin{cases} \sqrt{x}, & x \geq 0, \\ x, & x < 0, \end{cases} \\
f_2(x) &= \begin{cases} \frac{1}{2} \sqrt{x}, & x \geq 0, \\ -x, & x < 0, \end{cases}
\end{align*}
\]
x_0 = 0 and \( \eta : \Omega \times \Omega \to \mathbb{R} \) defined as
\[ \eta(x, y) = x - y. \]

We obtain that
\[
\partial_L f_1(x) = \begin{cases} \left( \frac{1}{2 \sqrt{x}}, \frac{1}{\sqrt{x}} \right), & x > 0, \\ [0, \infty] \times ([0, \infty] \cup \{-1\}), & x = 0, \\ (1, -1), & x < 0. \end{cases}
\]

It is not difficult to verify that \( f \) is weakly quasi-invex at \( x_0 \) with respect to \( \eta \), \( x_0 = 0 \) is solution of \((VOP)\), \( \eta \) is skew at \( x_0 \) and \( f_1 \) is not pseudo-invex with respect to \( \eta \) on \( \Omega \) because for \( x = -1, y = 0 \) there exists \( \xi^* = 0 \in \partial_L f(y) \) such that \( \langle \xi^*, \eta(x, y) \rangle = 0 \) and \( f(x) < f(y) \).

**Theorem 3.13.** Suppose that \( x_0 \) is a solution of \((SVVLIP)\) and \( f \) is pseudo-invex with respect to \( \eta \) on \( \Omega \). Then, \( x_0 \) is a solution of \((VOP)\).

**Proof.** Suppose that \( x_0 \) is not a solution of \((VOP)\). Therefore, there exists \( \bar{x} \in \Omega \) such that
\[ f(\bar{x}) - f(x_0) \in -\mathbb{R}_+^n \setminus \{0\}. \]

Since \( f \) is pseudo-invex with respect to \( \eta \) on \( \Omega \), it follows that
\[ \langle \partial_L f(x_0), \eta(\bar{x}, x_0) \rangle \subseteq -\mathbb{R}_+^n \setminus \{0\}, \]
which contradicts the fact that \( x_0 \) is a solution of \((SVVLIP)\). Therefore, \( x_0 \) is a solution of \((VOP)\).
Remark 3.14. In [4] (Theorem 3.2) the authors obtained this result by assuming that $f_i(i = 1, ..., n)$ are invex with respect to $\eta$ on $\Omega$. Next, we provide an example which shows that a function $f = (f_1, ..., f_n)$ it can be pseudo-invex with respect to $\eta$ on $\Omega$ and there exists $k, 1 \leq k \leq n$, such that $f_k$ is not invex with respect to $\eta$ on $\Omega$.

Example 3.15. Let us consider $X = \mathbb{R}$, $\Omega = [-1, 1]$, $f = (f_1, f_2) : \Omega \to \mathbb{R}^2$ defined as

$$f_1(x) = \begin{cases} \sqrt{x}, & x \geq 0, \\ -x, & x < 0, \end{cases}$$

$$f_2(x) = x$$

and $\eta : \Omega \times \Omega \to \mathbb{R}$ defined as $\eta(x, y) = x - y$.

We have

$$\partial_L f_1(x) = \begin{cases} \left( \frac{1}{2\sqrt{x}}, 1 \right), & x > 0, \\ \left[0, \infty \right] \cup \{-1\} \times \{1\}, & x = 0, \\ (-1, 1), & x < 0. \end{cases}$$

It is not difficult to see that $x_0 = 0$ is solution of ($SVVLIP$), $f$ is pseudo-invex with respect to $\eta$. Function $f_1$ is not invex with respect to $\eta$ on $\Omega$ because for $x = 1, y = 0$ we obtain

$$f(1) - f(0) < \langle \xi^*, \eta(1, 0) \rangle,$$

for $\xi^* = 2$.

References


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