Note on a property of the Banach spaces

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Abstract. We show that we may consider a partial ordering $\leq$ in an infinite dimensional Banach space $(X, \|\|)$, which we obtain through any normed Hamel base of the space, such that $(X, \|\|, \leq)$ is a Banach lattice.

Mathematics Subject Classification (2010): 46B20, 46B30.

Keywords: Order, norm, lattice.

1. Introduction

Why trying to see, concerning a Banach space $X$, whether there exists or not a partial ordering in $X$ that is compatible with the topology? The particular geometric properties of Banach lattices and, the contrast concerning the continuity properties of the coordinate linear functionals associated either to a Schauder basis or to a Hamel base in a Banach space ([2], Chapter 4 and [3]), we decided to consider these matters altogether. We prove in Theorem 3.1 that $(X, \|\|)$ being an infinite dimensional real Banach space and the normed vectors $x_\alpha$ ($\alpha \in \mathcal{A}$) determining a Hamel base $\mathcal{H}$ of $X$, we may consider a partial order $\leq_{\mathcal{H}}$ in $X$ such that the triple $(X, \|\|_{\mathcal{H}}, \leq_{\mathcal{H}})$ is a Banach lattice where $\|\|_{\mathcal{H}}$ is an equivalent norm to $\|\|$ in $X$. In the Preliminaries, paragraph 2., we briefly set the notations. We consider real Banach spaces $X$ and we say that a linear isomorphism which is a homeomorphism between two topological vector spaces is a linear homeomorphism ([4], II.1, p. 53 in a definition). Also in [4], we can find the algebraic Hamel base of a vector space $X$ not reducing to $\{0\}$ namely (p. 42), $\mathcal{H} = \{x_\alpha : \alpha \in \mathcal{A}\}$ is Hamel base of $X$ if $\mathcal{H}$ is an infinite linearly independent set which spans $X$, as we consider in paragraph 2.

2. Preliminaries

In what follows we consider a real Banach space $(X, \|\|)$. Recall that $(X, \leq)$ is a Riesz space through a partial order $\leq$ in $X$ if and only if $\leq$ is compatible with the linear structure that is, $x + z \leq y + z$ whenever $x \leq y$, $x, y, z \in X$, we have that $\alpha x \geq 0$ for each $x \geq 0$, $\alpha \geq 0$ where $x \in X$ and $\alpha$ is a scalar and, further, there exist $x \lor y =$
sup \{x, y\}, x \wedge y = \inf \{x, y\} for each x, y \in X. We write \((X, \|\cdot\|_\leq)\) meaning that 
\((X, \|\cdot\|)\) is a Banach space, \((X, \leq)\) is a Riesz space and \(\|x\| \leq \|y\|\) whenever \(|x| \leq |y|\), so that 
\((X, \|\cdot\|, \leq)\) (or just \(X\)) is a Banach lattice. Hence, we put \(|x| = x \vee (-x)\) We write \(x^+ = x \vee 0, x^- = x \wedge 0\). We see easily that \(x^- = (-x) \vee 0 = -(x \wedge 0)\). For \(x \wedge y = -(\langle x \rangle \wedge -(\langle y \rangle))\). We have that \(x = x^+ - x^-, \langle x \rangle = x^+ + x^-\). Notice that 
\(|x| = x \vee (-x) = (x^+ - x^-) = (y^+ - y^-) - ((-x) \vee (-y)) + y - y =
(x^+ + y^+ - x^- - y^-) - ((y - x) \vee 0) - y\) (\cite{2}, Theorem 1.1.1. i), ii), p. 3) hence for \(\leq\) a partial order compatible with the linear structure of \(X\), \(X\) is a Riesz space provided that \(x^+\) exists for each \(x \in X\).

**Definition 2.1.** (Following \cite{4}) For \(A\) a nonempty set of indices, we say that the family \((\lambda_\alpha)\) in \(\mathbb{R}^A\) is summable, \(\sum_A \lambda_\alpha = s\) if it holds that 
\(\sum_{\alpha \in A} |\lambda_\alpha| - s |\leq \varepsilon\) for each finite superset \(\alpha \in \mathcal{F}(A)\), the class of all nonempty finite subsets of \(A\), \(\varepsilon > 0\) a priori given. The family \((\lambda_\alpha)\) is said to be absolutely summable if \((|\lambda_\alpha|)\) is a summable family.

**Notation 2.2.** We let \(l_F(A) = \{(\lambda_\alpha) \in \mathbb{R}^A : \lambda_\alpha = 0\ for \ all \ \alpha \notin A\ and \ some \ A \in \mathcal{F}(A)\}\).

**Notation 2.3.** We write \(l_1(A)\) for the Banach space determined by the absolutely summable families \((\lambda_\alpha)\) equipped with the norm \(\|(\lambda_\alpha)\|_1 = \sum_A |\lambda_\alpha|\).

**Remark 2.4.** The space \(l_1(A)\) is a Banach lattice when equipped with the partial ordering \((\lambda_\alpha) \leq (\mu_\alpha)\) if and only if \(\lambda_\alpha \leq \mu_\alpha\) (\(\alpha \in A\)). \(l_1(A)\) is the completion of \((l_F(A), \|\cdot\|_1)\).

**Proof.** This follows from \cite{4}. The partial ordering is extended the obvious way.

Letting \(\{x_\alpha : \alpha \in A\}\) be a normed Hamel base of \(X\), \(\|x_\alpha\| = 1, \alpha \in A\), putting \(\sum_A s_\alpha x_\alpha \prec_H \sum_A t_\alpha x_\alpha\) if and only if \(s_\alpha \leq t_\alpha\) (\(\alpha \in A\), the finite sms are understood), we have that \((X, \prec_H)\) is a Riesz space. Notice that the linear operator \(T(\lambda_\alpha) = \sum_A \lambda_\alpha x_\alpha\) on \(l_F(A)\) to \((X, \|\cdot\|, \prec_H)\) is injective, continuous of norm 1. We may consider the linear homeomorphism \((\bar{T}/K) : (l_1(A)/K, \|\cdot\| : l_1(A)/K\|) \rightarrow (X, \|\cdot\|), \bar{T}\) for the linear extension to \(l_1(A)\) of \(T\), where \(K = Ker(\bar{T})\).

3. The results

Following \cite{1}, \((X, \|\cdot\|, \leq)\) being a Banach lattice we say that a subspace \(Y\) of \(X\) has the solid property if \(x \in Y\) whenever \(|x| \leq |y|\), \(x \wedge y \leq \|y\|\) and \(y \in Y\). \(Y\) being closed, we then may consider the partial ordering \([x] \leq [y]\) in the quotient \(X/Y\) if and only if \(y - x \in P\) where \(P = \cup \{\pi(x) : x \geq 0\}\), \(\pi(x) = [x]\), \(\pi\) for the canonical map. Clearly that \(\leq\) is compatible with the linear structure. Also \([x]^+ = [x^+]\), \((X, Y, \leq)\) is a Riesz space such that \([x] \vee [y] = [x \vee y], [x] \wedge [y] = [x \wedge y]\) and \([0] = [x]\) \((\|1\|, 14G, p. 13)\). \(X/Y\) has the solid property if \(w \in X/Y\) whenever \(w \leq v\). Hence also \(x \leq [y]\) if and only if for each \(v \in [y]\), there is some \(w \in [x]\) such that \(w \leq v\). It follows that \(\|x\| \leq \|y\|\) implies that for each \(v \in [y]\) there exists \(w \in [x]\), \(|w| \leq |v|\) hence \(\|[x] : X/Y\| \leq \|[y] : X/Y\|\), \((Y/X, \leq)\) is a Banach lattice. We see easily
that $K = Ker(\tilde{T})$ as above in the Preliminaries is a closed subspace of $l_1(A)$ having the solid property, hence $(l_1(A)/K, \leq)$ is a Banach lattice where we keep denoting the ordering in the quotient by the same symbol $\leq$.

Clearly that $\theta : (E, \|\cdot\|_E, \leq_{E}) \to (F, \|\cdot\|_F)$ being a linear homeomorphism between Banach spaces such that $E$ is a Banach lattice, putting $\theta(a) \leq_\theta \theta(b)$ if and only if $a \leq_{E} b$ in $E$ we obtain that $(F, \|\cdot\|_\theta)$ is a Riesz space. We have that $\theta(a \lor b) = \theta(a) \lor \theta(b)$ and, more generally, $\theta$ preserves the lattice operations. Further, if we put $\|\theta(a)\|_\theta = \|a : E\|$ for $\theta(a) \in F$ we have that $(F, \|\cdot\|_\theta)$ is a Banach space and it follows from the open mapping theorem that the norms $\|\cdot\|_F, \|\cdot\|_\theta$ are equivalent in $F$. Also for $| \theta(a) | \leq_\theta | \theta(b) |$ we find that $| a | \leq_{E} | b |$ hence $\|a : E\| \leq \|b : E\|$, $\|\theta(a)\|_\theta \leq \|\theta(b)\|_\theta$, we obtain that $(F, \|\cdot\|_\theta)$ is a Banach lattice.

Denoting $\theta = \tilde{T}/K : (l_1(A)/K, \|\cdot\|_{l_1(A)/K}) \to (X, \|\cdot\|)$ in the above sense (we have that each $x \in X$ is a unique image $\theta([\lambda_\alpha(x)])$, $(\lambda_\alpha(x)) \in l_1(A)$) we have

**Theorem 3.1.** The elements $\theta([\lambda_\alpha(x)]) = x$ determine the Banach space $(X, \|\cdot\|_\theta)$ where the norm $\|\cdot\|_\theta$ is equivalent to the original norm of $X$.

**Proof.** This follows from above. □

**Corollary 3.2.** Given an infinite dimensional real Banach space $(X, \|\cdot\|)$ and a normed Hamel base $\mathcal{H} = \{x_\alpha : \alpha \in A \}$ of $X$, there exist an equivalent norm $\|\cdot\|_\mathcal{H}$ in $X$ and a partial ordering $\leq_\mathcal{H}$ in $X$ associated to $\mathcal{H}$ such that the triple $(X, \|\cdot\|_\mathcal{H}, \leq_\mathcal{H})$ is a Banach lattice.

**Proof.** This follows from above theorem where we denote $\|\cdot\|_\mathcal{H} = \|\cdot\|_\theta$, $\leq_\mathcal{H} = \leq_\theta$ following the above definition. □

**Acknowledgement** This work was developed in CIMA-UE with financial support form FCT (Programa TOCTI-FEDER)

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