Stud. Univ. Babeş-Bolyai Math. 58(2013), No. 1, 87–95

On a bounded critical point theorem of Schechter

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Abstract. A new proof based on Bishop-Phelps' variational principle is given to a critical point theorem of Schechter for extrema in a ball of a Hilbert space. The same technique is used to obtain a similar result in annular domains. Comments on the involved boundary conditions and an application to a two-point boundary value problem are included. An alternative variational approach to the compression-expansion Krasnoselskii's fixed point method is thus provided. In addition, estimations from below are obtained here for the first time, in terms of the energetic norm.

Mathematics Subject Classification (2010): 47J30, 58E05, 34B15.

Keywords: Critical point, extremum point, Palais-Smale condition, two-point boundary value problem.

1. Introduction

Let X be a Hilbert space with inner product $(.,.)$ and norm $|.|$, identified to its dual, let X_R denote the closed ball of X of radius R centered at the origin and let ∂X_R be its boundary. In [13], [14], the following critical point theorem for minima located in X_R, in a slightly different form, was proved by using pseudogradients and deformation arguments.

Theorem 1.1 (Schechter's theorem for minima). *Let $F : X_R \to \mathbf{R}$ be a C^1-functional, bounded from below. There exists a sequence (x_n), $x_n \in X_R$, such that $F(x_n) \to \inf F(X_R)$ and one of the following two situations holds:*

(a) $F'(x_n) \to 0$;

(b) $|x_n| = R$, $(F'(x_n), x_n) \le 0$ *for all n, and*

$$F'(x_n) - \frac{(F'(x_n), x_n)}{R^2} x_n \to 0. \tag{1.1}$$

If in addition $(F'(x), x) \ge -a > -\infty$ for all $x \in \partial X_R$, F satisfies a Palais-Smale type compactness condition guarantying that any sequence as above has a convergent subsequence, and the boundary condition

$$F'(x) + \mu x \ne 0 \text{ for all } x \in \partial X_R \text{ and } \mu > 0 \tag{1.2}$$

holds, then there exists $x \in X_R$ *with*

$$F(x) = \inf F(X_R), \quad F'(x) = 0.$$

It is easy to see that under the assumption $(F'(x), x) \geq -a > -\infty$ for all $x \in \partial X_R$, Theorem 1.1 yields Schechter's original statement [14, Theorem 5.3.3]: There exists a sequence (x_n), $x_n \in X_R \setminus \{0\}$, such that

$$F(x_n) \to \inf F(X_R), \; (F'(x_n), x_n) \to b \leq 0 \text{ and } F'(x_n) - \frac{(F'(x_n), x_n)}{|x_n|^2} x_n \to 0.$$

The dual result for maxima in X_R is the following theorem, a slightly modified form of Theorem 5.5.5 in [14].

Theorem 1.2 (Schechter's theorem for maxima). *Let* $F : X_R \to \mathbf{R}$ *be a* C^1*-functional, bounded from above. There exists a sequence* (x_n), $x_n \in X_R$, *such that* $F(x_n) \to \sup F(X_R)$ *and one of the following two situations holds:*

(a) $F'(x_n) \to 0$;

(b) $|x_n| = R$, $(F'(x_n), x_n) \geq 0$ *for all* n, *and*

$$F'(x_n) - \frac{(F'(x_n), x_n)}{R^2} x_n \to 0. \tag{1.3}$$

If in addition $(F'(x), x) \leq a < \infty$ *for all* $x \in \partial X_R$, F *satisfies a Palais-Smale type compactness condition guarantying that any sequence as above has a convergent subsequence, and the boundary condition*

$$F'(x) + \mu x \neq 0 \text{ for all } x \in \partial X_R \text{ and } \mu < 0$$

holds, then there exists $x \in X_R$ *with*

$$F(x) = \sup F(X_R), \quad F'(x) = 0.$$

In this paper we first present a simple and direct proof of these results using Bishop-Phelps' variational principle. Similar results are then obtain for the localization of critical points of extremum in annular domains. Comments on the involved boundary conditions and an application to a two-point boundary value problem are included. The results are related to those from our previous papers [10] and [11].

We finish this introductory section by the statement of Bishop-Phelps' theorem [3], [6].

Theorem 1.3 (Bishop–Phelps' theorem). *Let* (M, d) *be a complete metric space,* $\varphi : M \to \mathbf{R}$ *lower semicontinuous and bounded from below and* $\varepsilon > 0$. *Then for any* $x_0 \in M$, *there exists* $x \in M$ *such that*

(i) $\varphi(x) \leq \varphi(x_0) - \varepsilon d(x_0, x)$;

(ii) $\varphi(x) < \varphi(y) + \varepsilon d(y, x)$ *for every* $y \neq x$.

Notice the equivalence of Bishop-Phelps' theorem and Ekeland's variational principle (see, e.g. [5], [9]).

2. New proof of Schechter's theorem

Proof of Theorem 1.1. We apply Bishop-Phelps' theorem to $M = X_R$, $\varphi = F$, $\varepsilon = \frac{1}{n}$ $(n \in \mathbf{N} \setminus \{0\})$ and $x_0 \in X_R$ with $F(x_0) \leq \inf F(X_R) + \frac{1}{n}$. It follows that there exists $x_n \in X_R$ such that

$$F(x_n) \leq F(x_0) - \frac{1}{n}|x_n - x_0| \leq F(x_0) \leq \inf F(X_R) + \frac{1}{n} \tag{2.1}$$

and

$$F(x_n) < F(y) + \frac{1}{n}|y - x_n| \quad \text{for every } y \in X_R \text{ with } y \neq x_n. \tag{2.2}$$

Clearly, (2.1) implies $F(x_n) \to \inf F(X_R)$. Two cases are possible: (c1) There is a subsequence of (x_n) with $|x_n| < R$; (c2) The terms of the sequence (x_n), except possibly a finite number, belong to ∂X_R.

In case (c1), we may suppose that $|x_n| < R$ for all n. For a fixed n and any $z \in X$ with $|z| = 1$, we take $y := x_n - tz$ which still belongs to X_R, for $t > 0$ small enough. Then (2.2) gives us

$$-t(F'(x_n), z) + o(t) + \frac{t}{n} > 0.$$

Dividing by t and letting t tend to zero, we obtain $(F'(x_n), z) \leq \frac{1}{n}$, whence $|F'(x_n)| \leq \frac{1}{n}$, that is $F'(x_n) \to 0$. Thus, in case (c1), property (a) holds.

In case (c2) we may assume that $|x_n| = R$ for all n. The key remark is that an element of the form $y = x_n - tz$ with $|z| = 1$, still belongs to the ball for $t > 0$ small enough, whenever $(x_n, z) > 0$. Indeed

$$|x_n - tz|^2 = t^2 - 2t(x_n, z) + R^2 \leq R^2$$

for $0 < t \leq 2(x_n, z)$. Hence, for such a z, we still have de conclusion $(F'(x_n), z) \leq \frac{1}{n}$. By density, the same inequality holds even if $(x_n, z) = 0$. Therefore

$$(F'(x_n), z) \leq \frac{1}{n} \text{ for every } z \in X \text{ with } |z| = 1 \text{ and } (x_n, z) \geq 0. \tag{2.3}$$

Now two subcases are possible: in the first one $(F'(x_n), x_n) > 0$ for a subsequence, when in view of the above remark, $(F'(x_n), F'(x_n)) \leq \frac{1}{n}|F'(x_n)|$, that is $|F'(x_n)| \leq \frac{1}{n}$. Thus, in this case, $F'(x_n) \to 0$ and (a) holds. In the second subcase, $(F'(x_n), x_n) \leq 0$ for all n except possibly a finite number of indices. Then we take in (2.3) $z = \frac{1}{|w_n|}w_n$, where $w_n = F'(x_n) - \frac{(F'(x_n), x_n)}{R^2}x_n$. Clearly $(x_n, w_n) = 0$. Hence

$$(F'(x_n), w_n) \leq \frac{1}{n}|w_n|.$$

Also, since $(x_n, w_n) = 0$, one has

$$(F'(x_n), w_n) = \left(F'(x_n) - \frac{(F'(x_n), x_n)}{R^2}x_n,\ w_n\right) = |w_n|^2.$$

Hence $|w_n|^2 \leq \frac{1}{n}|w_n|$, whence $|w_n| \leq \frac{1}{n}$ and so $w_n \to 0$. Therefore (b) holds.

For the last part of the theorem, it suffices to see that in case (b), under the additional assumption that $(F'(x), x) \geq -a > -\infty$ for all $x \in \partial X_R$, we may suppose that the sequence of real numbers $(F'(x_n), x_n)$ converges to some $b \leq 0$. Then, in view of the Palais-Smale condition, if at least for a subsequence $x_n \to x$, we have $F'(x) + \mu x = 0$, where $x \in \partial X_R$ since $|x_n| = R$, and $\mu = -b/R^2 \geq 0$. The case $\mu > 0$ being excluded by the boundary condition (1.2), it remains that $F'(x) = 0$. □

Proof of Theorem 1.2. Apply Theorem 1.1 to the functional $-F$. □

3. Critical points of extremum in annular domains

In this section, the same technique based on Bishop-Phelps' theorem is used in order to localize critical points of extremum in the annular domain

$$X_{r,R} := \{x \in X : r \leq |x| \leq R\},$$

where $0 < r < R$.

Theorem 3.1. *Let $F : X_{r,R} \to \mathbf{R}$ be a C^1-functional, bounded from below. Then there exists a sequence (x_n), $x_n \in X_{r,R}$ such that $F(x_n) \to \inf F(X_{r,R})$, and one of the following three situations holds:*

(a) $F'(x_n) \to 0$;

(b) $|x_n| = r$, $(F'(x_n), x_n) \geq 0$ *for all* n, *and*

$$F'(x_n) - \frac{(F'(x_n), x_n)}{r^2} x_n \to 0; \tag{3.1}$$

(c) $|x_n| = R$, $(F'(x_n), x_n) \leq 0$ *for all* n, *and*

$$F'(x_n) - \frac{(F'(x_n), x_n)}{R^2} x_n \to 0. \tag{3.2}$$

If in addition

$$(F'(x), x) \leq a < \infty \quad \textit{for } |x| = r, \;\; (F'(x), x) \geq -a > -\infty \quad \textit{for } |x| = R, \tag{3.3}$$

F satisfies a Palais-Smale type compactness condition guaranteeing that any sequence as above has a convergent subsequence, and the boundary conditions

$$\begin{aligned} F'(x) + \mu x &\neq 0 \quad \textit{for } |x| = r \textit{ and } \mu < 0; \\ F'(x) + \mu x &\neq 0 \quad \textit{for } |x| = R \textit{ and } \mu > 0, \end{aligned} \tag{3.4}$$

hold, then there exists $x \in X_{r,R}$ with

$$F(x) = \inf F(X_{r,R}), \quad F'(x) = 0.$$

Proof. Applying Bishop-Phelps' theorem on $M = X_{r,R}$ we find a sequence (x_n) of elements of $X_{r,R}$ such that $F(x_n) \leq \inf F(X_{r,R}) + \frac{1}{n}$ and

$$F(x_n) < F(y) + \frac{1}{n}|x_n - y| \quad \text{for } y \neq x_n. \tag{3.5}$$

Then, there must be a subsequence in one of the following cases: (c1) $r < |x_n| \leq R$; (c2) $|x_n| = r$. In case (c1) we may repeat the proof of Theorem 1.1 since in view of the strict inequality $|x_n| > r$, all choices of y of the form $x_n - tz$ from that proof are still possible, that is satisfy $|y| \geq r$ too, provided that $t > 0$ is small enough. Hence in case (c1) one of the situations (a), (c) holds. Assume now that we are in case (c2).

There exist two subcases: (1) $(F'(x_n), x_n) < 0$ for a subsequence. Then we can apply (3.5) to $y = x_n - tF'(x_n)$ since

$$|y|^2 = r^2 - 2t(F'(x_n), x_n) + t^2|F'(x_n)|^2 \geq r^2 + t^2|F'(x_n)|^2 \geq r^2$$

for all $t > 0$. In addition $|y| \leq R$ if t is small enough. Thus $y \in X_{r,R}$ for sufficiently small $t > 0$. Now (3.5) implies

$$-t(F'(x_n), F'(x_n)) + o(t) + \frac{1}{n}|F'(x_n)| \geq 0,$$

whence after dividing by t and passing to the limit with $t \to 0$, we deduce $|F'(x_n)| \leq \frac{1}{n}$, that is (a) also holds in this subcase. Assume now subcase: (2) $(F'(x_n), x_n) \geq 0$ for all n, except possibly a finite number of indices. Then in (3.5), we take $y = x_n - tz_n$, where $t > 0$, $z_n = F'(x_n) - \mu_n x_n$ and $\mu_n = \frac{(F'(x_n), x_n)}{r^2}$. Since $(x_n, z_n) = 0$, we have

$$|y|^2 = r^2 - 2t(x_n, z_n) + t^2|z_n|^2 = r^2 + t^2|z_n|^2 \geq r^2$$

for every $t > 0$. Hence $|y| \geq r$. In addition, from $|x_n| < R$, we have $|y| \leq R$ for all $t > 0$ small enough. Thus $y \in X_{r,R}$ and (3.5) applies and yields

$$(F'(x_n), z_n) \leq \frac{1}{n}|z_n|.$$

Consequently

$$|z_n|^2 = (F'(x_n) - \mu_n x_n, z_n) = (F'(x_n), z_n) \leq \frac{1}{n}|z_n|,$$

whence $|z_n| \leq \frac{1}{n}$ and so $z_n \to 0$, that is situation (b) holds. The proof of the last part of the theorem is similar to that of Theorem 1.1. □

Obviously, a dual result of Theorem 3.1 for maxima in annular domains can easily be stated.

Notice that *cone versions* of Theorems 1.1, 1.2 and 3.1 can be stated in sets of the form $K_R = \{x \in K : |x| \leq R\}$ and $K_{r,R} = \{x \in K : r \leq |x| \leq R\}$, respectively, where K is a wedge in X, and in particular, a cone (see the forthcoming paper [12]). Their statement is similar, with the additional cone invariance condition

$$x - F'(x) \in K \text{ for every } x \in K. \tag{3.6}$$

4. Comments on the boundary conditions

4.1. Meaning of the boundary conditions

The meaning of boundary conditions (3.4) and the natural way they are related to the minimization problem we are concerning with, can be well understood in one-dimension. Thus, for a C^1-function $F : D = [-R, -r] \cup [r, R] \to \mathbf{R}$, conditions (3.4) reduce to

$$\begin{aligned} F'(-r) &\geq 0, \quad F'(r) \leq 0, \\ F'(-R) &\leq 0, \quad F'(R) \geq 0. \end{aligned} \tag{4.1}$$

Indeed, if for example, $F'(-r) < 0$, then taking $\mu = \frac{F'(-r)}{r}$ we have $F'(-r)+\mu\cdot(-r) = 0$ where $\mu < 0$, which contradicts (3.4). Similar explanations can be done for the other three inequalities. Now remembering that we are interested in critical points of F which minimizes F, that is in points x with $F'(x) = 0$ and $F(x) = \inf F(D)$, we immediately can see that under conditions (4.1), if at some point x from the boundary of D, i.e. $x \in \{-R, -r, r, R\}$, functional F attains its minimum over D, then x must be a critical point of F, i.e. $F'(x) = 0$.

Therefore, conditions (4.1), and in general conditions (3.4), exclude the possibility for F to attain its minimum at a noncritical point of the boundary.

4.2. Connection with fixed point theory

In case that F' has the representation $F'(x) = x - N(x)$, when the critical points of F are the fixed points of N, conditions (3.4) read as follows

$$\begin{aligned} N(x) &\neq \lambda x \ \text{ for } |x| = r \text{ and } \lambda < 1, \\ N(x) &\neq \lambda x \ \text{ for } |x| = R \text{ and } \lambda > 1, \end{aligned}$$

and can be seen as a compression property of N over the annular domain $X_{r,R}$. The corresponding boundary conditions for the dual result for maxima are

$$\begin{aligned} N(x) &\neq \lambda x \ \text{ for } |x| = r \text{ and } \lambda > 1, \\ N(x) &\neq \lambda x \ \text{ for } |x| = R \text{ and } \lambda < 1, \end{aligned}$$

and expresses an expansion property of N over $X_{r,R}$.

Thus we may say that compression is related to minima, while expansion is related to maxima.

5. Application to boundary value problems

In this section we present a variational method for localization in annular domains of the solutions of boundary value problems, as an alternative approach to Krasnoselskii's fixed point method intensively used in the literature (see, e.g. [1], [2], [4], [7], [8], [15]). By our knowledge, this is for the first time that estimations from below are obtained in terms of the energetic norm. We shall discuss the two-point boundary value

$$\begin{cases} -u''(t) = f(u(t)), & t \in [0,1] \\ u(0) = u(1) = 0, \end{cases} \tag{5.1}$$

where f is a continuous function on $\mathbf{R}$, nonnegative and nondecreasing on $\mathbf{R}_+$. We seek positive solutions which are symmetric with respect to the middle of the interval $[0,1]$, that is $u(1-t) = u(t)$ for every $t \in \left[0, \frac{1}{2}\right]$. Thus we shall consider the Hilbert space

$$X = \widehat{H}_0^1(0,1) := \left\{ u \in H_0^1(0,1) : u(1-t) = u(t) \ \text{ for all } t \in \left[0, \frac{1}{2}\right] \right\},$$

endowed with the inner product

$$(u,v) = \int_0^1 u'(t)\, v'(t)\, dt$$

and norm

$$|u| = \left(\int_0^1 u'(t)^2\,dt\right)^{1/2} = \left(2\int_0^{\frac{1}{2}} u'(t)^2\,dt\right)^{1/2}.$$

We also consider the functional

$$F : \widehat{H}_0^1(0,1) \to \mathbf{R},\ \ F(u) = \int_0^1 \left(\frac{1}{2}u'(t)^2 - g(u(t))\right)dt,$$

where $g(\tau) = \displaystyle\int_0^\tau f(s)\,ds$. Clearly F is a C^1-functional and

$$F'(u) = u - N(u), \tag{5.2}$$

where

$$N(u)(t) = \int_0^1 G(t,s)\,f(u(s))\,ds,$$

and G is the Green function $G(t,s) = s(1-t)$ for $0 \le s \le t \le 1$, $G(t,s) = t(1-s)$ for $0 \le t < s \le 1$. Hence the solutions of (5.1) are critical points of F. Note that F is bounded from below on each ball of its domain. Indeed, if $|u| \le R$, then

$$|u(t)| = \left|\int_0^t u'(s)\,ds\right| \le \left(\int_0^1 1^2 ds\right)^{1/2}\left(\int_0^1 u'(s)^2\,ds\right)^{1/2} \le R \tag{5.3}$$

for all $t \in [0,1]$. Hence

$$F(u) \ge -\int_0^1 g(u(t))\,dt \ge -\max_{|\tau|\le R} g(\tau) > -\infty.$$

Theorem 5.1. *Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function, nondecreasing on $\mathbf{R}_+$, with $f(\tau) > 0$ for all $\tau > 0$. Assume that there are two numbers $0 < r < R$ such that*

$$f\left(\frac{3r}{25}\right) \ge \frac{125r}{9}, \tag{5.4}$$

$$f(R) \le R. \tag{5.5}$$

Then (5.1) has a nonnegative concave solution u with $\frac{3r}{25} \le u\left(\frac{1}{5}\right) \le R$, which minimizes F on the set of all nonnegative functions $v \in \widehat{H}_0^1(0,1)$ satisfying $r \le |v| \le R$.

Proof. We shall apply the cone version of Theorem 3.1, where the cone is $K = \widehat{H}_0^1(0,1;\mathbf{R}_+)$. First note that the boundedness conditions (3.3) are satisfied in view of (5.2) and of the property of N of sending bounded sets into bounded sets. Also, the Palais-Smale condition holds due to the complete continuity of the operator N, and the invariance condition (3.6) is guaranteed by the positivity of N. Thus it remains to check the boundary conditions (3.4). Assume first that the condition corresponding to the sphere $|u| = R$ does not hold. Then there is $u \in \widehat{H}_0^1(0,1;\mathbf{R}_+)$ with $|u| = R$ and $\mu > 0$ such that $F'(u) + \mu u = 0$. Then $N(u) = (1+\mu)u$, that is

$$-u''(t) = \frac{1}{1+\mu} f(u(t)) \ \ \text{on}\ \ [0,1]\ \text{and}\ u(0) = u(1) = 0. \tag{5.6}$$

Clearly, since f is nonnegative, u is concave. In addition since u is symmetric with respect to $1/2$, u is increasing on $[0,1/2]$ and decreasing on $[1/2,1]$. If we multiply by $u(t)$, we integrate over $[0,1]$, and we take into account (5.3), we obtain

$$R^2 = |u|^2 = \frac{1}{1+\mu}\int_0^1 f(u(t))\,u(t)\,dt < \int_0^1 f(u(t))\,u(t)\,dt \le f(R)\,R,$$

which contradicts (5.5). Next assume that the boundary condition on the sphere $|u| = r$ does not hold. Then, for some $u \in \widehat{H}_0^1(0,1;\mathbf{R}_+)$ with $|u| = r$ and $\mu < 0$, we have $F'(u) + \mu u = 0$, that is $N(u) = (1+\mu)\,u$, or equivalently (5.6), where this time $\frac{1}{1+\mu} > 1$. Now fix any number $a \in (0,1/2)$ and as above, after multiplication and integration, obtain

$$\begin{aligned} r^2 &= |u|^2 = \frac{1}{1+\mu}\int_0^1 f(u(t))\,u(t)\,dt > \int_0^1 f(u(t))\,u(t)\,dt \qquad (5.7)\\ &= 2\int_0^{\frac12} f(u(t))\,u(t)\,dt \ge 2\int_a^{\frac12} f(u(t))\,u(t)\,dt \ge 2\left(\frac12 - a\right) f(u(a))\,u(a). \end{aligned}$$

On the other hand, u' being decreasing, we have

$$u(a) = \int_0^a u'(t)\,dt \ge a u'(a). \tag{5.8}$$

In addition it is not difficult to prove the inequality

$$u'(a) \ge (1-2a)\,u'(0). \tag{5.9}$$

Indeed, if we let $\phi(t) = u'(t) - (1-2t)\,u'(0)$ for $t \in \left[0,\frac12\right]$, then

$$\phi'(t) = u''(t) + 2u'(0) = -\frac{1}{1+\mu} f(u(t)) + 2u'(0).$$

Since $f(u(t))$ is increasing on $[0,1/2]$, we deduce that ϕ' is decreasing, so ϕ is concave. In addition $\phi(0) = \phi(1/2) = 0$. Hence $\phi(t) \ge 0$ for all $t \in [0,1/2]$. Thus (5.9) is true. An other remark is that

$$r^2 = \int_0^1 u'(t)^2\,dt = 2\int_0^{\frac12} u'(t)^2\,dt \le u'(0)^2,$$

whence

$$u'(0) \ge r. \tag{5.10}$$

Now (5.8), (5.9) and (5.10) give $u(a) \ge a(1-2a)\,r$. This together with (5.7) implies

$$r^2 > a(1-2a)^2 f(a(1-2a)\,r)\,r,$$

that is

$$f(a(1-2a)\,r) < \frac{r}{a(1-2a)^2}.$$

For $a = 1/5$ this contradicts (5.4). Therefore all the assumptions of the cone version of Theorem 3.1 hold. $\square$

Finally we note that Theorem 3.1 in abstract setting and Theorem 5.1 for a concrete application, immediately yield multiplicity results of solutions if their hypotheses are satisfied for several finitely or infinitely many pairs of numbers r, R.

Acknowledgement. This paper was supported by a grant of the Romanian National Authority for Scientific Research, CNCSIS–UEFISCSU, project PCCE-55/2008.

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