Tripled fixed point theorems in partially ordered metric spaces

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Abstract. The notion of tripled fixed point is introduced by Berinde and Borcut [1]. In this manuscript, some new tripled fixed point theorems are obtained by using a generalization of the results of Luong and Thuang [11].


Keywords: Tripled fixed point, coupled fixed point, partially ordered set.

1. Introduction

Existence and uniqueness of a fixed point for contraction type mappings in partially ordered metric spaces were discussed first by Ran and Reurings [15] in 2004. Later, so many results were reported on existence and uniqueness of a fixed point and its applications in partially ordered metric spaces (see e.g. [1]-[18]).

In 1987, Guo and Lakshmikantham [6] introduced the notion of the coupled fixed point. The concept of coupled fixed point reconsidered in partially ordered metric spaces by Bhaskar and Lakshmikantham [5] in 2006. In this remarkable paper, by introducing the notion of a mixed monotone mapping the authors proved some coupled fixed point theorems for mixed monotone mapping and considered the existence and uniqueness of solution for periodic boundary value problem.

The triple $(X, d, \leq)$ is called partially ordered metric spaces if $(X, \leq)$ is a partially ordered set and $(X, d)$ is a metric space. Further, if $(X, d)$ is a complete metric space, then the triple $(X, d, \leq)$ is called partially ordered complete metric spaces. Throughout the manuscript, we assume that $X \neq \emptyset$ and

$$X^k = \underbrace{X \times X \times \cdots X}_{k \text{-many}}.$$ 

Then the mapping $\rho_k : X^k \times X^k \rightarrow [0, \infty)$ such that

$$\rho_k(x, y) := d(x_1, y_1) + d(x_2, y_2) + \cdots + d(x_k, y_k),$$

forms a metric on $X^k$ where $x = (x_1, x_2, \ldots, x_k), y = (y_1, y_2, \ldots, y_k) \in X^k$, $k \in \mathbb{N}$. 

We state the notions of a mixed monotone mapping and a coupled fixed point as follows.

**Definition 1.1.** ([5]) Let \((X, \leq)\) be a partially ordered set and \(F : X \times X \to X\). The mapping \(F\) is said to have the mixed monotone property if \(F(x, y)\) is monotone non-decreasing in \(x\) and is monotone non-increasing in \(y\), that is, for any \(x, y \in X\),

\[
x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y)
\]

and

\[
y_1, y_2 \in X, y_1 \leq y_2 \Rightarrow F(x, y_1) \geq F(x, y_2).
\]

**Definition 1.2.** ([5]) An element \((x, y) \in X \times X\) is called a coupled fixed point of the mapping \(F : X \times X \to X\) if

\[
x = F(x, y) \text{ and } y = F(y, x).
\]

In [5] Bhaskar and Lakshmikantham proved the existence of coupled fixed points for an operator \(F : X \times X \to X\) having the mixed monotone property on \((X, d, \leq)\) by supposing that there exists a \(k \in [0, 1)\) such that

\[
d(F(x, y), F(u, v)) \leq \frac{k}{2} [d(x, u) + d(y, v)], \text{ for all } u \leq x, y \leq v. \tag{1.1}
\]

under the assumption one of the following condition:

1. Either \(F\) is continuous, or
2. (i) if a non-decreasing sequence \(\{x_n\} \to x\), then \(x_n \leq x, \forall n\);
   (ii) if a non-increasing sequence \(\{y_n\} \to y\), then \(y \leq y_n, \forall n\).

Very recently, Borcut and Berinde [1] gave the natural extension of Definition 1.1 and Definition 1.2.

**Definition 1.3.** Let \((X, \leq)\) be a partially ordered set and \(F : X \times X \times X \to X\). The mapping \(F\) is said to have the mixed monotone property if for any \(x, y, z \in X\)

\[
x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow F(x_1, y, z) \leq F(x_2, y, z),
\]

\[
y_1, y_2 \in X, y_1 \leq y_2 \Rightarrow F(x, y_1, z) \geq F(x, y_2, z),
\]

\[
z_1, z_2 \in X, z_1 \leq z_2 \Rightarrow F(x, y, z_1) \leq F(x, y, z_2),
\]

**Definition 1.4.** Let \(F : X^3 \to X\). An element \((x, y, z)\) is called a tripled fixed point of \(F\) if

\[
F(x, y, z) = x, \quad F(y, x, y) = y, \quad F(z, y, x) = z.
\]

We recall the main theorem of Borcut and Berinde [1] which is inspired by the main theorem in [5].
Theorem 1.5. Let $(X, \leq, d)$ be a partially ordered set and suppose there is a metric $d$ on $X$ such that $(X, d)$ is a complete metric space. Suppose $F : X \times X \times X \rightarrow X$ such that $F$ has the mixed monotone property and

$$d(F(x, y, z), F(u, v, w)) \leq jd(x, u) + kd(y, v) + ld(z, w),$$

for any $x, y, z \in X$ for which $x \leq u$, $v \leq y$ and $z \leq w$. Suppose either $F$ is continuous or $X$ has the following properties:

1. if a non-decreasing sequence $x_n \rightarrow x$, then $x_n \leq x$ for all $n$,
2. if a non-increasing sequence $y_n \rightarrow y$, then $y \leq y_n$ for all $n$,
3. if a non-decreasing sequence $z_n \rightarrow z$, then $z \leq z_n$ for all $n$.

If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \leq F(x_0, y_0, z_0)$, $y_0 \geq F(y_0, x_0, z_0)$ and $z_0 \leq F(z_0, y_0, x_0)$, then there exist $x, y, z \in X$ such that $F(x, y, z) = x$, $F(y, x, y) = y$, $F(z, y, x) = z$, that is, $F$ has a tripled fixed point.

In this paper, we prove the existence and uniqueness of a tripled fixed point of $F : X^3 \rightarrow X$ satisfying nonlinear contractions in the context of partially ordered metric spaces.

2. Existence of a tripled fixed point

In this section we show the existence of a tripled fixed point. For this purpose, we state the following technical lemma which will be used in the proof of the main theorem efficiently.

Throughout the paper $M = [m_{ij}]$ is a matrix of real numbers and $M^t = [m_{ji}]$ denotes the transpose of $M$.

Lemma 2.1. Let $M = \begin{bmatrix} a & b & c \\ b & a + c & 0 \\ c & b & a \end{bmatrix}$. Then for $M^n = \begin{bmatrix} m_{11}^n & m_{12}^n & m_{13}^n \\ m_{21}^n & m_{22}^n & m_{23}^n \\ m_{31}^n & m_{32}^n & m_{33}^n \end{bmatrix}$ we have

$m_{11}^n + m_{12}^n + m_{13}^n = m_{21}^n + m_{22}^n + m_{23}^n = m_{31}^n + m_{32}^n + m_{33}^n = (a + b + c)^n < 1.$

Proof. We use mathematical induction. For $n = 1$,

$$M = \begin{bmatrix} a & b & c \\ b & a + c & 0 \\ c & b & a \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix},$$

then by assumption

$m_{11} + m_{12} + m_{13} = m_{21} + m_{22} + m_{23} = m_{31} + m_{32} + m_{33} = a + b + c < 1.$
For $n = 2$,

\[
M^2 = \begin{bmatrix}
  a & b & c \\
  b & a + c & 0 \\
  c & b & a
\end{bmatrix} \begin{bmatrix}
  a & b & c \\
  b & a + c & 0 \\
  c & b & a
\end{bmatrix}
\]

\[
= \begin{bmatrix}
  a^2 + b^2 + c^2 & b(a + c) + ab + bc & 2ac \\
  b(a + c) + ab & (a + c)^2 + b^2 & bc \\
  b^2 + 2ac & b(a + c) + ab + bc & a^2 + c^2
\end{bmatrix}
\]

\[
M^2 = \begin{bmatrix}
  m_{11}^2 & m_{12}^2 & m_{13}^2 \\
  m_{21}^2 & m_{22}^2 & m_{23}^2 \\
  m_{31}^2 & m_{32}^2 & m_{33}^2
\end{bmatrix}
\]

Where

\[
m_{11}^2 + m_{12}^2 + m_{13}^2 = m_{21}^2 + m_{22}^2 + m_{23}^2 = m_{31}^2 + m_{32}^2 + m_{33}^2
\]

Since $(a + b + c)^2 = a^2 + 2ab + 2ac + b^2 + 2bc + c^2$ then

\[
m_{11}^2 + m_{12}^2 + m_{13}^2 = (a + b + c)^2 < 1.
\]

Suppose it is true for an arbitrary $n$, that is, for $M^n = \begin{bmatrix}
  m_{11}^n & m_{12}^n & m_{13}^n \\
  m_{21}^n & m_{22}^n & m_{23}^n \\
  m_{31}^n & m_{32}^n & m_{33}^n
\end{bmatrix}$ we have

\[
m_{11}^n + m_{12}^n + m_{13}^n = m_{21}^n + m_{22}^n + m_{23}^n
\]

Then,

\[
M^{n+1} = M^n M = \begin{bmatrix}
  m_{11}^{n+1} & m_{12}^{n+1} & m_{13}^{n+1} \\
  m_{21}^{n+1} & m_{22}^{n+1} & m_{23}^{n+1} \\
  m_{31}^{n+1} & m_{32}^{n+1} & m_{33}^{n+1}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
  m_{11}^{n+1} + m_{12}^{n+1} + m_{13}^{n+1} & m_{11}^{n+1} + m_{12}^{n+1} + m_{13}^{n+1} & m_{11}^{n+1} + m_{12}^{n+1} + m_{13}^{n+1} \\
  m_{21}^{n+1} + m_{22}^{n+1} + m_{23}^{n+1} & m_{21}^{n+1} + m_{22}^{n+1} + m_{23}^{n+1} & m_{21}^{n+1} + m_{22}^{n+1} + m_{23}^{n+1} \\
  m_{31}^{n+1} + m_{32}^{n+1} + m_{33}^{n+1} & m_{31}^{n+1} + m_{32}^{n+1} + m_{33}^{n+1} & m_{31}^{n+1} + m_{32}^{n+1} + m_{33}^{n+1}
\end{bmatrix}
\]

\[
m_{11}^{n+1} + m_{12}^{n+1} + m_{13}^{n+1} = m_{21}^{n+1} + m_{22}^{n+1} + m_{23}^{n+1} = m_{31}^{n+1} + m_{32}^{n+1} + m_{33}^{n+1}
\]

Analogously we get that

\[
m_{21}^{n+1} + m_{22}^{n+1} + m_{23}^{n+1} = (a + b + c)^{n+1} < 1.
\]

\[\square\]

**Theorem 2.2.** Let $(X, d, \leq)$ be a partially ordered complete metric space. Let $F : X^3 \rightarrow X$ be a mapping having the mixed monotone property on $X$. Assume that there exist
non-negative numbers \(a, b, c\) and \(L\) such that \(a + b + c < 1\) for which

\[
d(F(x, y, z), F(u, v, w)) \leq \alpha d(x, u) + \beta d(y, v) + \gamma d(z, w)
\]

\[
+ L \min \left\{ \begin{array}{l}
d(F(x, y, z), x), d(F(x, y, z), y), d(F(x, y, z), z), \\
d(F(x, y, z), u), d(F(x, y, z), v), d(F(x, y, z), w), \\
d(F(u, v, w), x), d(F(u, v, w), y), d(F(u, v, w), z), \\
d(F(u, v, w), u), d(F(u, v, w), v), d(F(u, v, w), w) \\
\end{array} \right\}
\]

(2.1)

for all \(x \geq u, y \leq v, z \geq w\). Assume that \(X\) has the following properties:

(a) \(F\) is continuous, or,

(b) (i) if non-decreasing sequence \(x_n \to x\) (respectively, \(z_n \to z\)), then \(x_n \leq x\) (respectively, \(z_n \leq z\)), for all \(n\),

(ii) if non-increasing sequence \(y_n \to y\), then \(y_n \geq y\) for all \(n\).

If there exist \(x_0, y_0, z_0 \in X\) such that

\[
x_0 \leq F(x_0, y_0, z_0), \quad y_0 \geq F(y_0, x_0, y_0), \quad z_0 \leq F(x_0, y_0, z_0)
\]

then there exist \(x, y, z \in X\) such that

\[
F(x, y, z) = x \text{ and } F(y, x, y) = y \text{ and } F(z, y, x) = z
\]

**Proof.** We construct a sequence \(\{(x_n, y_n, z_n)\}\) in the following way: Set

\[
x_1 = F(x_0, y_0, z_0) \geq x_0, \quad y_1 = F(y_0, x_0, y_0) \leq y_0, \quad z_1 = F(z_0, y_0, x_0) \geq z_0,
\]

and by the mixed monotone property of \(F\), for \(n \geq 1\), inductively we get

\[
x_n = F(x_{n-1}, y_{n-1}, z_{n-1}) \geq x_{n-1} \geq \cdots \geq x_0, \\
y_n = F(y_{n-1}, x_{n-1}, y_{n-1}) \leq y_{n-1} \leq \cdots \leq y_0, \\
z_n = F(z_{n-1}, y_{n-1}, x_{n-1}) \geq z_{n-1} \geq \cdots \geq z_0.
\]

(2.2)

Moreover,

\[
d(x_n, x_{n+1}) \leq \alpha d(x_{n-1}, x_n) + \beta d(y_{n-1}, y_n) + \gamma d(z_{n-1}, z_n)
\]

\[
+ L \min \left\{ \begin{array}{l}
d(x_n, x_{n-1}), d(x_n, y_{n-1}), d(x_n, z_{n-1}), \\
d(x_n, x_n), d(x_n, y_n), d(x_n, z_n), \\
d(x_{n+1}, x_{n-1}), d(x_{n+1}, y_{n-1}), d(x_{n+1}, z_{n-1}), \\
d(x_{n+1}, x_n), d(x_{n+1}, y_n), d(x_{n+1}, z_n), \\
\end{array} \right\}
\]

(2.3)

\[
d(y_n, y_{n+1}) \leq \alpha d(y_{n-1}, y_n) + \beta d(x_{n-1}, x_n) + \gamma d(y_{n-1}, y_n)
\]

\[
+ L \min \left\{ \begin{array}{l}
d(y_n, y_{n-1}), d(y_n, y_{n-1}), d(y_n, y_{n-1}), \\
d(y_n, y_n), d(y_n, x_n), d(y_n, y_n), \\
d(y_{n+1}, y_{n-1}), d(y_{n+1}, x_{n-1}), d(y_{n+1}, y_{n-1}), \\
d(y_{n+1}, y_n), d(y_{n+1}, y_n), d(y_{n+1}, y_n), \\
\end{array} \right\}
\]

(2.4)

\[
\leq (a + c)d(y_{n-1}, y_n) + \beta d(x_{n-1}, x_n)
\]
and
\[
d(z_n, z_{n+1}) \leq ad(z_{n-1}, z_n) + bd(y_{n-1}, y_n) + cd(x_{n-1}, x_n) + L \min \left\{ d(z_n, x_{n-1}), d(z_n, y_{n-1}), d(z_n, z_{n-1}), d(z_{n+1}, x_n), d(z_{n+1}, y_n), d(z_{n+1}, z_n) \right\}
\] (2.5)

Thus, from by (2.3)-(2.5) and Lemma 2.1, we obtain that
\[
D_{n+1} \leq M D_n \leq \cdots \leq M^n D_1
\] (2.6)

where
\[
D_{n+1} = \begin{bmatrix} d(x_n, x_{n+1}) \\ d(y_n, y_{n+1}) \\ d(z_n, z_{n+1}) \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} a & b & c \\ b & a+c & 0 \\ c & b & a \end{bmatrix} = [m_{ij}] \text{ as in Lemma 2.1.}
\]

We show that \(\{x_n\}, \{y_n\}\) and \(\{z_n\}\) are Cauchy sequences.

Due to Lemma 2.1 and (2.6), we have
\[
d(x_p, x_q) \leq d(x_p, x_{p-1}) + d(x_{p-1}, x_{p-2}) + \cdots + d(x_{q+1}, x_p)
\]
\[
\leq \begin{bmatrix} m_{11}^{q+1} \\ m_{12}^{q+1} \\ m_{13}^{q+1} \end{bmatrix} D_1 + \begin{bmatrix} m_{11}^{p-1} \\ m_{12}^{p-1} \\ m_{13}^{p-1} \end{bmatrix} D_1 + \cdots + \begin{bmatrix} m_{11}^{q+1} \\ m_{12}^{q+1} \\ m_{13}^{q+1} \end{bmatrix} D_1
\]
\[
= \begin{bmatrix} m_{11}^{q+1} + \cdots + m_{11}^{p-1} + m_{11}^p \\ m_{12}^{q+1} + \cdots + m_{12}^{p-1} + m_{12}^p \\ m_{13}^{q+1} + \cdots + m_{13}^{p-1} + m_{13}^p \end{bmatrix} D_1
\]
\[
\leq (m_{11}^{q+1} + \cdots + m_{11}^{p-1} + m_{11}^p) d(1, x_0)
\]
\[
+ (m_{12}^{q+1} + \cdots + m_{12}^{p-1} + m_{12}^p) d(y_1, y_0)
\]
\[
+ (m_{13}^{q+1} + \cdots + m_{13}^{p-1} + m_{13}^p) d(z_1, z_0)
\]
\[
\leq (k^q + k^{q+1} + \cdots + k^{p-1}) d(1, x_0)
\]
\[
+ (k^q + k^{q+1} + \cdots + k^{p-1}) d(y_1, y_0)
\]
\[
+ (k^q + k^{q+1} + \cdots + k^{p-1}) d(z_1, z_0)
\]
\[
= (k^q + k^{q+1} + \cdots + k^{p-1})(d(x_1, x_0) + d(y_1, y_0) + d(z_1, z_0))
\]
\[
\leq k^q \frac{1-k^{p-q}}{1-k} (d(x_1, x_0) + d(y_1, y_0) + d(z_1, z_0))
\] (2.7)

where \(k = a+b+c < 1\). Thus (2.7) yields that \(\{x_n\}\) is a Cauchy sequence. Analogously, one can show \(\{y_n\}\) and \(\{z_n\}\) are Cauchy sequences.

Since \(X\) is a complete metric space, there exist \(x, y, z \in X\) such that
\[
\lim_{n \to \infty} x_n = x, \quad \lim_{n \to \infty} y_n = y \quad \text{and} \quad \lim_{n \to \infty} z_n = z.
\] (2.8)

Now, suppose that assumption (a) holds. Taking the limit as \(n \to \infty\) in (2.2) and by (2.8), we get
\[
x = \lim_{n \to \infty} x_n = \lim_{n \to \infty} F(x_{n-1}, y_{n-1}, z_{n-1})
\]
\[
= F(\lim_{n \to \infty} x_{n-1}, \lim_{n \to \infty} y_{n-1}, \lim_{n \to \infty} z_{n-1})
\]
\[
= F(x, y, z)
\]
and

\[
\begin{align*}
y &= \lim_{n \to \infty} y_n = \lim_{n \to \infty} F(y_{n-1}, x_{n-1}, y_{n-1}) \\
&= F(\lim_{n \to \infty} y_{n-1}, \lim_{n \to \infty} x_{n-1}, \lim_{n \to \infty} y_{n-1}) \\
&= F(y, x, y)
\end{align*}
\]

\[
\begin{align*}
z &= \lim_{n \to \infty} z_n = \lim_{n \to \infty} F(z_{n-1}, y_{n-1}, x_{n-1}) \\
&= F(\lim_{n \to \infty} z_{n-1}, \lim_{n \to \infty} y_{n-1}, \lim_{n \to \infty} x_{n-1}) \\
&= F(z, y, x).
\end{align*}
\]

Thus we proved that \( x = F(x, y, z), y = F(y, x, y) \) and \( z = F(z, y, x) \).

Finally, suppose that (b) holds. Since \( \{x_n\} \) (respectively, \( \{z_n\} \)) is non-decreasing sequence and \( x_n \to x \) (respectively, \( z_n \to z \)) and as \( \{y_n\} \) is non-increasing sequence and \( y_n \to y \), by assumption (b), we have \( x_n \geq x \) (respectively, \( z_n \geq z \)) and \( y_n \leq y \) for all \( n \). We have

\[
d(F(x_n, y_n, z_n), F(x, y, z)) + L \min \left\{ \begin{array}{l}
d(x_n, x) + d(y_n, y) + d(z_n, z) \\
d(F(x_n, y_n, z_n), x), d(F(x_n, y_n, z_n), y), \\
d(F(x_n, y_n, z_n), z), d(F(x_n, y_n, z_n), x), \\
d(F(x_n, y_n, z_n), y_n), d(F(x_n, y_n, z_n), z_n), \\
d(F(x_n, y_n, z_n), z), d(F(x, y, z), y), \\
d(F(x, y, z), z), d(F(x, y, z), x), \\
d(F(x, y, z), z), d(F(x, y, z), z_n), \\
\end{array} \right\}
\] (2.9)

Taking \( n \to \infty \) in (2.9) we get \( d(x, F(x, y, z)) \leq 0 \) which implies \( F(x, y, z) = x \).

Analogously, we can show that \( F(y, x, y) = y \) and \( F(z, y, x) = z \).

Therefore, we proved that \( F \) has a tripled fixed point. \( \square \)

**Corollary 2.3.** *(Main Theorem of [1])* Let \((X, d, \leq)\) be a partially ordered complete metric space. Suppose \(F : X \times X \times X \to X\) such that \(F\) has the mixed monotone property and

\[
d(F(x, y, z), F(u, v, w)) \leq jd(x, u) + kd(y, v) + ld(z, w), \tag{2.10}
\]

for any \(x, y, z \in X\) for which \(x \leq u, v \leq y\) and \(z \leq w\). Suppose either \(F\) is continuous or \(X\) has the following properties:

1. if a non-decreasing sequence \(x_n \to x\), then \(x_n \leq x\) for all \(n\),
2. if a non-increasing sequence \(y_n \to y\), then \(y_n \leq y\) for all \(n\),
3. if a non-decreasing sequence \(z_n \to z\), then \(z_n \leq z\) for all \(n\).

If there exist \(x_0, y_0, z_0 \in X\) such that \(x_0 \leq F(x_0, y_0, z_0)\), \(y_0 \geq F(y_0, x_0, z_0)\) and \(z_0 \leq F(z_0, y_0, x_0)\), then there exist \(x, y, z \in X\) such that

\[
F(x, y, z) = x, \quad F(y, x, y) = y, \quad F(z, y, x) = z,
\]

that is, \(F\) has a tripled fixed point.

**Proof.** Taking \(L = 0\) in Theorem (2.2), we obtain Corollary 2.3. \( \square \)
3. Uniqueness of tripled fixed point

In this section we shall prove the uniqueness of tripled fixed point. For a partially ordered set \((X, \leq)\), we endow the product \(X \times X \times X\) with the following partial order relation: for all \((x, y, z), (u, v, w) \in (X \times X)\),

\[(x, y, z) \leq (u, v, w) \iff x \leq u, y \geq v \text{ and } z \leq w.\]

We say that \((x, y, z)\) is equal to \((u, v, r)\) if and only if \(x = u, y = v, z = r\).

**Theorem 3.1.** In addition to hypotheses of Theorem 2.2, suppose that for all \((x, y, z), (u, v, r) \in X \times X \times X\), there exists \((a, b, c) \in X \times X \times X\) that is comparable to \((x, y, z)\) and \((u, v, r)\), then \(F\) has a unique triple fixed point.

**Proof.** The set of triple fixed point of \(F\) is not empty due to Theorem 2.2. Assume, now, \((x, y, z)\) and \((u, v, r)\) are the triple fixed points of \(F\), that is,

\[
F(x, y, z) = x, \quad F(u, v, r) = u,
\]

\[
F(y, x, y) = y, \quad F(v, u, v) = v,
\]

\[
F(z, y, x) = z, \quad F(r, v, u) = r,
\]

We shall show that \((x, y, z)\) and \((u, v, r)\) are equal. By assumption, there exists \((p, q, s) \in X \times X \times X\) that is comparable to \((x, y, z)\) and \((u, v, r)\). Define sequences \(\{p_n\}, \{q_n\}\) and \(\{s_n\}\) such that

\[
p = p_0, \quad q = q_0, \quad s = s_0, \quad \text{and}
\]

\[
p_n = F(p_{n-1}, q_{n-1}, s_{n-1}),
q_n = F(q_{n-1}, p_{n-1}, q_{n-1}),
\]

\[
s_n = F(s_{n-1}, q_{n-1}, p_{n-1}),
\]

for all \(n\). Since \((x, y, z)\) is comparable with \((p, q, s)\), we may assume that \(x \geq (p, q, s) = (p_0, q_0, s_0)\). Recursively, we get that

\[
(x, y, z) \geq (p_n, q_n, s_n) \quad \text{for all } n.
\]

By (3.2) and (2.1), we have

\[
d(x, p_{n+1}) = d(F(x, y, z), F(p_n, q_n, s_n)) \\
\leq ad(x, p_n) + bd(y, q_n) + cd(z, s_n) \\
+ L \min \left\{ \begin{aligned}
&d(F(x, y, z), F(x, y, z)), d(F(x, y, z), y), \\
&d(F(x, y, z), z), d(F(x, y, z), p_n), \\
&d(F(x, y, z), q_n), d(F(x, y, z), s_n), \\
&d(F(p_n, q_n, s_n), x), d(F(p_n, q_n, s_n), y), \\
&d(F(p_n, q_n, s_n), z), d(F(p_n, q_n, s_n), p_n), \\
&d(F(p_n, q_n, s_n), q_n), d(F(p_n, q_n, s_n), s_n)
\end{aligned} \right\}
\]

\[
\leq \min \left\{ \begin{aligned}
ad(x, p_0) + bd(y, q_0) + cd(z, s_0) \\
L \min \left\{ \begin{aligned}
&d(F(x, y, z), F(x, y, z)), d(F(x, y, z), y), \\
&d(F(x, y, z), z), d(F(x, y, z), p_n), \\
&d(F(x, y, z), q_n), d(F(x, y, z), s_n), \\
&d(F(p_n, q_n, s_n), x), d(F(p_n, q_n, s_n), y), \\
&d(F(p_n, q_n, s_n), z), d(F(p_n, q_n, s_n), p_n), \\
&d(F(p_n, q_n, s_n), q_n), d(F(p_n, q_n, s_n), s_n)
\end{aligned} \right\}
\end{aligned} \right\}
\]

\[
\leq k^n [d(x, p_0) + d(y, q_0) + d(z, s_0)]
\]

where
\[d(q_{n+1}, y) = d(F(q_n, p_n, q_n), F(y, x, y))\]
\[\leq ad(y, q_n) + bd(x, p_n) + cd(y, q_n)\]
\[+ L \min \left\{ d(F(y, x, y), y), d(F(y, x, y), x),
\quad d(F(y, x, y), y), d(F(y, x, y), q_n),
\quad d(F(y, x, y), p_n), d(F(y, x, y), q_n),
\quad d(F(q_n, p_n, q_n), y), d(F(q_n, p_n, q_n), x),
\quad d(F(q_n, p_n, q_n), y), d(F(q_n, p_n, q_n), q_n),
\quad d(F(q_n, p_n, q_n), p_n), d(F(p_n, q_n, s_n), q_n)\right\}\]
\[\leq ad(y, q_n) + bd(x, p_n) + cd(y, q_n)\]
\[\leq k^n[d(x, p_0) + d(y, q_0) + d(z, s_0)]\]

\[d(z, s_{n+1}) = d(F(x, y, z), F(p_n, q_n, s_n))\]
\[\leq ad(x, p_n) + bd(y, q_n) + cd(z, s_n)\]
\[+ L \min \left\{ d(F(z, x, y), z), d(F(z, x, y), y),
\quad d(F(z, x, y), y), d(F(z, x, y), s_n),
\quad d(F(z, x, y), q_n), d(F(z, x, y), p_n),
\quad d(F(s_n, q_n, p_n), z), d(F(s_n, q_n, p_n), y),
\quad d(F(s_n, q_n, p_n), y), d(F(s_n, q_n, p_n), s_n),
\quad d(F(s_n, q_n, p_n), s_n), d(F(s_n, q_n, p_n), p_n)\right\}\]
\[\leq ad(x, p_n) + bd(y, q_n) + cd(z, s_n)\]
\[\leq k^n[d(x, p_0) + d(y, q_0) + d(z, s_0)]\]

Letting \(n \to \infty\) in (3.3) - (3.5), we get
\[\lim_{n \to \infty} d(x, p_{n+1}) = 0, \quad \lim_{n \to \infty} d(y, q_{n+1}) = 0, \quad \lim_{n \to \infty} d(z, s_{n+1}) = 0.\] (3.6)

Analogously, we obtain
\[\lim_{n \to \infty} d(u, p_{n+1}) = 0, \quad \lim_{n \to \infty} d(v, q_{n+1}) = 0, \quad \lim_{n \to \infty} d(r, s_{n+1}) = 0.\] (3.7)

By (3.6) and (3.7) we have \(x = u, y = v, z = r\).

\section*{4. Examples}

In this sections we give some examples to show that our results are effective.

**Example 4.1.** Let \(X = [0, 1]\) with the metric \(d(x, y) = |x - y|\), for all \(x, y \in X\) and the usual ordering.

Let \(F : X^3 \to X\) be given by
\[F(x, y, z) = \frac{4x^2 - 4y^2 + 8z^2 + 8}{65}, \text{ for all } x, y, z, w \in X\]

It is easy to check that all the conditions of Corollary 2.3 are satisfied and \((\frac{1}{8}, \frac{1}{8}, \frac{1}{8})\) is the unique tripled fixed point of \(F\) where \(a = b = \frac{8}{65}\) and \(c = \frac{16}{65}\).

But if we changed the space \(X = [0, 1]\) with \(X = [0, \infty)\), then conditions of Corollary 2.3 are not satisfied anymore. Indeed, \(F(8, 0, 8) = \frac{776}{65}\) and \(F(7, 0, 7) = \frac{596}{65}\).

Thus, \(d(F(8, 0, 8), F(7, 0, 7)) = \frac{36}{13}\). On the other hand, \(d(8, 7) = 1\). Thus, there exist
no non-negative real number \(a, b, c\) with \(a + b + c < 1\) and satisfies the conditions of Corollary 2.3. But, for \(L = 2\) and \(a = b = \frac{8}{65}, c = \frac{16}{65}\), the conditions of Theorem 2.2 are satisfied. Moreover, \((\frac{1}{8}, \frac{1}{8})\) is the unique tripled fixed point of \(F\).

**Example 4.2.** Let \(X = [0, \infty)\) with the metric \(d(x, y) = |x - y|\), for all \(x, y \in X\) and the following order relation:

\[ x, y \in X, \; x \leq y \iff x = y = 0 \text{ or } (x, y \in (0, \infty) \text{and } x \leq y), \]

where \(\leq\) be the usual ordering.

Let \(F : X^3 \to X\) be given by

\[
F(x, y, z) = \begin{cases} 
1, & \text{if } xyz \neq 0 \\
0, & \text{if } xyz = 0 
\end{cases}
\]

for all \(x, y, z \in X\).

It is easy to check that all the conditions of Corollary 2.3 are satisfied. Applying Corollary 2.3 we conclude that \(F\) has a tripled fixed point. In fact, \(F\) has two tripled fixed points. They are \((0, 0, 0)\) and \((1, 1, 1)\). Therefore, the conditions of Corollary 2.3 are not sufficient for the uniqueness of a tripled fixed point.

**References**


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