Analysis of a viscoelastic unilateral and frictional contact problem with adhesion

Arezki Touzaline

Abstract. We consider a mathematical model which describes the quasistatic frictional contact between a viscoelastic body with long memory and a foundation. The contact is modelled with a normal compliance condition in such a way that the penetration is limited and restricted to unilateral constraint and associated to the nonlocal friction law with adhesion, where the coefficient of friction is solution-independent. The bonding field is described by a first order differential equation. We derive a variational formulation written as the coupling between a variational inequality and a differential equation. The existence and uniqueness result of the weak solution under a smallness assumption on the coefficient of friction is established. The proof is based on arguments of time-dependent variational inequalities, differential equations and Banach fixed point theorem.

Mathematics Subject Classification (2010): 54AXX.

Keywords: Viscoelastic, normal compliance, adhesion, friction, variational inequality, weak solution.

1. Introduction

Contact problems involving deformable bodies are quite frequent in the industry as well as in daily life and play an important role in structural and mechanical systems. Contact processes involve complicated surface phenomena, and are modelled by highly nonlinear initial boundary value problems. Taking into account various contact conditions associated with more and more complex behavior laws lead to the introduction of new and non standard models, expressed by the aid of evolution variational inequalities. An early attempt to study frictional contact problems within the framework of variational inequalities was made in [10]. The mathematical, mechanical and numerical state of the art can be found in [23]. In this reference we find a detailed analysis and numerical studies of the adhesive contact problems. Recently a new book [25] introduces to the reader the theory of variational inequalities with emphasis on the study of contact mechanics and, more specifically, on antiplane frictional contact problems. Also, recently existence results were established in [1, 5, 6, 8, 11, 20, 26, 29, 30, 31] in the
study of unilateral and frictional contact problems with or without adhesion. In [31]
a quasistatic viscoelastic unilateral contact problem with adhesion and friction was
studied and an existence and uniqueness result was proved for a coefficient of friction
sufficiently small. Also in [7] a dynamic contact problem with nonlocal friction and ad-
hesion between two viscoelastic bodies of Kelvin-Voigt type was studied. An existence
result was proved without condition on the coefficient of friction. Here as in [16] we
study a mathematical model which describes a frictional and adhesive contact prob-
lem between a viscoelastic body with long memory and a foundation. The contact is
modelled with a normal compliance condition associated to unilateral constraint and
the nonlocal friction law with adhesion. Recall that models for dynamic or quasistatic
processes of frictionless adhesive contact between a deformable body and a foundation
have been studied in [2, 3, 4, 5, 7, 8, 12, 19, 21, 23, 24, 27, 28]. Following [13, 14] we use
the bonding field as an additional state variable \( \beta \), defined on the contact surface of
the boundary. The variable satisfies the restrictions \( 0 \leq \beta \leq 1 \). At a point on the
boundary contact surface, when \( \beta = 1 \) the adhesion is complete and all the bonds
are active; when \( \beta = 0 \) all the bonds are inactive, severed, and there is no adhesion;
when \( 0 < \beta < 1 \) the adhesion is partial and only a fraction \( \beta \) of the bonds is active.
However, according to [17], the method presented here considers a compliance model
in which the compliance term does not represent necessarily a compact perturbation
of the original problem without contact. This leads us to study such models, where
a strictly limited penetration is permitted with the limit procedure to the Signorini
contact problem. In this work as in [31] we derive a variational formulation of the
mechanical problem written as the coupling between a variational inequality and a
differential equation. We prove the existence of a unique weak solution if the coef-
cient of friction is sufficiently small, and obtain a partial regularity result for the
solution.

The paper is structured as follows. In section 2 we present some notations and pre-
liminaries. In section 3 we state the mechanical problem and give a variational for-
mulation. In section 4 we establish the proof of our main existence and uniqueness
result, Theorem 4.1.

2. Notations and preliminaries

Everywhere in this paper we denote by \( S^d \) the space of second order symmetric tensors
on \( \mathbb{R}^d \) \((d = 2, 3)\) while \(|.|\) represents the Euclidean norm on \( \mathbb{R}^d \) and \( S^d \). Thus, for every
\( u, v \in \mathbb{R}^d \), \( u.v = u_i v_i \), \(|v| = (v.v)^{\frac{1}{2}}\), and for every \( \sigma, \tau \in S^d \), \( \sigma.\tau = \sigma_{ij} \tau_{ij}\), \(|\tau| = (\tau.\tau)^{\frac{1}{2}}\).
Here and below, the indices \( i \) and \( j \) run between 1 and \( d \) and the summation convention
over repeated indices is adopted. Let \( \Omega \subset \mathbb{R}^d \) be a bounded domain with a Lipschitz
boundary \( \Gamma \) and let \( \nu \) denote the unit outer normal on \( \Gamma \). We shall use the notation:

\[
H = \left( L^2(\Omega) \right)^d, \quad H_1 = \left( H^1(\Omega) \right)^d, \quad Q = \left\{ \sigma = (\sigma_{ij}) : \sigma_{ij} = \sigma_{ji} \in L^2(\Omega) \right\},
\]

\[
Q_1 = \{ \sigma \in Q : \text{div} \sigma \in H \}.
\]
Note that $H$ and $Q$ are real Hilbert spaces endowed with the respective canonical inner products
\[
(u, v)_H = \int_\Omega u_i v_i dx, \quad \langle \sigma, \tau \rangle_Q = \int_\Omega \sigma_{ij} \tau_{ij} dx.
\]
The strain tensor is
\[
\varepsilon (u) = (\varepsilon_{ij} (u)) = \frac{1}{2} (u_{i,j} + u_{j,i});
\]
\[
div \sigma = (\sigma_{ij,j}) \text{ is the divergence of } \sigma.
\]
For every $v \in H_1$ we also use the notation $v$ for the trace of $v$ on $\Gamma$ and we denote by $v_\nu$ and $v_\tau$ the normal and tangential components of $v$ on the boundary $\Gamma$, given by
\[
v_\nu = v \nu, \quad v_\tau = v - v_\nu \nu.
\]
We define, similarly, by $\sigma_\nu$ and $\sigma_\tau$ the normal and the tangential traces of a function $\sigma \in Q_1$, and when $\sigma$ is a regular function then
\[
\sigma_\nu = (\sigma \nu) \nu, \quad \sigma_\tau = \sigma \nu - \sigma_\nu \nu,
\]
and the following Green’s formula holds:
\[
\langle \sigma, \varepsilon (v) \rangle_Q + (\text{div} \sigma, v)_H = \int_\Gamma \sigma_\nu v da \quad \forall v \in H_1,
\]
where $da$ is the surface measure element. Let $T > 0$. For every real Hilbert space $X$ we employ the usual notation for the spaces $L^p (0, T; X)$, $1 \leq p \leq \infty$, and $W^{1, \infty} (0, T; X)$. Recall that the norm on the space $W^{1, \infty} (0, T; X)$ is given by
\[
\| u \|_{W^{1, \infty} (0, T; X)} = \| u \|_{L^\infty (0, T; X)} + \| \dot{u} \|_{L^\infty (0, T; X)},
\]
where $\dot{u}$ denotes the first derivative of $u$ with respect to time. Finally, we denote by $C ([0, T]; X)$ the space of continuous functions from $[0, T]$ to $X$, with the norm
\[
\| x \|_{C ([0, T]; X)} = \max_{t \in [0, T]} \| x (t) \|_X.
\]
Moreover, for a real number $r$, we use $r_+$ to represent its positive part, that is $r_+ = \max \{ r, 0 \}$.

3. Problem statement and variational formulation

We consider the following physical setting. A viscoelastic body with long memory occupies a bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) with a regular boundary $\Gamma$ that is partitioned into three disjoint measurable parts $\Gamma_1, \Gamma_2, \Gamma_3$ such that $\text{meas} (\Gamma_1) > 0$. The body is acted upon by a volume force of density $\varphi_1$ on $\Omega$ and a surface traction of density $\varphi_2$ on $\Gamma_2$. The body is in unilateral contact with adhesion following the nonlocal friction law with a foundation, over the potential contact surface $\Gamma_3$.

Thus, the classical formulation of the mechanical problem is written as follows. **Problem $P_1$.** Find a displacement $u : \Omega \times [0, T] \to \mathbb{R}^d$, a stress field $\sigma : \Omega \times [0, T] \to \mathbb{S}^d$ and a bonding field $\beta : \Gamma_3 \times [0, T] \to [0, 1]$ such that for all $t \in [0, T]$, 
\[
\sigma (t) = F \varepsilon (u (t)) + \int_0^t \mathcal{F} (t - s) \varepsilon (u (s)) ds \quad \text{in } \Omega,
\]
\[ \text{div} \sigma (t) + \varphi_1 (t) = 0 \text{ in } \Omega, \]  
\[ u (t) = 0 \quad \text{on } \Gamma_1, \]  
\[ \sigma (t) \chi = \varphi_2 (t) \quad \text{on } \Gamma_2, \]  
\[ u_\nu (t) \leq g, \ \sigma_\nu (t) + p (u_\nu (t)) - c_\nu \beta^2 (t) R_\nu (u_\nu (t)) \leq 0 \]  
\[ \left\{ \begin{array}{l}
(\sigma_\nu (t) + p (u_\nu (t)) - c_\nu \beta^2 (t) R_\nu (u_\nu (t)))(u_\nu (t) - g) = 0 \\
|\sigma_\tau (t) + c_\tau \beta^2 (t) R_\tau (u_\tau (t))| \leq \mu |R\sigma_\nu (u (t))|
\end{array} \right\} \text{ on } \Gamma_3, \]  
\[ \left\{ \begin{array}{l}
|\sigma_\tau (t) + c_\tau \beta^2 (t) R_\tau (u_\tau (t))| < \mu |R\sigma_\nu (u (t))| \Rightarrow u_\tau (t) = 0 \\
|\sigma_\tau (t) + c_\tau \beta^2 (t) R_\tau (u_\tau (t))| = \mu |R\sigma_\nu (u (t))| \Rightarrow \\
\exists \lambda \geq 0 \ s.t. \ u_\tau (t) = -\lambda (\sigma_\tau (t) + c_\tau \beta^2 (t) R_\tau (u_\tau (t))) \\
\dot{\beta} (t) = -\left[ \beta (t) (c_\nu (R_\nu (u_\nu (t)))^2 + c_\tau |R_\tau (u_\tau (t))|^2) - \varepsilon_a \right]_+ \text{ on } \Gamma_3,
\end{array} \right\}
\]  
\[ \beta (0) = \beta_0 \quad \text{on } \Gamma_3. \]  

Equation (3.1) represents the viscoelastic constitutive law with long memory of the material; \( F \) is the elasticity operator and \( \int_0^t \mathcal{F} (t - s) \varepsilon (u (s)) \, ds \) is the memory term in which \( \mathcal{F} \) denotes the tensor of relaxation; the stress \( \sigma (t) \) at current instant \( t \) depends on the whole history of strains up to this moment of time. Equation (3.2) represents the equilibrium equation while (3.3) and (3.4) are the displacement and traction boundary conditions, respectively, in which \( \sigma \chi \) represents the Cauchy stress vector. The conditions (3.5) represent the unilateral contact with adhesion in which \( c_\nu \) is a given adhesion coefficient which may dependent on \( x \in \Gamma_3 \) and \( R_\nu, R_\tau \) are truncation operators defined by
\[ R_\nu (s) = \begin{cases} 
L & \text{if } s < -L \\
-\nu \text{ if } -L \leq s \leq 0 \\
0 & \text{if } s > 0
\end{cases}, \quad R_\tau (v) = \begin{cases} 
v & \text{if } |v| \leq L, \\
L & \text{if } |v| > L
\end{cases}. \]

Here \( L > 0 \) is the characteristic length of the bond, beyond which the latter has no additional traction (see [23]) and \( p \) is a normal compliance function which satisfies the assumption (3.19); \( g \) denotes the maximum value of the penetration which satisfies \( g \geq 0 \). When \( u_\nu < 0 \) i.e. when there is separation between the body and the foundation then the condition (3.5) combined with hypothesis (3.19) and definition of \( R_\nu \) shows that \( \sigma_\nu = c_\nu \beta^2 R_\nu (u_\nu) \) and does not exceed the value \( L \|c_\nu\|_{L^\infty (\Gamma_3)} \). When \( g > 0 \), the body may interpenetrate into the foundation, but the penetration is limited that is \( u_\nu \leq g \). In this case of penetration (i.e. \( u_\nu \geq 0 \)), when \( 0 \leq u_\nu < g \) then \( -\sigma_\nu = p (u_\nu) \) which means that the reaction of the foundation is uniquely determined by the normal displacement and \( \sigma_\nu \leq 0 \). Since \( p \) is an increasing function then the reaction is increasing with the penetration. When \( u_\nu = g \) then \( -\sigma_\nu \geq p (g) \) and \( \sigma_\nu \) is not uniquely determined. When \( g > 0 \) and \( p = 0 \), conditions (3.5) become the Signorini’s contact conditions with a gap and adhesion
\[ u_\nu \leq g, \ \sigma_\nu - c_\nu \beta^2 R_\nu (u_\nu) \leq 0, \ (\sigma_\nu - c_\nu \beta^2 R_\nu (u_\nu))(u_\nu - g) = 0. \]
When \( g = 0 \), the conditions (3.5) combined with hypothesis (3.19) lead to the Signorini contact conditions with adhesion, with zero gap, given by

\[
u_{\nu} \leq 0, \ \sigma_{\nu} - c_{\nu} \beta^2 R_{\nu} (u_{\nu}) \leq 0, \ \left( \sigma_{\nu} - c_{\nu} \beta^2 R_{\nu} (u_{\nu}) \right) u_{\nu} = 0.
\]

These contact conditions were used in [26, 29]. It follows from (3.5) that there is no penetration between the body and the foundation, since \( u_{\nu} \leq 0 \) during the process. Also, note that when the bonding field vanishes, then the contact conditions (3.5) become the classical Signorini contact conditions with zero gap, that is,

\[
u_{\nu} \leq 0, \ \sigma_{\nu} \leq 0, \ \sigma_{\nu} u_{\nu} = 0.
\]

Conditions (3.6) represent Coulomb’s law of dry friction with adhesion where \( \mu \) denotes the coefficient of friction. Equation (3.7) represents the ordinary differential equation which describes the evolution of the bonding field and it was already used in [26]. Since \( \dot{\beta} \leq 0 \) on \( \Gamma_3 \times (0, T) \), once debonding occurs bonding cannot be reestablished and, indeed, the adhesive process is irreversible. Also from [18] it must be pointed out clearly that condition (3.7) does not allow for complete debonding in finite time.

We turn now to the variational formulation of Problem \( P_1 \). We denote by \( V \) the closed subspace of \( H_1 \) defined by

\[
V = \{ v \in H_1 : v = 0 \text{ on } \Gamma_1 \},
\]

and let the convex subset of admissible displacements given by

\[
K = \{ v \in V : v_{\nu} \leq g \text{ a.e. on } \Gamma_3 \}.
\]

Since \( \text{meas} (\Gamma_1) > 0 \), the following Korn’s inequality holds [10],

\[
\| \epsilon (v) \|_Q \geq c_\Omega \| v \|_{H_1} \quad \forall v \in V,
\]

where \( c_\Omega > 0 \) is a constant which depends only on \( \Omega \) and \( \Gamma_1 \). We equip \( V \) with the inner product

\[
(u, v)_V = \langle \epsilon (u), \epsilon (v) \rangle_Q
\]

and \( \| . \|_V \) is the associated norm. It follows from Korn’s inequality (3.9) that the norms \( \| . \|_{H_1} \) and \( \| . \|_V \) are equivalent on \( V \). Then \( (V, \| . \|_V) \) is a real Hilbert space. Moreover by Sobolev’s trace theorem, there exists \( d_\Omega > 0 \) which only depends on the domain \( \Omega, \Gamma_1 \) and \( \Gamma_3 \) such that

\[
\| v \|_{L^2 (\Gamma_3)}^d \leq d_\Omega \| v \|_V \quad \forall v \in V.
\]

We suppose that the body forces and surface tractions have the regularity

\[
\varphi_1 \in C ([0, T] ; H), \quad \varphi_2 \in C \left( [0, T] ; \left( L^2 (\Gamma_2) \right)^d \right).
\]

We define the function \( f : [0, T] \rightarrow V \) by

\[
(f(t), v)_V = \int_\Omega \varphi_1 (t) \cdot vdx + \int_{\Gamma_2} \varphi_2 (t) \cdot vda \quad \forall v \in V, \ t \in [0, T],
\]

and we note that (3.11) and (3.12) imply

\[
f \in C ([0,T] ; V).
\]
In the study of the mechanical problem $P_1$ we assume that the elasticity operator $F$ satisfies

\begin{enumerate}[(a)]  
  
  \item $F : \Omega \times S^d \rightarrow S^d$;  
  
  \item there exists $M > 0$ such that  
  
  \[ |F(x, \varepsilon_1) - F(x, \varepsilon_2)| \leq M |\varepsilon_1 - \varepsilon_2| \quad \forall \varepsilon_1, \varepsilon_2 \in S^d, \text{ a.e. } x \in \Omega; \]

  \item there exists $m > 0$ such that  
  
  \[ (F(x, \varepsilon_1) - F(x, \varepsilon_2)) \cdot (\varepsilon_1 - \varepsilon_2) \geq m |\varepsilon_1 - \varepsilon_2|^2, \quad \forall \varepsilon_1, \varepsilon_2 \in S^d, \text{ a.e. } x \in \Omega; \]

  \item the mapping $x \rightarrow F(x, \varepsilon)$ is Lebesgue measurable on $\Omega$ for any $\varepsilon \in S^d$;  

  \item the mapping $x \rightarrow F(x, 0) \in Q$.  
\end{enumerate}

We also need to introduce the space of the tensors of fourth order defined by

\[ Q_\infty = \{ \mathcal{E} = (\mathcal{E}_{ijkl}) : \mathcal{E}_{ijkl} = \mathcal{E}_{jikl} = \mathcal{E}_{klij} \in L^\infty(\Omega), 1 \leq i, j, k, l \leq d \}, \]

which is the real Banach space with the norm

\[ \|\mathcal{E}\|_{Q_\infty} = \max_{1 \leq i, j, k, l \leq d} \|\mathcal{E}_{ijkl}\|_{L^\infty(\Omega)}. \]

We assume that the tensor of relaxation $\mathcal{F}$ satisfies

\[ \mathcal{F} \in C([0, T]; Q_\infty). \tag{3.14} \]

The adhesion coefficients $c_\nu$, $c_\tau$ and $\varepsilon_a$ satisfy

\[ c_\nu, c_\tau \in L^\infty(\Gamma_3), \varepsilon_a \in L^2(\Gamma_3) \text{ and } c_\nu, c_\tau, \varepsilon_a \geq 0 \text{ a.e. on } \Gamma_3, \tag{3.15} \]

and we assume that the initial bonding field satisfies

\[ \beta_0 \in L^2(\Gamma_3); 0 \leq \beta_0 \leq 1 \text{ a.e. on } \Gamma_3. \tag{3.16} \]

Next, we consider the subset $W$ of $H_1$ defined as

\[ W = \{ v \in H_1 : \text{div} \sigma(v) \in H \} \]

and let $j_c : V \times V \rightarrow \mathbb{R}$, $j_f : (V \cap W) \times V \rightarrow \mathbb{R}$ be the functionals given by

\[ j_c(u, v) = \int_{\Gamma_3} p(u_\nu) v_\nu da \quad \forall (u, v) \in V \times V, \]

\[ j_f(u, v) = \int_{\Gamma_3} \mu |R\sigma_\nu(u)| |v_\tau| da \quad \forall (u, v) \in (V \cap W) \times V, \]

where

\[ R : H^{-\frac{1}{2}}(\Gamma) \rightarrow L^2(\Gamma_3) \text{ is a linear and continuous mapping (see [9]).} \tag{3.17} \]

The coefficient of friction $\mu$ is assumed to satisfy

\[ \mu \in L^\infty(\Gamma_3) \text{ and } \mu \geq 0 \text{ a.e. on } \Gamma_3. \tag{3.18} \]
Next we let
\[ j = j_c + j_f. \]

We also define the functional
\[ h : L^2(\Gamma_3) \times V \times V \to \mathbb{R} \]
by
\[ h(\beta, u, v) = \int_{\Gamma_3} (-c\beta^2 R_\nu (u_\nu) v_\nu + c\tau \beta^2 \tau R_\tau (u_\tau) . v_\tau) da, \quad \forall (\beta, u, v) \in L^2(\Gamma_3) \times V \times V, \]
where the normal compliance function \(p\) satisfies:

\begin{align}
(\alpha) & \quad p : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+; \\
(\beta) & \quad \text{there exists } L_p > 0 \text{ such that} \\
& \quad |p(x, r_1) - p(x, r_2)| \leq L_p |r_1 - r_2| \quad \forall r_1, r_2 \in \mathbb{R}, \text{ a.e. } x \in \Gamma_3; \\
(\gamma) & \quad (p(x, r_1) - p(x, r_2)) (r_1 - r_2) \geq 0 \quad \forall r_1, r_2 \in \mathbb{R}, \text{ a.e. } x \in \Gamma_3; \\
(\delta) & \quad \text{the mapping } x \to p(x, r) \text{ is Lebesgue measurable on } \Gamma_3, \\
& \quad \text{for any } r \in \mathbb{R}; \\
(\epsilon) & \quad p(x, r) = 0 \quad \forall r \leq 0, \text{ a.e. } x \in \Gamma_3.
\end{align}

Finally, we need to introduce the following set of the bonding field:

\[ B = \{ \theta : [0, T] \to L^2(\Gamma_3) : 0 \leq \theta(t) \leq 1, \forall t \in [0, T], \text{ a.e. on } \Gamma_3 \}. \]

By a standard procedure based on Green’s formula we derive the following variational formulation of Problem P1, in terms of displacement and bonding field.

**Problem P2.** Find a displacement field \(u \in C([0, T]; V)\) and a bonding field \(\beta \in W^{1,\infty}(0, T; L^2(\Gamma_3)) \cap B\) such that

\[ u(t) \in K \cap W, \quad \langle F \varepsilon(u(t)), \varepsilon(v) - \varepsilon(u(t)) \rangle_Q \]

\[ + \left( \int_0^t F(t-s) \varepsilon(u(s)) ds, \varepsilon(v) - \varepsilon(u(t)) \right) \leq f(t), \quad \forall v \in K, \quad t \in (0, T), \]

\[ \dot{\beta}(t) = - \left[ \beta(t) (c\nu (R_\nu (u_\nu (t)))^2 + c\tau |R_\tau (u_\tau (t))|^2) - \varepsilon_a \right] \quad \text{a.e. } t \in (0, T), \quad \beta(0) = \beta_0. \]
4. Existence and uniqueness of solution

Our main result in this section is the following theorem.

**Theorem 4.1.** Let (3.11), (3.13), (3.14), (3.15), (3.16), (3.17), (3.18) and (3.19) hold. Then, there exists a constant \( \mu_0 > 0 \) such that Problem \( P_2 \) has a unique solution if

\[
\| \mu \|_{L^\infty(\Gamma_3)} < \mu_0.
\]

The proof of Theorem 4.1 is carried out in several steps. In the first step, we consider the closed subset \( Z \) of the space \( C \left( [0, T]; L^2(\Gamma_3) \right) \) defined as

\[
Z = \{ \theta \in C \left( [0, T]; L^2(\Gamma_3) \right) \cap B : \theta (0) = \beta_0 \},
\]

where the Banach space \( C \left( [0, T]; L^2(\Gamma_3) \right) \) is endowed with the norm

\[
\| \beta \|_k = \max_{t \in [0, T]} \left[ \exp \left( -kt \right) \| \beta (t) \|_{L^2(\Gamma_3)} \right], \quad k > 0.
\]

Next for a given \( \beta \in Z \), we consider the following variational problem. **Problem \( P_{1, \beta} \).** Find \( u_{\beta} \in C \left( [0, T]; V \right) \) such that

\[
u_{\beta} (t) \in K \cap W, \quad \langle F \xi (u_{\beta} (t)), \varepsilon (v) - \varepsilon (u_{\beta} (t)) \rangle_Q + \left\langle J_1^t F (t-s) \varepsilon (u_{\beta} (s)) ds, \varepsilon (v) - \varepsilon (u_{\beta} (t)) \right\rangle_Q + h (\beta (t), u_{\beta} (t), v - u_{\beta} (t)) + j (u_{\beta} (t), v) - j (u_{\beta} (t), u_{\beta} (t)) \geq (f(t), v - u_{\beta} (t))_V \quad \forall v \in K, \ t \in [0, T].
\]

We have the following result. **Theorem 4.2.** There exists a constant \( \mu_1 > 0 \) such that Problem \( P_{1, \beta} \) has a unique solution if

\[
\| \mu \|_{L^\infty(\Gamma_3)} < \mu_1.
\]

To prove this theorem, for \( \eta \in C \left( [0, T]; Q \right) \) we consider the following intermediate problem. **Problem \( P_{1, \beta, \eta} \).** Find \( u_{\beta, \eta} \in C \left( [0, T]; V \right) \) such that

\[
u_{\beta, \eta} (t) \in K \cap W, \quad \langle F \xi (u_{\beta, \eta} (t)), \varepsilon (v - u_{\beta, \eta} (t)) \rangle_Q + \langle \eta (t), \varepsilon (v - u_{\beta, \eta} (t)) \rangle_Q + h (\beta (t), u_{\beta, \eta} (t), v - u_{\beta, \eta} (t)) + j (u_{\beta, \eta} (t), v) - j (u_{\beta, \eta} (t), u_{\beta, \eta} (t)) \geq (f(t), v - u_{\beta, \eta} (t))_V \quad \forall v \in K, \ t \in [0, T].
\]

Since Riesz’s representation theorem implies that there exists an element \( f_\eta \in C \left( [0, T]; V \right) \) such that

\[
(f_\eta (t), v)_V = (f(t), v)_V - \langle \eta (t), \varepsilon (v) \rangle_Q,
\]

then Problem \( P_{1, \beta, \eta} \) is equivalent to the following problem.
Viscoelastic unilateral and frictional contact problem

**Problem** $P_{2\beta\eta}$. Find $u_{\beta\eta} \in C([0,T] ; V)$ such that

$$u_{\beta\eta} (t) \in K \cap W, \; \langle F \varepsilon (u_{\beta\eta} (t)), \varepsilon (v - u_{\beta\eta} (t)) \rangle_Q + h (\beta (t), u_{\beta\eta} (t), v - u_{\beta\eta} (t))$$

$$+ j (u_{\beta\eta} (t), v) - j (u_{\beta\eta} (t) , u_{\beta\eta} (t)) \geq (f_q (t), v - u_{\beta\eta} (t))_V \; \forall v \in K, \; t \in [0,T]. \quad (4.3)$$

We now show the proposition below.

**Proposition 4.3.** There exists a constant $\mu_1 > 0$ such that Problem $P_{2\beta\eta}$ has a unique solution if

$$\|\mu\|_{L^\infty (\Gamma)} < \mu_1.$$ 

We shall prove Proposition 4.3 in several steps by using arguments on Banach fixed point theorem. Indeed, let $q \in C_+$ where $C_+$ is a non-empty closed subset of $L^2 (\Gamma_3)$ defined as

$$C_+ = \{ s \in L^2 (\Gamma_3) ; \; s \geq 0 \; a.e. \; on \; \Gamma_3 \}$$

and let the functional $j_q : V \rightarrow \mathbb{R}$ given by

$$j_q (v) = \int_{\Gamma_3} \mu q |v| da \; \forall v \in V.$$ 

We consider the following auxiliary problem.

**Problem** $P_{2\beta\eta q}$. Find $u_{\beta\eta q} \in C([0,T] ; V)$ such that

$$u_{\beta\eta q} (t) \in K, \; \langle F \varepsilon (u_{\beta\eta q} (t)), \varepsilon (v - u_{\beta\eta q} (t)) \rangle_Q + h (\beta (t), u_{\beta\eta q} (t), v - u_{\beta\eta q} (t))$$

$$+ j_c (u_{\beta\eta q} (t), v - u_{\beta\eta q} (t)) + j_q (v) - j_q (u_{\beta\eta q} (t)) \geq (f_q (t), v - u_{\beta\eta q} (t))_V,$$

$$\forall v \in K, \; t \in [0,T]. \quad (4.4)$$

We have the following lemma.

**Lemma 4.4.** Problem $P_{2\beta\eta q}$ has a unique solution.

**Proof.** Let $t \in [0,T]$ and let $A_{\beta(t)} : V \rightarrow V$ be the operator defined by

$$A_{\beta(t)} (u, v)_V = \langle F \varepsilon (u), \varepsilon (v) \rangle_Q + h (\beta (t), u, v) + j_c (u, v) \; \forall u, v \in V.$$ 

As in [28], using (3.13) (b), (3.13) (c), (3.15), (3.19) (b), (3.19) (c) and the properties of $R_\nu$ and $R_\tau$, we see that the operator $A_{\beta(t)}$ is Lipschitz continuous and strongly monotone. On the other hand the functional $j_q : V \rightarrow \mathbb{R}$ is a continuous seminorm; using standard arguments on elliptic variational inequalities (see [25]), it follows that there exists a unique element $u_{\beta\eta q} (t) \in K$ which satisfies the inequality (4.4).

Now, for each $t \in [0,T]$, we define the map $\Psi_t : C_+ \rightarrow C_+$ by

$$\Psi_t (q) = |R_\sigma (u_{\beta\eta q} (t))|.$$ 

We show the following lemma.

**Lemma 4.5.** There exists a constant $\mu_1 > 0$ such that the mapping $\Psi_t$ has a unique fixed point $q^*$ and $u_{\beta\eta q^*} (t)$ is a unique solution of the inequality (4.3) if

$$\|\mu\|_{L^\infty (\Gamma)} < \mu_1.$$
Proof. Let $q_1, q_2 \in C_+$. Using (3.17), it follows that there exists a constant $c_0 > 0$ such that

$$
\|\Psi_t (q_1) - \Psi_t (q_2)\|_{L^2(\Gamma_3)} \leq c_0 \|\sigma_v (u_{\beta\eta q_1} (t)) - \sigma_v (u_{\beta\eta q_2} (t))\|_{H^{-1/2}(\Gamma)}.
$$

(4.5)

Moreover using (3.13) (b) yields

$$
\|\sigma_v (u_{\beta\eta q_1} (t)) - \sigma_v (u_{\beta\eta q_2} (t))\|_{H^{-1/2}(\Gamma)} \leq M \|u_{\beta\eta q_1} (t) - u_{\beta\eta q_2} (t)\|_V.
$$

(4.6)

We also use (3.10), (3.13) (c), (3.19) (c) and the properties of $R_{\nu}$ and $R_{\tau}$ to find after some calculus algebra that

$$
\|u_{\beta\eta q_1} (t) - u_{\beta\eta q_2} (t)\|_V \leq \frac{\|\mu\|_{L^\infty(\Gamma_3)} d \Omega}{m} \|q_1 - q_2\|_{L^2(\Gamma_3)}.
$$

(4.7)

Hence, taking into account (3.18), we combine (4.5), (4.6) and (4.7) to deduce that

$$
\|\Psi_t (q_1) - \Psi_t (q_2)\|_{L^2(\Gamma_3)} \leq \frac{c_0 M d \Omega}{m} \|q_1 - q_2\|_{L^2(\Gamma_3)}.
$$

Take $\mu_1 = m/c_0 M d \Omega$, then this inequality shows that if $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_1$, $\Psi$ is a contraction; thus it has a unique fixed point $q^*$ and $u_{\beta\eta q^*} (t)$ is a unique solution of (4.3).

Denote $u_{\beta\eta q^*} = u_{\beta\eta}$. We now shall see that $u_{\beta\eta} \in C ([0, T] ; V)$. Indeed, let $t_1, t_2 \in [0, T]$. Take $v = u_{\beta\eta} (t_2)$ in (4.3) written for $t = t_1$ and then $v = u_{\beta\eta} (t_1)$ in the same inequality written for $t = t_2$. Using (3.13) (c), (3.17), (3.19) (c) and the properties of $R_{\nu}$ and $R_{\tau}$, and adding the resulting inequalities, it follows that there exists a constant $c_1 > 0$ such that

$$
\|u_{\beta\eta} (t_2) - u_{\beta\eta} (t_1)\|_V \leq \frac{c_1}{m - \|\mu\|_{L^\infty(\Gamma_3)} c_0 M d \Omega} \|\beta (t_2) - \beta (t_1)\|_{L^2(\Gamma_3)} + \|\eta (t_2) - \eta (t_1)\|_Q + \|f (t_2) - f (t_1)\|_V.
$$

Then, as $\beta \in C ([0, T] ; L^2 (\Gamma_3))$, $\eta \in C ([0, T] ; Q)$ and $f \in C ([0, T] ; V)$, we immediately conclude. We also have that $u_{\beta\eta} (t) \in W, \forall t \in [0, T]$. Indeed, for each $t \in [0, T]$, denote $\sigma (u_{\beta\eta} (t)) = F_{\varepsilon} (u_{\beta\eta} (t)) + \eta (t)$, take $v = u_{\beta\eta} (t) \pm \varphi$ in inequality (4.3) where $\varphi \in (C_0^\infty (\Omega))^d$ and use Green’s formula with the regularity $\varphi_1 (t) \in H$ leads to $\text{div} \sigma (u_{\beta\eta} (t)) \in H$ and then $u_{\beta\eta} (t) \in W$. \hfill \Box

Now we introduce the operator

$$
\Lambda_\beta : C ([0, T] ; Q) \rightarrow C ([0, T] ; Q)
$$

deﬁned by

$$
\Lambda_\beta \eta (t) = \int_0^t \mathcal{F} (t - s) \varepsilon (u_{\beta\eta} (s)) ds \quad \forall \eta \in C ([0, T] ; Q), \ t \in [0, T].
$$

(4.8)

We have the lemma below.

**Lemma 4.6.** The operator $\Lambda_\beta$ has a unique fixed point $\eta_\beta$. 
As in [31] we deduce
$$\|\Lambda_{\beta \eta_1} (t) - \Lambda_{\beta \eta_2} (t)\|_Q \leq c_2 \int_0^t \|\eta_1 (s) - \eta_2 (s)\|_Q \, ds \quad \forall t \in [0, T],$$
where $c_2 > 0$. Reiterating this inequality $n$ times, yields
$$\|\Lambda_{\beta \eta_1}^n - \Lambda_{\beta \eta_2}^n\|_{C([0,T];Q)} \leq \frac{(c_2 T)^n}{n!} \|\eta_1 - \eta_2\|_{C([0,T];Q)}.$$
As $\lim_{n \to +\infty} \frac{(c_2 T)^n}{n!} = 0$, it follows that for a positive integer $n$ sufficiently large, $\Lambda_{\beta}^n$ is a contraction; then, by using the Banach fixed point theorem, it has a unique fixed point $\eta_{\beta}$ which is also a unique fixed point of $\Lambda_{\beta}$ i.e.,
$$\Lambda_{\beta \eta_{\beta}} (t) = \eta_{\beta} (t) \quad \forall t \in [0, T]. \tag{4.9}$$
Then by (4.3) and (4.9) we conclude that $u_{\beta \eta_{\beta}}$ is the unique solution of Problem $P_{1\beta}$.

Next denote $u_{\beta} = u_{\beta \eta_{\beta}}$. In the second step we state the following problem.

**Problem $P_{ad}$**. Find $\beta^* : [0,T] \to L^2 (\Gamma_3)$ such that
$$\dot{\beta^*} (t) = - \left[ \beta^* (t) \left( c_{\nu} (R_{\nu} (u_{\beta \nu} (t)))^2 + c_{\tau} |R_{\tau} (u_{\beta \tau} (t))|^2 \right) - \varepsilon_a \right]_+ \quad a.e. \ t \in (0, T), \tag{4.10}$$
$$\beta^* (0) = \beta_0. \tag{4.11}$$

We obtain the following result.

**Proposition 4.7.** Problem $P_{ad}$ has a unique solution $\beta^*$ which satisfies
$$\beta^* \in W^{1,\infty} (0,T; L^2 (\Gamma_3)) \cap B.$$

**Proof.** Let $t \in [0,T]$ and consider the mapping $\Phi : Z \to Z$ defined by
$$\Phi \beta (t) = \beta_0 - \int_0^t \left[ \beta (s) \left( c_{\nu} (R_{\nu} (u_{\beta \nu} (s)))^2 + c_{\tau} |R_{\tau} (u_{\beta \tau} (s))|^2 \right) - \varepsilon_a \right]_+ \, ds,$$
where $u_{\beta}$ is the solution of Problem $P_{1\beta}$. For $\beta_1, \beta_2 \in Z$, there exists a constant $c_2 > 0$ such that
$$\|\Phi \beta_1 (t) - \Phi \beta_2 (t)\|_{L^2(\Gamma_3)} \leq c_2 \int_0^t \left\| \beta_1 (s) \left( R_{\nu} (u_{\beta_1 \nu} (s)))^2 - \beta_2 (s) \left( R_{\nu} (u_{\beta_2 \nu} (s)))^2 \right. \right\|_{L^2(\Gamma_3)} \, ds$$
$$+ c_2 \int_0^t \left\| \beta_1 (s) \left| R_{\tau} (u_{\beta_1 \tau} (s)))^2 - \beta_2 (s) \left| R_{\tau} (u_{\beta_2 \tau} (s)))^2 \right. \right\|_{L^2(\Gamma_3)} \, ds.$$
As in [31] we deduce
$$\|\Phi \beta_1 (t) - \Phi \beta_2 (t)\|_{L^2(\Gamma_3)} \leq c_3 \left( \int_0^t \|\beta_1 (s) - \beta_2 (s)\|_{L^2(\Gamma_3)} \, ds + \int_0^t \|u_{\beta_1} (s) - u_{\beta_2} (s)\|_V \, ds \right), \tag{4.12}$$
for some constant $c_3 > 0$. Now to continue the proof we have needed to prove the following lemma.

**Lemma 4.8.** There exists a constant $\mu_0 > 0$ such that

$$
\| u_{\beta_1}(t) - u_{\beta_2}(t) \|_V \leq c \| \beta_1(t) - \beta_2(t) \|_{L^2(\Gamma_3)} \quad \forall t \in [0, T],
$$

if

$$
\| \mu \|_{L^\infty(\Gamma_3)} < \mu_0.
$$

**Proof.** Let $t \in [0, T]$. Take $u_{\beta_2}(t)$ in (4.1) satisfied by $u_{\beta_1}(t)$, then take $u_{\beta_1}(t)$ in the same inequality satisfied by $u_{\beta_2}(t)$; by adding the resulting inequalities we obtain

$$
\langle F \varepsilon(u_{\beta_1}(t)) - F \varepsilon(u_{\beta_2}(t)) , \varepsilon(u_{\beta_1}(t)) - \varepsilon(u_{\beta_2}(t)) \rangle_Q
$$

$$
\leq \left\langle \int_0^t \mathcal{F}(t-s) (\varepsilon(u_{\beta_1}(s)) - \varepsilon(u_{\beta_2}(s))) ds \varepsilon(u_{\beta_2}(t)) - \varepsilon(u_{\beta_1}(t)) \right\rangle_Q
$$

$$
+ h(\beta_1(t), u_{\beta_1}(t), u_{\beta_2}(t) - u_{\beta_1}(t)) + h(\beta_2(t), u_{\beta_2}(t), u_{\beta_1}(t) - u_{\beta_2}(t))
$$

$$
+ j(u_{\beta_1}(t), u_{\beta_2}(t)) - j(u_{\beta_1}(t), u_{\beta_1}(t)) + j(u_{\beta_2}(t), u_{\beta_1}(t)) - j(u_{\beta_2}(t), u_{\beta_2}(t)).
$$

Using (3.13) (b) this inequality implies

$$
m \| u_{\beta_1}(t) - u_{\beta_2}(t) \|^2_V \leq
$$

$$
\left\langle \int_0^t \mathcal{F}(t-s) (\varepsilon(u_{\beta_1}(s)) - \varepsilon(u_{\beta_2}(s))) ds \varepsilon(u_{\beta_2}(t)) - \varepsilon(u_{\beta_1}(t)) \right\rangle_Q
$$

$$
+ h(\beta_1(t), u_{\beta_1}(t), u_{\beta_2}(t) - u_{\beta_1}(t)) + h(\beta_2(t), u_{\beta_2}(t), u_{\beta_1}(t) - u_{\beta_2}(t))
$$

$$
+ j(u_{\beta_1}(t), u_{\beta_2}(t)) - j(u_{\beta_1}(t), u_{\beta_1}(t)) + j(u_{\beta_2}(t), u_{\beta_1}(t)) - j(u_{\beta_2}(t), u_{\beta_2}(t)).
$$

(4.13)

We have

$$
\left\langle \int_0^t \mathcal{F}(t-s) (\varepsilon(u_{\beta_1}(s)) - \varepsilon(u_{\beta_2}(s))) ds \varepsilon(u_{\beta_2}(t)) - \varepsilon(u_{\beta_1}(t)) \right\rangle_Q
$$

$$
\leq \left( \int_0^t \| \mathcal{F}(t-s) \|_{Q_\infty} \| u_{\beta_2}(s) - u_{\beta_1}(s) \|_V ds \right) \| u_{\beta_1}(t) - u_{\beta_2}(t) \|_V
$$

$$
\leq c_4 \left( \int_0^t \| u_{\beta_1}(s) - u_{\beta_2}(s) \|_V ds \right) \| u_{\beta_1}(t) - u_{\beta_2}(t) \|_V,
$$

for some positive constant $c_4$. Using Young’s inequality, we find

$$
\left\langle \int_0^t \mathcal{F}(t-s) (\varepsilon(u_{\beta_1}(s)) - \varepsilon(u_{\beta_2}(s))) ds \varepsilon(u_{\beta_2}(t)) - \varepsilon(u_{\beta_1}(t)) \right\rangle_Q
$$

$$
\leq \frac{c_4^2}{2m} \left( \int_0^t \| u_{\beta_2}(s) - u_{\beta_1}(s) \|_V ds \right)^2 + \frac{m}{2} \| u_{\beta_1}(t) - u_{\beta_2}(t) \|^2_V.
$$

(4.14)
Using the properties of $R_\nu$ and $R_\nu$ (see [28]), we have
\[
h(\beta_1(t), u_{\beta_1}(t), u_{\beta_2}(t) - u_{\beta_1}(t)) + h(\beta_2(t), u_{\beta_2}(t), u_{\beta_1}(t) - u_{\beta_2}(t)) \leq c_5 \|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma_3)} \|u_{\beta_1}(t) - u_{\beta_2}(t)\|_V,
\]
where $c_5 > 0$. Using also (3.10), (3.17) and (3.19) (c) yields
\[
j(u_{\beta_1}(t), u_{\beta_2}(t)) - j(u_{\beta_1}(t), u_{\beta_1}(t)) + j(u_{\beta_2}(t), u_{\beta_1}(t)) - j(u_{\beta_2}(t), u_{\beta_2}(t)) \leq c_0 Md_\Omega \|\mu\|_{L^\infty(\Gamma_3)} \|u_{\beta_1}(t) - u_{\beta_2}(t)\|^2_V.
\]
(4.15)

We now combine inequalities (4.13), (4.14) and (4.15) to deduce
\[
m \|u_{\beta_1}(t) - u_{\beta_2}(t)\|^2_V \leq c_0 Md_\Omega \|\mu\|_{L^\infty(\Gamma_3)} \|u_{\beta_1}(t) - u_{\beta_2}(t)\|^2_V + \frac{c_4^2}{2m} \left(\int_0^t \|u_{\beta_1}(s) - u_{\beta_2}(s)\|_V ds\right)^2 + \frac{m}{2} \|u_{\beta_1}(t) - u_{\beta_2}(t)\|^2_V
\]
\[
+ c_5 \|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma_3)} \|u_{\beta_1}(t) - u_{\beta_2}(t)\|_V.
\]
(4.16)

Using Young’s inequality we get
\[
c_5 \|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma_3)} \|u_{\beta_1}(t) - u_{\beta_2}(t)\|_V \leq c_6 \|\beta_1(t) - \beta_2(t)\|^2_{L^2(\Gamma_3)} + \frac{m}{4} \|u_{\beta_1}(t) - u_{\beta_2}(t)\|^2_V
\]
(4.17)

for some constant $c_6 > 0$. Then (4.16) and (4.17) imply that
\[
\frac{m}{4} \|u_{\beta_1}(t) - u_{\beta_2}(t)\|^2_V \leq c_0 Md_\Omega \|\mu\|_{L^\infty(\Gamma_3)} \|u_{\beta_1}(t) - u_{\beta_2}(t)\|^2_V + \frac{c_4^2}{2m} \left(\int_0^t \|u_{\beta_1}(s) - u_{\beta_2}(s)\|_V ds\right)^2 + c_6 \|\beta_1(t) - \beta_2(t)\|^2_{L^2(\Gamma_3)}.
\]

Let now
\[
\mu_0 = \mu_1/4.
\]

Then if
\[
\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0,
\]
we deduce that there exists a constant $c_7 > 0$ such that
\[
\|u_{\beta_1}(t) - u_{\beta_2}(t)\|^2_V \leq c_7 \left(\int_0^t \|u_{\beta_1}(s) - u_{\beta_2}(s)\|_V ds + \|\beta_1(t) - \beta_2(t)\|^2_{L^2(\Gamma_3)}\right).
\]

Then using Gronwall’s argument, it follows that there exists a constant $c > 0$ such that
\[
\|u_{\beta_1}(t) - u_{\beta_2}(t)\|_V \leq c \|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma_3)}.
\]
(4.18)
Now to end the proof of Proposition 4.7 we use (4.12) and (4.18) to deduce
\[ \| \Phi \beta_1 (t) - \Phi \beta_2 (t) \|_{L^2(\Gamma_3)} \leq c_8 \int_0^t \| \beta_1 (s) - \beta_2 (s) \|_{L^2(\Gamma_3)} \, ds \quad \forall t \in [0, T], \]
where \( c_8 > 0 \), and then we obtain
\[ \| \Phi \beta_1 - \Phi \beta_2 \|_k \leq \frac{c_8}{k} \| \beta_1 - \beta_2 \|_k. \]
This inequality shows that for \( k > c_8 \), \( \Phi \) is a contraction. Then it has a unique fixed point \( \beta^* \) which satisfies (4.10) and (4.11). We now have all ingredients to prove Theorem 4.1.

**Proof of Theorem 4.1.** *Existence.* Let \( \beta = \beta^* \) and let \( u_{\beta^*} \) the solution of Problem \( P_{1\beta} \). We conclude by (4.1), (4.10) and (4.11) that \( (u_{\beta^*}, \beta^*) \) is a solution to Problem \( P_2 \).

**Uniqueness.** Suppose that \( (u, \beta) \) is a solution of Problem \( P_2 \) which satisfies (3.20), (3.21) and (3.22). It follows from (3.20) that \( u \) is a solution to Problem \( P_{1\beta} \), and from Theorem 4.2 that \( u = u_{\beta^*} \). Take \( u = u_{\beta^*} \) in (3.20) and use the initial condition (3.22), we deduce that \( \beta \) is a solution to Problem \( P_{ad} \). Therefore, we obtain from Proposition 4.7 that \( \beta = \beta^* \) and then we conclude that \( (u_{\beta^*}, \beta^*) \) is a unique solution to Problem \( P_2 \). \( \square \)

Let now \( \sigma^* \) be the function defined by (3.1) which corresponds to the function \( u_{\beta^*} \). Then, it results from (3.13) and (3.14) that \( \sigma^* \in C ([0, T]; Q) \). Using also a standard argument, it follows from the inequality (3.20) that
\[ \text{div} \sigma^* (t) + \varphi_1 (t) = 0 \text{ in } \Omega, \text{ for all } t \in [0, T]. \]

Therefore, using the regularity \( \varphi_1 \in C ([0, T]; H) \), we deduce that \( \text{div} \sigma^* \in C ([0, T]; H) \) which implies that \( \sigma^* \in C ([0, T]; Q_1) \). The triple \( (u_{\beta^*}, \sigma^*, \beta^*) \) which satisfies (3.1) and (3.20) – (3.22) is called a weak solution of Problem \( P_1 \). We conclude that under the stated assumptions, the problem \( P_1 \) has a unique weak solution \( (u_{\beta^*}, \sigma^*, \beta^*) \). Moreover, the regularity of the weak solution is \( u_{\beta^*} \in C ([0, T]; V) \), \( \sigma^* \in C ([0, T]; Q_1) \), \( \beta^* \in W^{1, \infty} (0, T; L^2 (\Gamma_3)) \cap B \).

**References**


Viscoelastic unilateral and frictional contact problem


Arezki Touzaline
Laboratoire de Systèmes Dynamiques
Faculté de Mathématiques, USTHB
BP 32 EL ALIA
Bab-Ezzouar, 16111, Algérie

e-mail: ttouzaline@yahoo.fr