Higher order iterates of Szasz-Mirakyan-Baskakov operators

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Abstract. In this paper, we discuss the generalization of Szasz-Mirakyan-Baskakov type operators defined in [7], using the iterative combinations in ordinary and simultaneous approximations. We have better estimates in higher order modulus of continuity for these operators in simultaneous approximation.


Keywords: Iterative combinations, Szasz-Mirakyan operators, simultaneous approximation, Steklov mean, modulus of continuity.

1. Introduction

Lebesgue integrable functions $f$ on $[0, \infty)$ are defined by

$$H[0, \infty) = \left\{ f : \int_0^\infty \frac{|f(t)|}{(1+t)^n} dt < \infty, n \in \mathbb{N} \right\}$$

A new sequence of linear positive operators was introduced by Gupta-Srivastava [4] in 1995. They combined Szasz-Mirakyan and Baskakov operators as

$$S_n(f; x) = (n - 1) \sum_{v=0}^\infty q_{n,v}(x) \int_0^\infty p_{n,v}(t)f(t)dt, \quad \forall x \in [0, \infty),$$

where

$$p_{n,v}(t) = \frac{(n + v - 1)!}{v!(n - 1)!} \frac{t^v}{(1 + t)^{n+v}},$$

$$q_{n,v}(x) = \frac{e^{-nx} (nx)^v}{v!}, \quad 0 \leq x < \infty.$$ 

We define the norm $\| \cdot \|$ on $C_\gamma[0, \infty)$ by

$$\|f\|_\gamma = \sup_{0 \leq t < \infty} |f(t)|t^{-\gamma},$$
where $C_{\gamma}[0, \infty) = \{ f \in C[0, \infty) : |f(t)| \leq Mt^{\gamma}, \gamma > 0 \}$. It can be noticed that the order of approximation by these operators (1.1) is at best of $O(n^{-1})$, howsoever smooth the function may be. So in order to improve the rate of convergence, we consider the iterative combinations $R_{n,v} : H[0, \infty) \rightarrow C^\infty[0, \infty)$ of the operators $S_{n}(f, x)$ described as below

$$R_{n,v}(f(t), x) = (I - (I - S_{n})^{v})(f; x) = \sum_{r=1}^{v} (-1)^{r+1} \binom{v}{r} S_{n}^{r}(f(t); x), \quad (1.2)$$

where $S_{n}^{0} = I$ and $S_{n}^{r} = S_{n}(S_{n}^{r-1})$ for $r \in N$.

The purpose of this paper is to obtain the corresponding general results in terms of $(2k+2)^{th}$ order modulus of continuity by using properties of linear approximating method, namely Steklov Mean. In the present paper, we use the notations $I \equiv [a, b], \quad 0 < a < b < \infty$, $I_{i} \equiv [a_{i}, b_{i}], \quad 0 < a_{1} < a_{2} < \ldots < b_{2} < b_{1} < \infty; \quad i = 1, 2, \ldots$

Also $\| \cdot \|_{C(I)}$ is sup-norm on the interval $I$ and having not same value in different cases by constant $C$. Some approximation properties for similar type operators were discussed in [3] and [7]. Very recently D. Sharma et al [8] obtained some results on similar type of operators.

2. Auxiliary results

In this section, we obtain some important lemmas which will be useful for the proof of our main theorem.

**Lemma 2.1.** [6] For $m \in N^{0}$, we define

$$U_{n,m}(x) = \sum_{v=0}^{\infty} p_{n,v}(x) \left( \frac{v}{n} - x \right)^{m},$$

then $U_{n,0} = 1$, $U_{n,1} = 0$. Further, there holds the recurrence formula

$$nU_{n,m+1}(x) = x \left[ U'_{n,m}(x) + mU_{n,m-1}(x) \right], \quad m \geq 1.$$ 

Consequently

1. $U_{n,m}(x)$ is a polynomial in $x$ of degree $\leq m$.
2. $U_{n,m}(x) = O(n^{-[m+1]/2})$, where $[\zeta]$ is integral part of $\zeta$.

**Lemma 2.2.** [4] There exists the polynomials $\phi_{i,j,r}(x)$ independent of $n$ and $v$ such that

$$x^{r}(1 + x)^{r} \frac{d^{r}}{dx^{r}} p_{n,v}(x) = \sum_{2i+j \leq r: i, j \geq 0} n^{i}(v - nx)^{j} \phi_{i,j,r}(x) p_{n,v}(x);$$

$$x^{r} \frac{d^{r}}{dx^{r}} q_{n,v}(x) = \sum_{2i+j \leq r: i, j \geq 0} n^{i}(v - nx)^{j} \phi_{i,j,r}(x) q_{n,v}(x).$$
Lemma 2.3. [3] We assume that $0 < a_1 < a_2 < b_2 < b_1 < \infty$, for sufficiently small $\delta > 0$, then $(2k + 2)^{th}$ ordered Steklov mean $g_{2k+2,\delta}(t)$ which corresponds to $g(t) \in C_\gamma[0,\infty)$, is defined as

$$g_{2k+2,\delta}(t) = \delta^{-(2k+2)} \int_{-\delta/2}^{\delta/2} \int_{-\delta/2}^{\delta/2} \cdots \int_{-\delta/2}^{\delta/2} [g(t) - \Delta_{\eta}^{2k+2} g(t)] \prod_{i=1}^{2k+2} dt_i,$$

where

$$\eta = \frac{1}{2k+2} \sum_{i=1}^{2k+2} t_i, \quad \forall t \in [a,b].$$

It is easily checked in [1], [2] and [5] that

1. $g_{2k+2,\delta}$ has continuous derivatives upto order $(2k + 2)$ on $[a,b]$;
2. $\|g_{2k+2,\delta}\|_{C[a_1,b_1]} \leq K \delta^{-r} \omega_r(g,\delta,a,b), \quad r = 1, 2, \ldots (2k + 2)$;
3. $\|g - g_{2k+2,\delta}\|_{C[a_1,b_1]} \leq K \omega_{2k+2}(g,\delta,a,b)$;
4. $\|g_{2k+2,\delta}\|_{C[a_1,b_1]} \leq K \|g\|_\gamma$.

Here $K$ is a constant not necessarily same at different places.

Lemma 2.4. For the $m^{th}$ order moment $T_{n,m}(x), m \in N^0$ defined by

$$T_{n,m}(x) = (n-1) \sum_{v=0}^{\infty} q_{n,v}(x) \int_{0}^{\infty} p_{n,v}(t-x)^m dt, \quad \forall x \in [0,\infty)$$

we obtain

$$T_{n,0}(x) = 1 \quad (2.1)$$
$$T_{n,1}(x) = \frac{1 + 2x}{n-2}, \quad n > 2 \quad (2.2)$$
$$T_{n,2}(x) = \frac{(n+6)x^2 + 2x(n+3) + 2}{(n-2)(n-3)}, \quad n > 3 \quad (2.3)$$

and the recurrence relation for $n > (m+2)$

$$(n-m-2)T_{n,m+1}(x) = x[T_{n,m}'(x) + m(2+x)T_{n,m-1}(x)] + (m+1)(1+2x)T_{n,m}(x) \quad (2.4)$$

Further, for all $x \in [0,\infty)$, we have $T_{n,m}(x) = O(n^{-[m+1]/2})$.

Proof. Obviously (2.1)-(2.3) can be easily proved by using the definition of $T_{n,m}(x)$. To prove the recurrence relation (2.4), we proceed by taking

$$T_{n,m}'(x) = (n-1) \sum_{v=0}^{\infty} q_{n,v}'(x) \int_{0}^{\infty} p_{n,v}(t-x)^m dt - mT_{n,m-1}(x)$$
Multiplying by \( x \) on both sides and then using identity \( xq'_{n,v}(x) = (v - nx)q_{n,v}(x) \), we have

\[
x[T'_{n,m}(x) + mT_{n,m-1}(x)] = (n - 1) \sum_{v=0}^{\infty} (v - nx)q_{n,v}(x) \int_{0}^{\infty} p_{n,v}(t)(t - x)^{m} dt
\]

\[
= (n - 1) \sum_{v=0}^{\infty} q_{n,v}(x) \int_{0}^{\infty} (v - nx)p_{n,v}(t)(t - x)^{m} dt
\]

\[
= (n - 1) \sum_{v=0}^{\infty} q_{n,v}(x) \int_{0}^{\infty} (v - nt)p_{n,v}(t)(t - x)^{m} dt + (n - 1) \sum_{v=0}^{\infty} q_{n,v}(x) \int_{0}^{\infty} np_{n,v}(t)(t - x)^{m+1} dt
\]

\[
= (n - 1) \sum_{v=0}^{\infty} q_{n,v}(x) \int_{0}^{\infty} (v - nt)p_{n,v}(t)(t - x)^{m} dt + nT_{n,m+1}(x).
\]

Again, using identity \( t(1 + t)p'_{n,v}(t) = (v - nt)p_{n,v}(t) \) in RHS, we get

\[
x[T'_{n,m}(x) + mT_{n,m-1}(x)]
\]

\[
= (n - 1) \sum_{v=0}^{\infty} q_{n,v}(x) \int_{0}^{\infty} t(1 + t)p'_{n,v}(t)(t - x)^{m} dt + nT_{n,m+1}(x)
\]

\[
= (n - 1) \sum_{v=0}^{\infty} q_{n,v}(x) \int_{0}^{\infty} [(1 + 2x)(t - x) + (t - x)^{2} + x(1 + x)]p'_{n,v}(t)
\]

\[
\times (t - x)^{m} dt + nT_{n,m+1}(x)
\]

\[
= -(m + 1)(1 + 2x)T_{n,m}(x) - (m + 2)T_{n,m+1}(x) - mx(1 + x)T_{n,m-1}(x) + nT_{n,m+1}(x)
\]

This leads to our required result (2.4).

Further, for every \( m \in N^{0} \), the \( m^{th} \) order moment \( T_{n,m}^{(p)} \) for the operator \( S_{n}^{p} \) is defined by

\[
T_{n,m}^{(p)}(x) = S_{n}^{p}((t - x)^{m}, x).
\]

If we adopt the convention \( T_{n,m}^{(1)}(x) = T_{n,m}(x) \), obviously \( T_{n,m}^{(p)}(x) \) is of degree \( m \).

**Theorem 2.5.** Let \( f \in C_{\gamma}[0, \infty) \), if \( f^{(2v+p+2)} \) exists at a point \( x \in [0, \infty) \), then

\[
\lim_{n \to \infty} n^{v+1}[R_{n,v}^{(p)}(f; x) - f^{(p)}(x)] = \sum_{k=p}^{2v+p+2} Q(k, v, p, x)f^{(k)}(x), \tag{2.5}
\]

where \( Q(k, v, p, x) \) are certain polynomials in \( x \). Further if \( f^{(2v+p+2)} \) is continuous on \( (a - \eta, b + \eta) \subset [0, \infty) \) and \( \eta > 0 \), then this theorem holds uniformly in \([a, b]\).
3. Main results

In this section, we establish the direct theorem.

**Theorem 3.1.** Let \( f \in H[0, \infty) \) be bounded on every finite subinterval of \([0, \infty)\) and \( f(t) = O(t^\alpha) \) as \( t \to \infty \) for some \( \alpha > 0 \). If \( f^{(p)} \) exists and is continuous on \((a - \eta, b + \eta) \subset [0, \infty), \) for some \( \eta > 0 \). Then
\[
\| R_{n,v}^{(p)}(f(t); x) - f^{(p)}(x) \|_{C(I_2)} \leq K \{ n^{-\alpha} \| f \|_{C(I_1)} + \omega_{2v+2}(f^{(p)}, n^{-1/2}, I_1) \}
\]
where constant \( K \) is independent of \( f \) and \( n \).

**Proof.** We can write
\[
\| R_{n,v}^{(p)}(f(t); x) - f^{(p)}(x) \|_{C(I_2)} \leq \| R_{n,v}^{(p)}(f - f_{\eta,2v+2}; x) \|_{C(I_2)} + \| R_{n,v}^{(p)}(f_{\eta,2v+2}; x) - f_{\eta,2v+2}^{(p)}(x) \|_{C(I_2)}
\]
\[
+ \| f^{(p)}(x) - f_{\eta,2v+2}^{(p)}(x) \|_{C(I_2)} =: P_1 + P_2 + P_3.
\]
By the property of Steklov Mean and \( f_{\eta,2v+2}^{(p)}(x) = (f^{(p)})_{\eta,2v+2}(x) \), we get
\[
P_3 \leq K \omega_{2v+2}(f^{(p)}, \eta, I_1).
\]
To estimate \( P_2 \), applying Theorem 2.5 and interpolation property from [2], we have
\[
P_2 \leq Kn^{-(v+1)} \sum_{i=p}^{2v+p+2} \| f_{\eta,2v+2}^{(i)}(x) \|_{C(I_2)}
\]
\[
\leq Kn^{-(v+1)} \left( \| f_{\eta,2v+2} \|_{C(I_2)} + \| f_{\eta,2v+2}^{(p)} \|_{C(I_2)} \right).
\]
Hence by using properties (2) and (4) of Steklov Mean, we get
\[
P_2 \leq K n^{-(v+1)} [ \| f \|_{C(I_1)} + (\eta)^{-2v-2} \omega_{2v+2}(f^{(p)}, \eta, I_1) ].
\]
Suppose \( a^* \) and \( b^* \) be such that
\[
0 < a_1 < a^* < a_2 < b_2 < b^* < b_1 < \infty.
\]
In order to estimate \( P_1 \), let \( F = f - f_{\eta,2v+2} \). Then, by hypothesis, we have
\[
F(t) = \sum_{i=0}^{p} \frac{F^{(i)}(x)}{i!} (t-x)^i + \frac{F^{(p)}(\xi) - F^{(p)}(x)}{p!} (t-x)^p \psi(t) + h(t, x)(1 - \psi(t)), \quad (3.1)
\]
where \( \xi \) lies between \( t \) and \( x \), and \( \psi \) is the characteristic function of the interval \([a^*, b^*]\). For \( t \in [a^*, b^*] \) and \( x \in [a_2, b_2] \), we get
\[
F(t) = \sum_{i=0}^{p} \frac{F^{(i)}(x)}{i!} (t-x)^i + \frac{F^{(p)}(\xi) - F^{(p)}(x)}{p!} (t-x)^p,
\]
and for \( t \in [0, \infty) \setminus [a^*, b^*], \ x \in [a_2, b_2], \) we define 
\[
h(t, x) = F(t) - \sum_{i=0}^p \frac{F^{(i)}(x)}{i!} (t - x)^i.
\]

Now operating \( R_{n,v}^p \) on both the sides of (3.1), we have the three terms on right side namely \( E_1, E_2 \) and \( E_3 \) respectively. By using (1.2) and Lemma 2.4, we get 
\[
E_1 = \sum_{i=0}^p \frac{F^{(i)}(x)}{i!} \sum_{r=1}^v (-1)^{r+1} \binom{v}{r} D^p \left( S_n^r((t - x)^i; x) \right), \quad D \equiv \frac{d}{dx}
\]
\[
\rightarrow F^{(p)}(x),
\]
when \( n \to \infty \) uniformly in \( I_2 \). Therefore
\[
\|E_1\|_{C(I_2)} \leq K \left\| f^{(p)} - f^{(p)}_{n,2v+2} \right\|_{C(I_2)}.
\]
To obtain \( E_2 \), we have
\[
\|E_2\|_{C(I_2)} \leq \frac{2}{p!} \left\| f^{(p)} - f^{(p)}_{n,2v+2} \right\|_{C[a^*, b^*]} \sum_{r=1}^v (n - 1) \binom{v}{r} \sum_{v=0}^\infty |q_n^{(p)}(x)|
\]
\[
\times \int_0^\infty p_{n,v}(t) S_n^{r-1}(|t - x|^p, x) \ dt.
\]
Using Lemma 2.2, Cauchy Schwartz Inequality and Lemma 2.1
\[
(n - 1) \sum_{v=0}^\infty |q_n^{(p)}(x)| \int_0^\infty p_{n,v}(t) S_n^{r-1}(|t - x|^p, x) dt
\]
\[
\leq K \sum_{2r+j \leq p, \ r, j \geq 0} n^r \phi_{r,j,p}(x) x^{-p} (n - 1) \sum_{v=0}^\infty q_{n,v}(x)(v - nx)^j
\]
\[
\times \int_0^\infty p_{n,v}(t) S_n^{r-1}(|t - x|^p, x) dt
\]
\[
\leq K \sum_{2r+j \leq p, \ r, j \geq 0} n^r \left( \sum_{v=0}^\infty q_{n,v}(x)(v - nx)^2j \right)^{1/2}
\]
\[
\times \left( (n - 1) \sum_{v=0}^\infty q_{n,v}(x) \int_0^\infty p_{n,v}(t) S_n^{r-1}(|t - x|^{2p}, x) dt \right)^{1/2}
\]
\[
= K \sum_{2r+j \leq p, \ r, j \geq 0} n^r O(n^{j/2})O(n^{-p/2})
\]
\[
= O(1)
\]
as \( n \to \infty \) uniformly in \( I_2 \). Therefore,
\[
\|E_2\| \leq K \|f^{(p)} - f^{(p)}_{\eta,2v+2}\|_{C(a^*,b^*)}.
\]

Since \( t \in [0, \infty) \setminus [a^*,b^*] \) and \( x \in [a_2,b_2] \), we can choose a \( \delta > 0 \) in such a way that \( |t-x| \geq \delta \). If \( \beta \geq \max\{\alpha,p\} \) be an integer, we can find a positive constant \( Q \) such that
\[
|h(t,x)| \leq Q|t-x|^\beta \text{ whenever } |t-x| \geq \delta.
\]
Again applying Lemma 2.2, Cauchy Schwartz Inequality three times, Lemma 2.1 and Lemma 2.4, we get
\[
|E_3| \leq K \sum_{r=0}^{v} \binom{v}{r} \sum_{2r+j \leq p; \ r,j \geq 0} n^r \left( \sum_{v=0}^{\infty} q_{n,v}(x)(v-nx)^{2j} \right)^{1/2} \\
\times \left( (n-1) \sum_{v=0}^{\infty} q_{n,v}(x) \int_{0}^{\infty} p_{n,v}(t) S_{n}^{-1} \left( (1-\psi(t))(t-x)^{2\beta};t \right)dt \right)^{1/2} \\
\leq K \sum_{r=0}^{v} \binom{v}{r} \sum_{2r+j \leq p; \ r,j \geq 0} n^r \left( \sum_{v=0}^{\infty} q_{n,v}(x)(v-nx)^{2j} \right)^{1/2} \\
\times \left( (n-1) \sum_{v=0}^{\infty} q_{n,v}(x) \int_{0}^{\infty} p_{n,v}(t) S_{n}^{-1} \left( \frac{(t-x)^{2m}}{\delta^{2m-2\beta}};t \right)dt \right)^{1/2} \\
\leq K \sum_{2r+j \leq p; \ r,j \geq 0} n^r O(n^{j/2})O(n^{-m/2}), \quad m > \beta, \forall m \in I.
\]
Hence \( \|E_3\| = O(1) \), as \( n \to \infty \), uniformly in \( I_2 \). Combining the estimates of \( E_1, E_2 \) and \( E_3 \), we get
\[
P_1 \leq K \|f^{(p)} - f^{(p)}_{\eta,2v+2}\|_{C(a^*,b^*)} \leq K \omega_{2v+2}(f^{(p)}, \eta, I_1).
\]
Substituting \( \eta = n^{-1/2} \), we get the required theorem.

References


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