A subclass of analytic functions

Andreea-Elena Tudor

Abstract. In the present paper, by means of Carlson-Shaffer operator and a multiplier transformation, we consider a new class of analytic functions $BL(m, l, a, c, \mu, \alpha, \lambda)$. A sufficient condition for functions to be in this class and the angular estimates are provided.

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1. Introduction

Let $A_n$ denote the class of functions $f$ of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, \quad z \in U$$

which are analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$.

Let $H(U)$ be the space of holomorphic functions in $U$.

For $a \in \mathbb{C}$ and $n \in \mathbb{N}$, we denote by

$$H[a, n] = \{f \in H(U), \quad f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ..., z \in U\}$$

Let the function $\phi(a, c; z)$ be given by

$$\phi(a, c; z) := \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{k+1} \quad (c \neq 0, -1, -2, ...; z \in U)$$

where $(x)_k$ is the Pochhammer symbol defined by

$$(x)_k := \begin{cases} 1, & k = 0 \\ x(x+1)(x+2)...(x+k-1), & k \in \mathbb{N}^* \end{cases}$$

Carlson and Shaffer[1] introduced a linear operator $L(a, c)$, corresponding to the function $\phi(a, c; z)$, defined by the following Hadamard product:

$$L(a, c) := \phi(a, c; z) * f(z) = z + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} a_{k+1} z^{k+1}$$
We note that
\[ z \left( L(a, c) f(z) \right)' = aL(a + 1, c) f(z) - (a - 1) L(a, c) f(z) \]
For \( a = m + 1 \) and \( c = 1 \) we obtain the Ruscheweyh derivative of \( f \), (see [9]).

In [2], N.E. Cho and H. M. Srivastava introduced a linear operator of the form:
\[ \mathcal{I}(m, l) f(z) = z + \sum_{k=2}^{\infty} \left( \frac{l + k}{l + 1} \right)^m a_k z^k, \quad m \in \mathbb{Z}, \ l \geq 0 \]
We note that
\[ z \left( \mathcal{I}(m, l) f(z) \right)' = (l + 1) \mathcal{I}(m + 1, l) f(z) - l \mathcal{I}(m, l) f(z) \]
For \( l = 0 \) we obtain Sălăgean operator introduced in [10].

Let now \( L(m, l, a, c, \lambda) \) be the operator defined by:
\[ L(m, l, a, c, \lambda) f(z) = \lambda \mathcal{I}(m, l) f(z) + (1 - \lambda) L(a, c) f(z) \]
For \( \lambda = 0 \) we get Carlson-Shaffer operator introduced in [1], for \( \lambda = 1 \) we get linear operator in [2] and for \( a = m + 1, c = 1, l = 0 \) we get generalized Sălăgean and Ruscheweyh operator introduced by A. Alb Lupaş in [4] by means of operator \( L(m, l, a, c, \lambda) \) we introduce the following subclass of analytic functions:

**Definition 1.1.** We say that a function \( f \in \mathcal{A}_n \) is in the class \( \mathcal{B}(m, l, a, c, \mu, \alpha, \lambda) \), \( n, m \in \mathbb{N}, l, \mu, \lambda \geq 0, \alpha \in [0, 1) \) if
\[ \left| \frac{\mathcal{L}(m + 1, l, a + 1, c, \lambda) f(z)}{z} \left( \frac{z}{\mathcal{I}(m, l, a, c, \lambda) f(z)} \right)^\mu - 1 \right| < 1 - \alpha, \ z \in U. \]  

**Remark 1.2.** The class \( \mathcal{B}(m, l, a, c, \mu, \alpha, \lambda) \) includes various classes of analytic univalent functions, such as:
- \( \mathcal{B}(0, 0, a, c, 1, \alpha, 1) \equiv S^*(\alpha) \)
- \( \mathcal{B}(1, 0, a, c, 1, \alpha, 1) \equiv K(\alpha) \)
- \( \mathcal{B}(0, 0, a, c, 0, \alpha, 1) \equiv R(\alpha) \)
- \( \mathcal{B}(0, 0, a, c, 2, \alpha, 1) \equiv B(\alpha) \) introduced by Frasin and Darus in [7]
- \( \mathcal{B}(0, 0, a, c, \mu, \alpha, 1) \equiv B(\mu, \alpha) \) introduced by Frasin and Jahangiri in [6]
- \( \mathcal{B}(m, 0, a, c, \mu, \alpha, 1) \equiv BS(\mu, \alpha) \) introduced by A. Alb Lupaş and A. Cătaş in [5]
- \( \mathcal{B}(m, 0, m + 1, 1, \mu, \alpha, 0) \equiv BR(\mu, \alpha) \) introduced by A. Alb Lupaş and A. Cătaş in [4]
- \( \mathcal{B}(m, 0, m + 1, 1, \mu, \alpha, \lambda) \equiv BL(\mu, \alpha, \lambda) \) introduced by A. Alb Lupaş and A. Cătaş in [3].

To prove our main result we shall need the following lemmas:

**Lemma 1.3.** [6] Let \( p \) be analytic in \( U \), with \( p(0) = 1 \), and suppose that
\[ Re \left( 1 + \frac{zp'(z)}{p(z)} \right) > \frac{3\alpha - 1}{2\alpha}, \ z \in U \]  
Then \( Re p(z) > \alpha \) for \( z \in U \) and \( \alpha \in [-1/2, 1) \).
**Theorem 2.1.** Let $BL$ class

**2. Main results**

Using (2.1), we get

**Proof.** If we denote by $f$, then $f$ with $m, l, a, c, \mu, \alpha, \lambda$

If there exist a point $z_0 \in U$ such that

$$|\arg(p(z_0))| < \frac{\pi}{2} \alpha, \quad |z| < |z_0|, \quad |\arg(p(z_0))| = \frac{\pi}{2} \alpha$$

with $0 < \alpha \leq 1$, then we have

$$\frac{z_0p'(z_0)}{p(z_0)} = ik\alpha$$

where

$$k \geq \frac{1}{2} \left( a + \frac{1}{a} \right) \geq 1 \quad \text{when} \quad \arg(p(z_0)) = \frac{\pi}{2} \alpha,$$

$$k \leq \frac{1}{2} \left( a + \frac{1}{a} \right) \leq -1 \quad \text{when} \quad \arg(p(z_0)) = -\frac{\pi}{2} \alpha,$$

$$p(z_0)^{1/\alpha} = \pm ai, \quad a > 0.$$

**2. Main results**

In the first theorem we provide sufficient condition for functions to be in the class $BL(m, l, a, c, \mu, \alpha, \lambda)$

**Theorem 2.1.** Let $f \in A_n, \quad n, m \in \mathbb{N}, l, \mu, \lambda \geq 0, \alpha \in [1/2, 1)$. If

$$\frac{\lambda(l+1)I(m+2,l)f(z) - \lambda I(m+1,l)f(z) + (1-\lambda)(a+1)L(a+1,c)\mu f(z) - (1-\lambda)aL(a+1,c)f(z)}{L(m+1,l,a+1,c,\lambda)}$$

$$- \frac{\lambda(l+1)I(m+1,l)f(z) - \lambda I(m,l)f(z) + (1-\lambda)aL(a+1,c)f(z) - (1-\lambda)(a-1)L(a,c)f(z)}{L(m+1,l,a+1,c,\lambda)}$$

$$+ \mu < 1 + \frac{3\alpha - 1}{2\alpha} z, \quad z \in U \quad (2.1)$$

then $f \in BL(m, l, a, c, \mu, \alpha, \lambda)$

**Proof.** If we denote by

$$p(z) = L(m+1,l,a+1,c,\lambda)f(z) \left( \frac{z}{L(m,l,a,c,\lambda)f(z)} \right)^\mu \quad (2.2)$$

where

$$p(z) = 1 + p_1z + p_2z^2 + ..., \quad p(z) \in \mathcal{H}[1,1],$$

then, after a simple differentiation, we obtain

$$zp'(z) = \frac{\lambda(l+1)I(m+2,l)f(z) - \lambda I(m+1,l)f(z) + (1-\lambda)(a+1)L(a+2,c)\mu f(z)}{L(m+1,l,a+1,c,\lambda)} - \frac{(1-\lambda)aL(a+1,c)f(z)}{L(m+1,l,a+1,c,\lambda)} - \mu \frac{(1-\lambda)aL(a+1,c)f(z) - (1-\lambda)(a-1)L(a,c)f(z)}{L(m,l,a,c,\lambda)} - \frac{\lambda l I(m+1,l)f(z) - \lambda l I(m,l)f(z)}{L(m,l,a,c,\lambda)} - 1 + \mu$$

Using (2.1), we get

$$\text{Re} \left( 1 + \frac{zp'(z)}{p(z)} \right) > \frac{3\alpha - 1}{2\alpha}.$$
Thus, from Lemma 1.3, we get
\[ \Re \left\{ \frac{L(m + 1, l, a + 1, c, \lambda) f(z)}{z} \left( \frac{z}{L(m, l, a, c, \lambda) f(z)} \right)^\mu \right\} > \alpha. \]

Therefore, \( f \in BL(m, l, a, c, \mu, \alpha, \lambda) \), by Definition 1.1.

If we take \( a = l \) in Theorem 2.1, we obtain the following corollary:

**Corollary 2.2.** Let \( f \in A_n, n, m \in \mathbb{N}, l, \mu, \lambda \geq 0, \alpha \in [1/2, 1) \). If

\[
\frac{(l + 1)L(m + 2, l, a + 2, c, \lambda)}{L(m + 1, l, a + 1, c, \lambda)} - \mu \frac{(l + 1)L(m + 1, l, a + 1, c, \lambda)}{L(m, l, a, c, \lambda)} - l + \mu(l + 1) < \frac{3\alpha - 1}{2\alpha} z, \quad z \in U
\]

then \( f \in BL(m, l, a, c, \mu, \alpha, \lambda) \)

Next, we prove the following theorem:

**Theorem 2.3.** Let \( f(z) \in A \). If \( f(z) \in BL(m, l, l + 1, c, \mu, \alpha, \lambda) \), then

\[
\left| \arg \left( \frac{L(m, l + 1, c, \mu)}{z} \right) \right| < \frac{\pi}{2\alpha}
\]

for \( 0 < \alpha \leq 1 \) and \( 2/\pi \arctan(\alpha/(l + 1)) - \alpha(\mu - 1) = 1 \)

**Proof.** If we denote by

\[
p(z) = \frac{L(m, l, l + 1, c, \mu)}{z}
\]

where

\[
p(z) = 1 + p_1 z + p_2 z^2 + ..., \quad p(z) \in \mathbb{H}[1, 1],
\]

then, after a simple differentiation, we obtain

\[
\frac{L(m + 1, l, l + 2, c, \lambda) f(z)}{z} \left( \frac{z}{L(m, l + 1, c, \lambda) f(z)} \right)^\mu = \left( \frac{1}{p(z)} \right)^{\mu - 1} \left( 1 + \frac{1}{l + 1} \frac{zp'(z)}{p(z)} \right)
\]

Suppose there exists a point \( z_0 \in U \) such that

\[
|\arg(p(z))| < \frac{\pi}{2\alpha}, \quad |z| < |z_0|, \quad |\arg(p(z_0))| = \frac{\pi}{2\alpha}
\]

Then, from Lemma 1.4, we obtain:

- If \( \arg(p(z_0)) = \pi/2\alpha \), then

\[
\arg \left( \frac{L(m + 1, l, l + 2, c, \lambda) f(z_0)}{z_0} \left( \frac{z_0}{L(m, l + 1, c, \lambda) f(z_0)} \right)^\mu \right) = \arg \left( \frac{1}{p(z_0)} \right)^{\mu - 1} \left( 1 + \frac{1}{l + 1} \frac{zp'(z_0)}{p(z_0)} \right) = - (\mu - 1) \frac{\pi}{2\alpha} + \arg \left( 1 + \frac{1}{l + 1 + ik\alpha} \right)
\]

\[
\geq - (\mu - 1) \frac{\pi}{2\alpha} + \arctan \frac{\alpha}{l + 1} = \frac{\pi}{2} \left( \frac{2}{\pi} \arctan \frac{\alpha}{l + 1} - \alpha(\mu - 1) \right) = \frac{\pi}{2}
\]
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- If \( \arg(p(z_0)) = -\pi/2 \alpha \), then
  \[
  \arg \left( \frac{L(m+1,l,l+2,c,\lambda)f(z_0)}{z_0} \left( \frac{z_0}{L(m,l+l+1,c,\lambda)f(z_0)} \right)^\mu \right) \leq -\frac{\pi}{2}
  \]
  These contradict the assumption of the theorem.
  Thus, the function \( p(z) \) satisfy the inequality
  \[
  |\arg(p(z))| < \frac{\pi}{2} \alpha, \quad z \in U.
  \]

If we get, in Theorem 2.2, \( m = l = 0, \mu = 2 \) and \( \lambda = 1 \), we obtain the following corollary, proved by B. A. Frasin and M. Darus in [7]:

**Corollary 2.4.** Let \( f(z) \in A \). If \( f(z) \in B(\alpha) \), then
\[
\left| \arg \left( \frac{f(z)}{z} \right) \right| < \frac{\pi}{2} \alpha, \quad z \in U
\]
for some \( \alpha(0 < \alpha < 1) \) and \( (2/\pi) \tan^{-1} \alpha - \alpha = 1 \).

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**References**


Andreea-Elena Tudor
Babeș-Bolyai University
Faculty of Mathematics and Computer Sciences
1 Kogălniceanu Street, 400084 Cluj-Napoca, Romania
e-mail: tudor_andreea_elena@yahoo.com