On skew group algebras and symmetric algebras

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Abstract. We identify and define a class of algebras which we call inv-symm algebras and prove that are principally symmetric. Two important examples are given, and we prove that the skew group algebra associated to these algebras remains inv-symm.

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1. Inv-symm algebras

Following [2] we recall the concept of an inverse semigroup and we use basic results without comments. A semigroup \((S, \cdot)\) is inverse if for any \(s \in S\) there is a unique \(\hat{s}\) (named inverse) such that \(s \cdot \hat{s} \cdot s = s\) and \(\hat{s} \cdot s \cdot \hat{s} = \hat{s}\). By [2, 1.1, Theorem 3], if \((S, \cdot)\) is inverse then all idempotents of \(S\) commutes and we have \(\hat{s} = s\) and \(\hat{s} \cdot t = \hat{t} \cdot \hat{s}\) for any \(s \in S\). We denote usually by \(k\) a commutative ring and by \(A\) a \(k\)-algebra. If \(B\) is a subset of \(A\) with \(0 \notin B\), we denote by \(B^\sharp\) the set \(B \cup \{0\}\) and by \(\text{Idemp}(B)\) the set of all idempotents of \(B\). The following definition is suggested by the ideas from [3] and by methods used to prove that the group algebra is a symmetric algebra.

Definition 1.1. A \(k\)-algebra \(A\) is inv-symm if there is a finite \(k\)-basis \(B\) such that:

1. \((B^\sharp, \cdot)\) is an inverse semigroup.
2. For \(t, s \in B\) we have \(t \cdot s \neq 0\) if and only if \(s \cdot \hat{s} = \hat{t} \cdot t\).

Example 1.2. If \(A = kG\) is the group algebra over a finite group \(G\) then the finite set \(B = G\) is a \(k\)-basis which satisfies conditions from Definition 1.1. We have in this case \(\hat{s} = s^{-1}, t \cdot s \neq 0\) and \(s \cdot \hat{s} = \hat{t} \cdot t\) for any \(t, s \in B\).

Example 1.3. If \(A = \text{End}_k(M)\), where \(M\) is a \(kG\)-lattice (that is a finitely generated, free \(k\)-module with a \(G\)-stable finite basis \(X\)), then \(B = \{b_{x,y} \mid x, y \in X\}\) with \(b_{x,y} : M \to M, b_{x,y}(z) = x\) if \(z = y\), and \(b_{x,y}(z) = 0\) if \(z \neq y\), satisfies the conditions from 1.1. It requires some computation to verify that \(b_{x,y} \circ b_{x_1,y_1} = 0\) if \(y \neq x_1\), and \(b_{x,y} \circ b_{x_1,y_1} = b_{x,y_1}\) if \(y = x_1\). We have that \(b_{x,y} \in \text{Idemp}(B)\) if and only if \(x = y\).
Remark 1.4. Moreover the above two examples are also $G$-algebras with $G$-stable basis. This suggest that we can define a class of symmetric $G$-algebras and to analyze the skew group algebra in this case.

Lemma 1.5. Let $A$ be an inv-symm $k$-algebra with basis $B$ satisfying Definition 1.1 and $t, s \in B$. The following statements are true:

a) For $0 \in B^2$ we have $\hat{0} = 0$ and $s \in B$ if and only if $\hat{s} \in B$.

b) For all $s \in B$ we have $s \cdot \hat{s} \in \text{Idemp}(B)$ and $\hat{s} \cdot s \in \text{Idemp}(B)$. Particularly Idemp($B$) \(\neq \emptyset\).

c) If $t \cdot s \neq 0$ and $t \cdot s \in \text{Idemp}(B)$ then $t = \hat{s}$.

Proof. a) For $0$ is easy to check. Let $s \in B$, then there is a unique $\hat{s} \in B^2$ with the properties of the inverse element. Suppose that $\hat{s} = 0$ then $\hat{\hat{s}} = \hat{0}$, which gives $s = 0$, a contradiction.

b) For $s \in B$ we have $\hat{s} \in B^2$ such that $s \cdot \hat{s} = s$ and $\hat{s} \cdot s \cdot \hat{s} = \hat{s}$. Now $s \cdot \hat{s} \in B$ (since if $s \cdot \hat{s} = 0 \Rightarrow s = 0 \notin B$) and $(s \cdot \hat{s}) \cdot (s \cdot \hat{s}) = (s \cdot \hat{s}) \cdot \hat{s} = s \cdot \hat{s}$.

c) Suppose that $t \cdot s \neq 0$ and $t \cdot s \in \text{Idemp}(B)$. Then $s \cdot \hat{s} = \hat{\hat{s}} \cdot t$ and $t \cdot s \cdot t \cdot s = t \cdot s$.

We multiply the last relation with $\hat{s}$ on the right and obtain

$$t \cdot s \cdot t \cdot s \cdot \hat{s} = t \cdot s \cdot \hat{s} \Rightarrow t \cdot s \cdot t \cdot \hat{s} \cdot t = t \cdot \hat{s} \cdot t \Rightarrow t \cdot s \cdot t = t.$$

Similarly we obtain $s \cdot t \cdot s = t$, thus $t = \hat{s}$.

□

From [1] we recall the definition of a symmetric algebra. A $k$-algebra $A$ is called symmetric if it is finitely generated and projective as $k$-module and there is $\tau : A \to k$ a central form (that is $k$-linear map with $\tau(a \cdot a') = \tau(a' \cdot a)$ for all $a, a' \in A$), which induces an isomorphism of $A - A$-bimodules

$$\hat{\tau} : A \to A^*, \hat{\tau}(a)(b) = \tau(a \cdot b),$$

where $a, b \in A$ and $A^*$ is the $k$-dual. $\tau$ is called symmetric form of $A$ and $A$ is principally symmetric if $\tau$ is onto.

Theorem 1.6. If $A$ is an inv-symm $k$-algebra then $A$ is principally symmetric. In particular it is symmetric.

Proof. By Definition 1.1 $A$ is a finitely generated $k$-module and free, thus projective. We define the following $k$-linear form on the basis $B$

$$\tau_B : A \to k, \quad \tau_B(s) = \begin{cases} 1_k, & s \in \text{Idemp}(B) \\ 0, & s \notin \text{Idemp}(B) \end{cases}$$

From Lemma 1.5, b) it follows that $\tau_B$ is not the zero map and $\tau_B$ is a $k$-linear form. We prove that it is a central form, that is $\tau_B(s \cdot t) = \tau_B(t \cdot s)$ where $t, s \in B$, by considering the cases:

- If $t \cdot s \neq 0$ and $t \cdot s \in \text{Idemp}(B)$, by Lemma 1.5, c) it follows that $\hat{s} = t$ and then

$$\tau_B(s \cdot \hat{s}) = 1_k = \tau_B(\hat{s} \cdot s).$$

- If $t \cdot s \neq 0$ and $t \cdot s \in B \setminus \text{Idemp}(B)$ then $\tau_B(t \cdot s) = 0$. Now, if $s \cdot t \neq 0$ and $s \cdot t \in \text{Idemp}(B)$ by Lemma 1.5, c) we get that $s = \hat{t}$, which is a contradiction with
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\(\tau_B(t \cdot s) = 0\). So we have two possibilities: \(s \cdot t = 0\), or \(s \cdot t \neq 0\) and \(s \cdot t \notin \text{Idemp}(B)\). In both subcases \(\tau_B(s \cdot t) = 0\).

- If \(t \cdot s = 0\) then \(\tau_B(t \cdot s) = 0\), and the same analyze to the second case gives us equality.

\(\tau_B\) induces the following \(A - A\)-bimodule homomorphism \(\hat{\tau}_B: A \to A^*\) defined by

\[\hat{\tau}_B(t)(s) = \tau_B(t \cdot s)\]

for any \(t, s \in B\).

First we prove that \(\hat{\tau}_B\) is injective. Let \(t_1, t_2 \in B\) such that \(\tau_B(t_1 \cdot s) = \tau_B(t_2 \cdot s)\) for any \(s \in B\). We choose \(s = \hat{t}_1\) and obtain that \(\tau_B(t_2 \cdot \hat{t}_1) = 1_k\). It follows that \(t_2 \cdot \hat{t}_1 \neq 0\) and \(t_2 \cdot \hat{t}_1 \in \text{Idemp}(B)\). By Lemma 1.5, c) we obtain that \(t_2 = \hat{t}_1 = t_1\).

For surjectivity let \(\lambda \in A^*\) and define \(a = \sum_{t \in B} \lambda(t) \cdot \hat{t} \in A\). Then for \(s \in B\)

\[\hat{\tau}_B(a)(s) = \sum_{t \in B} \lambda(t) \tau_B(\hat{t} \cdot s)\]

Since \(\tau_B(\hat{t} \cdot s) = 1_k\) if and only if \(s = t\) we obtain that

\[\hat{\tau}_B(a)(s) = \lambda(s) \cdot \tau_B(\hat{s} \cdot s) = \lambda(s)\]

This concludes the proof. \(\square\)

2. Skew group algebras

In this section we will investigate the skew group algebra associated to a \(G\)-algebra which is an inv-symm algebra, where \(G\) is a finite group. The Remark 1.4 is the starting point of the next definition.

Definition 2.1. A \(G\)-algebra \(A\) is called \(G\)-inv-symm if it is inv-symm, with the basis \(B\) (from Definition 1.1) \(G\)-stable.

It is easy to show, using Theorem 1.6, that any \(G\)-inv-symm algebra is \(G\)-permutation and principally symmetric. If \(A\) is a \(G\)-algebra we denote the action of an \(g \in G\) on \(a \in A\) by \(g a\).

Theorem 2.2. Let \(G\) be a finite group and \(A\) a \(G\)-algebra. If \(A\) is \(G\)-inv-symm then the skew group algebra, denoted \(A \star G\), is inv-symm. In particular it is principally symmetric.

Proof. We remind the definition of a skew group algebra. The skew group algebra \(A \star G\) is the free \(A\)-module of basis

\[\{a \star g \mid a \in A, g \in G\}\]

where \(a \star g\) is a notation and the product is given by

\[(a \star g)(b \star h) = a \cdot g b \star gh\]

Since \(B\) is the \(k\)-basis of \(A\) it is easy to check that the set

\[B \star G = \{s \star g \mid s \in B, g \in G\}\]
is a \( k \)-basis of the skew group algebra. Moreover it is a finite semigroup with zero, with the product defined above, since \( B \) is \( G \)-stable. Next we verify the conditions from Definition 1.1:

(1). We prove that the inverse of \( s \star g \in B \star G \) is the element

\[
\widehat{s} \star g = g^{-1} \widehat{s} \star g^{-1} \in B \star G.
\]

We have

\[
(s \star g)(g^{-1} \widehat{s} \star g^{-1})(s \star g) = (s \cdot \widehat{s} \star 1_G)(s \star g) = s \cdot \widehat{s} \cdot 1_G s \star g = s \star g.
\]

Similarly we prove the other statement. Suppose now that there is \( t \star h \in B \star G \) such that \((s \star g)(t \star h)(s \star g) = s \star g\). Then we have that

\[
(s \cdot g t \cdot gh)(s \star g) = s \star g \Rightarrow s \cdot g t \cdot gh s \star gh = s \star g.
\]

We have that \( h = g^{-1} \) and \( t = g^{-1} \widehat{s} \), thus it is unique.

(2). Let \( s \star g, t \star h \in B \star G \). We have that \((t \star h)(s \star g) \neq 0\) if and only if \( t \cdot h s \neq 0\).

We also have that

\[
(s \star g)(g^{-1} \widehat{s} \star g^{-1}) = (h^{-1} \widehat{t} \star h^{-1})(t \star h) \Leftrightarrow s \cdot \widehat{s} \star g = h^{-1} \widehat{t} \cdot h^{-1} t \star 1_G \Leftrightarrow
\]

\[
s \cdot \widehat{s} = h^{-1} (\widehat{t} \cdot t) \Leftrightarrow h s \cdot h \widehat{s} = \widehat{t} \cdot t.
\]

But since \( A \) is \( G \)-inv-symm the last condition is equivalent to \( t \cdot h s \neq 0 \), by Definition 1.1, statement(2). \( \square \)

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