On $n$-weak amenability of a non-unital Banach algebra and its unitization

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Abstract. In [2] the authors asked if a non-unital Banach Algebra $\mathcal{A}$ is weakly amenable whenever its unitization $\mathcal{A}^\#$ is weakly amenable and whether $\mathcal{A}^\#$ is $2$-weakly amenable whenever $\mathcal{A}$ is $2$-weakly amenable. In this paper we give a partial solutions to these questions.

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1. Introduction

The notion of $n$-weak amenability for a Banach algebra was introduced by Dales, Ghahramani and Gronbæk in [2]. The Banach algebra $\mathcal{A}$ is called $n$-weakly amenable if $H^1(\mathcal{A}, \mathcal{A}^{(n)}) = (0)$, where $\mathcal{A}^{(n)}$ refers to the $n$-th dual of $\mathcal{A}$. Also $\mathcal{A}$ is permanently weakly amenable if $\mathcal{A}$ is $n$-weakly amenable for each $n \in \mathbb{N}$. In [2] the authors proved the following (Proposition 1.4):

Let $\mathcal{A}$ be a non-unital Banach algebra, and $n \in \mathbb{N}$.

(i) Suppose $\mathcal{A}^\#$ is $2n$-weakly amenable. Then $\mathcal{A}$ is $2n$-weakly amenable.

(ii) Suppose that $\mathcal{A}$ is $(2n - 1)$-weakly amenable. Then $\mathcal{A}^\#$ is $(2n - 1)$-weakly amenable.

(iii) Suppose that $\mathcal{A}$ is commutative. Then $\mathcal{A}^\#$ is $n$-weakly amenable if and only if $\mathcal{A}$ is $n$-weakly amenable.

In this paper we consider the converses to (i) and (ii) and give partial solutions to them. Let us recall some definitions.

Definition 1.1. ([6]) A Banach $\mathcal{A}$-module $X$ is called neo-unital if for each $x \in X$ there are $a, a' \in \mathcal{A}$ and $y, y' \in X$ with $x = ay = y'a'$.

Definition 1.2. ([3]) A Banach algebra $\mathcal{A}$ is called self-induced if $\mathcal{A}$ and $\mathcal{A} \hat{\otimes} \mathcal{A}$ are naturally isomorphic.
Here $\mathcal{A} \otimes_{\mathcal{A}} \mathcal{A} = \frac{\mathcal{A} \otimes \mathcal{A}}{K}$ where $K$ is the closed linear span of \{ab \otimes c - a \otimes bc : a, b, c \in \mathcal{A}\}.

Now we proceed to state and prove our theorem.

**Theorem 1.3.** Let $\mathcal{A}$ be a non-unital Banach algebra and suppose that $\mathcal{A}$ is self-induced.

(i) If $\mathcal{A}^\sharp$ is $(2n - 1)$-weakly amenable then $\mathcal{A}$ is $(2n - 1)$-weakly amenable.

(ii) If $\mathcal{A}$ is $2n$-weakly amenable then $\mathcal{A}^\sharp$ is $2n$-weakly amenable.

(iii) $H^2(\mathcal{A}, \mathcal{A}^{(2n)}) \cong H^2(\mathcal{A}^\sharp, \mathcal{A}^{\sharp(2n)})$

**Proof.** Clearly $\mathcal{A}$ is a closed two-sided ideal in $\mathcal{A}^\sharp$ with codimension one. We consider the corresponding short exact sequence and its iterated duals. That is,

\[ 0 \longrightarrow \mathcal{A} \overset{i}{\longrightarrow} \mathcal{A}^\sharp \overset{\varphi}{\longrightarrow} \mathcal{C} \longrightarrow 0 \]

where $i : \mathcal{A} \longrightarrow \mathcal{A}^\sharp$ defined by $a \mapsto (a, 0)$ and $\varphi : \mathcal{A}^\sharp \longrightarrow \mathcal{C}$ defined by $(a, \lambda) \mapsto \lambda$.

\[
\begin{align*}
0 & \longrightarrow \mathcal{C} \longrightarrow \mathcal{A}^{(2n-1)} \longrightarrow \mathcal{A}^{(2n-1)} \longrightarrow 0 \quad (1.1) \\
0 & \longrightarrow \mathcal{A}^{(2n)} \longrightarrow \mathcal{A}^{(2n)} \longrightarrow \mathcal{C} \longrightarrow 0 \quad (1.2)
\end{align*}
\]

It is easy to see that $i$ is an isometric isomorphism and $\varphi$ is a character on $\mathcal{A}^\sharp$ with ker $\varphi = \mathcal{A}$. Then we make $\mathcal{C}$ a module over $\mathcal{A}^\sharp$. Indeed,

\[ z \cdot (a, \lambda) = (a, \lambda) \cdot z = \varphi(a, \lambda)z = \lambda z \]

where $(a, \lambda)e\mathcal{A}^\sharp$ and $z\in\mathcal{C}$.

Now consider the long exact sequence of cohomology groups concerning to (1.1). That is,

\[
\begin{align*}
\cdots & \longrightarrow H^m(\mathcal{A}^\sharp, \mathcal{C}) \longrightarrow H^m(\mathcal{A}^\sharp, \mathcal{A}^{(2n-1)}) \\
& \quad \longrightarrow H^m(\mathcal{A}^\sharp, \mathcal{A}^{(2n-1)}) \longrightarrow H^{m+1}(\mathcal{A}, \mathcal{C}) \longrightarrow \cdots \quad (1.3)
\end{align*}
\]

Obviously $\mathcal{A}$, $\mathcal{A}^{(n)}$ and $\mathcal{C}$ are unital Banach $\mathcal{A}^\sharp$-bimodules. So by [4, Theorem 2.3] we have,

\[ H^m(\mathcal{A}^\sharp, \mathcal{C}) \cong H^m(\mathcal{A}, \mathcal{C}) \quad \text{and} \quad H^m(\mathcal{A}^\sharp, \mathcal{A}^{(2n-1)}) \cong H^m(\mathcal{A}, \mathcal{A}^{2n-1}) \quad (1.4) \]

Therefore by substituting (1.4) in (1.3) we get,

\[
\begin{align*}
\cdots & \longrightarrow H^m(\mathcal{A}, \mathcal{C}) \longrightarrow H^m(\mathcal{A}^\sharp, \mathcal{A}^{(2n-1)}) \\
& \quad \longrightarrow H^m(\mathcal{A}, \mathcal{A}^{(2n-1)}) \longrightarrow H^{m+1}(\mathcal{A}, \mathcal{C}) \longrightarrow \cdots \quad (1.5)
\end{align*}
\]

Since $\mathcal{A}$ is self-induced then $H^1(\mathcal{A}, \mathcal{C}) = H^2(\mathcal{A}, \mathcal{C}) = (0)$ [4,Lemma 2.5](note that $\mathcal{C}$ is an annihilator $\mathcal{A}$-bimodule). Hence by sequence (1.5) we obtain,

\[ H^1(\mathcal{A}^\sharp, \mathcal{A}^{(2n-1)}) \cong H^1(\mathcal{A}, \mathcal{A}^{(2n-1)}) \]
Obviously (i) holds.
For (ii) consider the long exact sequence of cohomology groups corresponding to the short exact sequence (1.2). That is,
\[
\begin{align*}
\ldots & \rightarrow \mathcal{H}^m(\mathcal{A}^z, \mathcal{A}^{(2n)}) \rightarrow \mathcal{H}^m(\mathcal{A}^z, \mathcal{A}^{(2n)}) \\
& \rightarrow \mathcal{H}^m(\mathcal{A}^z, \mathcal{C}) \rightarrow \mathcal{H}^{m+1}(\mathcal{A}^z, \mathcal{A}^{(2n)}) \rightarrow \ldots
\end{align*}
\] (1.6)
Like before we have,
\[
\mathcal{H}^m(\mathcal{A}^z, \mathcal{C}) \cong \mathcal{H}^m(\mathcal{A}, \mathcal{C}) \quad \text{and} \quad \mathcal{H}^m(\mathcal{A}^z, \mathcal{A}^{(2n)}) \cong \mathcal{H}^m(\mathcal{A}, \mathcal{A}^{(2n)}). \quad (1.7)
\]
By substituting (1.7) in (1.6) we get,
\[
\begin{align*}
\ldots & \rightarrow \mathcal{H}^m(\mathcal{A}, \mathcal{A}^{(2n)}) \rightarrow \mathcal{H}^m(\mathcal{A}^z, \mathcal{A}^{(2n)}) \rightarrow \mathcal{H}^m(\mathcal{A}, \mathcal{C}) \\
& \rightarrow \mathcal{H}^{m+1}(\mathcal{A}, \mathcal{A}^{(2n)}) \rightarrow \mathcal{H}^{m+1}(\mathcal{A}^z, \mathcal{A}^{(2n)}) \rightarrow \mathcal{H}^{m+1}(\mathcal{A}, \mathcal{C}) \rightarrow \ldots
\end{align*}
\] (1.8)
Now if \( \mathcal{A} \) is \( 2n \)-weakly amenable then self-inducement of \( \mathcal{A} \) and (1.8) imply
\[
\mathcal{H}^1(\mathcal{A}^z, \mathcal{A}^{(2n)}) = (0).
\]
So (ii) holds.
For (iii) self-inducement of \( \mathcal{A} \) and (1.8) imply
\[
\mathcal{H}^2(\mathcal{A}, \mathcal{A}^{(2n)}) \cong \mathcal{H}^2(\mathcal{A}^z, \mathcal{A}^{(2n)})
\]
\[\square\]

A special case occurs when the Banach algebra \( \mathcal{A} \) has a left(right) bounded approximate identity. In this case we have the following result.

**Proposition 1.4.** If the Banach algebra \( \mathcal{A} \) has a left(right) bounded approximate identity then the theorem holds.

**Proof.** By [5, Proposition II.3.13] \( \mathcal{A} \otimes_{\mathcal{A}} \mathcal{A} \rightarrow \mathcal{A}^2 \) given by \( a \otimes b \mapsto ab \) is a topological isomorphism. By [1,§11,corollary 11] \( \mathcal{A}^2 = \mathcal{A} \). So \( \mathcal{A} \otimes_{\mathcal{A}} \mathcal{A} \cong \mathcal{A} \). That is \( \mathcal{A} \) is self-induced. Hence the theorem holds. \[\square\]

**References**


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