Abstract. In this paper the notion of \(\mathcal{N}\)-I-ideals and \(\mathcal{N}\)-associative I-ideals in IS-algebra is introduced, as well as some of their properties are investigated. The relations between \(\mathcal{N}\)-I-ideals and \(\mathcal{N}\)-associative I-ideals are discussed. A characterization of \(\mathcal{N}\)-associative I-ideals is provided.

Mathematics Subject Classification (2010): 06F35, 03G25.

Keywords: IS-algebras, \(\mathcal{N}\)-structure, \(\mathcal{N}\)-I-ideal, \(\mathcal{N}\)-associative I-ideal.

1. Introduction

Imai and Iséki [1] in 1966 introduced the notion of a BCK-algebra. In the same year, Iséki [2] introduced BCI-algebras as a super class of the class of BCK-algebras. In 1993, Jun et al. [3] introduced a new class of algebras related to BCI-algebras and semigroups, called a BCI-semigroup/BCI-monoid/BCI-group. In 1998, for the convenience of study, Jun et al. [8] renamed the BCI-semigroup (respectively, BCI-monoid and BCI-group) as the IS-algebra (respectively, IM-algebra and IG-algebra) and studied further properties of these algebras (see [7]).

A (crisp) set \(A\) in a universe \(X\) can be defined in the form of its characteristic function \(\mu_A : X \to \{0, 1\}\) yielding the value 1 for elements belonging to the set \(A\) and the value 0 for elements excluded from the set \(A\). So far most of the generalization of the crisp set have been conducted on the unit interval \([0, 1]\) and they are consistent with the asymmetry observation. In other words, the generalization of the crisp set to fuzzy sets relied on spreading positive information that fit the crisp point \(\{1\}\) into the interval \([0, 1]\). Because no negative meaning of information is suggested, we now feel a need to deal with negative information. To do so, we also feel a need to supply mathematical tool. To attain such object, Jun et al. [5] introduced a new function which is called negative-valued function, and constructed \(\mathcal{N}\)-structures. They applied \(\mathcal{N}\)-structures to BCK/BCI-algebras, and discussed \(\mathcal{N}\)-subalgebras and \(\mathcal{N}\)-ideals in BCK/BCI-algebras. Jun et al. [6] considered closed ideals in BCH-algebras based on
\(N\)-structures. Jun et al. [4] introduced the notion of a (created) \(N\)-ideal of subtraction algebras, and investigated several characterizations of \(N\)-ideals.

In this paper, we introduced the notion of \(N\)-\(I\)-ideals and \(N\)-associative \(I\)-ideals in IS-algebras, and studied several related properties.

2. Basic results on IS-algebras

The following necessary elementary aspects of IS-algebras will be used throughout this paper.

By a BCI-algebra we mean an algebra \((X, *, 0)\) of type \((2, 0)\) satisfying the following axioms: for every \(x, y, z \in X\) [2],

\[(I)\] \((x * y) * (x * z) * (z * y) = 0,\]

\[(II)\] \((x * (x * y)) * y = 0,\]

\[(III)\] \(x * x = 0,\]

\[(IV)\] \(x * y = 0\) and \(y * x = 0\) imply \(x = y.\)

A BCI-algebra \(X\) satisfying \(0 \leq x\) for all \(x \in X\) is called a BCK-algebra. In any BCI-algebra \(X\) one can define a partial order \(\preceq\) by putting \(x \preceq y\) if and only if \(x * y = 0.\)

A BCI-algebra \(X\) has the following properties for any \(x, y, z \in X\) [2]:

\[(A1)\] \(x * 0 = x,\]

\[(A2)\] \((x * y) * z = (x * z) * y,\]

\[(A3)\] \(x \preceq y\) implies that \((x * z) \preceq (y * z)\) and \((z * y) \preceq (z * x),\]

\[(A4)\] \((x * z) * (y * z) \preceq x * y,\]

\[(A5)\] \(x * (x * (x * y)) = x * y,\]

\[(A6)\] \(0 * (x * y) = (0 * x) * (0 * y),\]

\[(A7)\] \(0 * (0 * ((x * z) * (y * z))) = (0 * y) * (0 * x).\]

A non-empty subset \(I\) of a BCI-algebra \(X\) is called an ideal of \(X\) if \((S1): 0 \in I,\)

\[(S2): x * y \in I \text{ and } y \in I \text{ imply that } x \in I.\]

A non-empty subset \(I\) of \(X\) is called an associative ideal of \(X\) if it satisfies \((S1)\) and \((S3): ((x * y) * z) \in I, (y * z) \in I \text{ imply that } x \in I.\)

**Definition 2.1.** [8]. An IS-algebra is a non-empty set \(X\) with two binary operations \("*"\) and \(\cdot\) and constant 0 satisfying the axioms

\[(B1)\] \((X, *, 0)\) is a BCI-algebra,

\[(B2)\] \((X, \cdot)\) is a semigroup,

\[(B3)\] the operation \("\cdot\) is distributive (on both sides) over the operation \("*\), that is,

\(x \cdot (y * z) = (x \cdot y) \cdot (x \cdot z)\) and \((x * y) \cdot z = (x * z) \cdot (y * z)\) for all \(x, y, z \in X.\)

Note that, the IS-algebra is a generalization of the ring (see [8]).

**Proposition 2.2.** [3]. Let \(X\) be an IS-algebra. Then we have

\[(1)\] \(0 * x = x * 0 = 0,\]

\[(2)\] \(x \preceq y\) implies that \(x \cdot z \preceq y \cdot z \text{ and } z \cdot x \preceq z \cdot y,\) for all \(x, y, z \in X.\)
Definition 3.2. [8]. A non-empty subset $A$ of an IS-algebra $X$ is called a left (resp. right) $\mathcal{I}$-ideal of $X$ if
1. $x \cdot a \in A$ (resp. $a \cdot x \in A$) whenever $x \in X$ and $a \in A$,
2. for any $x, y \in X$, $x \ast y \in A$ and $y \in A$ imply that $x \in A$.

Both a left and right $\mathcal{I}$-ideal is called $\mathcal{I}$-ideal.

Definition 3.1. Let $X$ be an IS-algebra. An $\mathcal{N}$-structure $(X, \xi)$ is called a left $\mathcal{N}$-$\mathcal{I}$-ideal (resp. a right $\mathcal{N}$-$\mathcal{I}$-ideal) of $X$ if
1. $(\xi(xy) \leq \xi(y))$ (resp. $\xi(xy) \leq \xi(x))$ for all $x, y \in X$;
2. $\xi(x) \leq \max \{\xi(x \ast y), \xi(y)\}$ for all $x, y \in X$.

An $\mathcal{N}$-structure $(X, \xi)$ is called an $\mathcal{N}$-$\mathcal{I}$-ideal of $X$ if it is both a left $\mathcal{N}$-$\mathcal{I}$-ideal and a right $\mathcal{N}$-$\mathcal{I}$-ideal of $X$.

Definition 3.2. Let $X$ be an IS-algebra. An $\mathcal{N}$-structure $(X, \xi)$ is called a left $\mathcal{N}$-associative $\mathcal{I}$-ideal (resp. a right $\mathcal{N}$-associative $\mathcal{I}$-ideal) of $X$ if it satisfies (C1) and (C3)
1. $\xi(x) \leq \max \{\xi((x \ast y) \ast z), \xi(y \ast z)\}$ for all $x, y, z \in X$.

An $\mathcal{N}$-structure $(X, \xi)$ is called an $\mathcal{N}$-associative $\mathcal{I}$-ideal of $X$ if it is both a left $\mathcal{N}$-associative $\mathcal{I}$-ideal and a right $\mathcal{N}$-associative $\mathcal{I}$-ideal of $X$.

Example 3.3. Consider an IS-algebra $X = \{0, a, b, c\}$ with Cayley tables as follows:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>b</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>c</td>
<td>b</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>b</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>b</td>
<td>a</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>0</td>
<td>b</td>
<td>b</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
</tbody>
</table>

(1) Let $(X, \xi)$ be an $\mathcal{N}$-structure in which $\xi$ is given by
\[ \xi = \begin{pmatrix} 0 & a & b & c \\ t_0 & t_1 & t_0 & t_1 \end{pmatrix}, \] where $t_0 < t_1$ in $[-1, 0]$.
Then $(X, \xi)$ is an $\mathcal{N}$-$\mathcal{I}$-ideal of $X$.
(2) Let $(X, \zeta)$ be an $\mathcal{N}$-structure in which $\zeta$ is given by
\[ \zeta = \begin{pmatrix} 0 & a & b & c \\ t_0 & t_1 & t_0 & t_1 \end{pmatrix}, \] where $t_0 < t_1$ in $[-1, 0]$.
Then \((X, \zeta)\) is an \(\mathcal{N}\)-associative \(\mathcal{I}\)-ideal of \(X\).

**Proposition 3.4.** Every left (resp. right) \(\mathcal{N}\)-associative \(\mathcal{I}\)-ideal \((X, \xi)\) satisfies the following inequality:

\[
(\forall x \in X) \ (\xi(0) \leq \xi(x)) \tag{3.1}
\]

**Theorem 3.5.** Every left (resp. right) \(\mathcal{N}\)-associative \(\mathcal{I}\)-ideal is a left (resp. right) \(\mathcal{N}\)-\(\mathcal{I}\)-ideal.

**Proof.** Let \((X, \xi)\) be a left (resp. right) \(\mathcal{N}\)-associative \(\mathcal{I}\)-ideal of \(X\). Then, \(\xi(xy) \leq \xi(y)\) (resp. \(\xi(xy) \leq \xi(x)\)) for all \(x, y \in X\). Now, let \(z = 0\) in (C3), we have \(\xi(x) \leq \max \{\xi((x \cdot y) \cdot 0), \xi(y \cdot 0)\}\) for all \(x, y \in X\). So, \(\xi(x) \leq \max \{\xi((x \cdot y)), \xi(y)\}\). Therefore, \((X, \xi)\) is a left (resp. right) \(\mathcal{N}\)-\(\mathcal{I}\)-ideal of \(X\). \(\square\)

The next example shows that the converse of Theorem 3.5 is not always true.

**Example 3.6.** Consider the \(\mathcal{N}\)-\(\mathcal{I}\)-ideal \((X, \xi)\) given in Example 3.3. By routine calculations, it is easy to check that \((X, \xi)\) is not an \(\mathcal{N}\)-associative \(\mathcal{I}\)-ideal of \(X\).

**Proposition 3.7.** Every left (resp. right) \(\mathcal{N}\)-associative \(\mathcal{I}\)-ideal \((X, \xi)\) satisfies the following inequality:

\[
(\forall x, y \in X) \ (\xi(x) \leq \xi((x \cdot y) \cdot y)) \tag{3.2}
\]

**Proof.** Let \((X, \xi)\) be a left (resp. right) \(\mathcal{N}\)-associative \(\mathcal{I}\)-ideal of \(X\). If we let \(z := y\) in (C3), then we have \(\xi(x) \leq \max \{\xi((x \cdot y) \cdot y), \xi(y \cdot y)\}\) for all \(x, y \in X\). Using 3.1 and (III), it follows that, \(\xi(x) \leq \xi((x \cdot y) \cdot y)\) for all \(x, y \in X\). \(\square\)

**Proposition 3.8.** If \((X, \xi)\) is a left (resp. right) \(\mathcal{N}\)-associative \(\mathcal{I}\)-ideal of \(X\), then

\[
(\forall x, y \in X) \ (x \leq y \Rightarrow \xi(x) \leq \xi(y)) \tag{3.3}
\]

**Proof.** Let \(x, y \in X\) be such that \(x \leq y\). If we let \(z := 0\) in (C3), then we have \(\xi(x) \leq \max \{\xi((x \cdot y) \cdot 0), \xi(y \cdot 0)\}\) for all \(x, y \in X\). Since, \(x \leq y\) implies \(x \cdot y = 0\), \(\xi(x) \leq \max \{\xi(0 \cdot 0), \xi(y \cdot 0)\}\). It follows from axiom (III) and (A1) that \(\xi(x) \leq \xi(y)\). \(\square\)

**Proposition 3.9.** Let \((X, \xi)\) be a left (resp. right) \(\mathcal{N}\)-\(\mathcal{I}\)-ideal of \(X\). Then, \(x \cdot y \leq z\) implies \(\xi(x) \leq \max \{\xi(z), \xi(y)\}\) for all \(x, y, z \in X\).

**Theorem 3.10.** Let \((X, \xi)\) be a left (resp. right) \(\mathcal{N}\)-associative \(\mathcal{I}\)-ideal of \(X\). Then, for any \(x, y, z \in X\),

1. \(x \cdot y \leq z\) implies \(\xi(x) \leq \xi(y \cdot z)\).
2. \(\xi(x) \leq \xi(0 \cdot x)\).
3. \(\xi((x \cdot y) \cdot (x \cdot z)) \leq \xi(y \cdot z)\) (resp. \(\xi((x \cdot z) \cdot (y \cdot z)) \leq \xi(x \cdot y)\)).
\(\mathcal{N}\)-structures applied to associative-\(\mathcal{I}\)-ideals in IS-algebras

**Proof.** (i) Suppose that \((X, \xi)\) is a left (resp. right) \(\mathcal{N}\)-associative \(\mathcal{I}\)-ideal of \(X\), by (C3) we have \(\xi(x) \leq \max \{ \xi((x \ast y) \ast w), \xi(y \ast w) \}\) for all \(x, y, w \in X\). Since, \(x \ast y \leq z\) implies \((x \ast y) \ast w \leq z \ast w\), by (3.3), it follows that \(\xi((x \ast y) \ast w) \leq \xi(z \ast w)\). Hence, \(\xi(x) \leq \max \{ \xi(z \ast w), \xi(y \ast w) \}\). If we let \(w = z\), then we have, \(\xi(x) \leq \max \{ \xi(0), \xi(y \ast z) \}\).

(ii) Let \(z = x \ast y\) in (C3), then

\[\xi(x) \leq \max \{ \xi(0), \xi(y \ast (x \ast y)) \} = \xi(y \ast (x \ast y)) \quad (3.4)\]

If we let \(y = 0\) in (3.4), then we obtain also

\[\xi(x) \leq \xi(0 \ast (x \ast 0)) = \xi(0 \ast x) \quad \text{by (A1)}\]

(iii) It follows directly from (B3) and (C1).

**Definition 3.11.** [5]. Let \((X, \xi)\) and \((X, \zeta)\) be two \(\mathcal{N}\)-structures.

1. The union, \(\xi \cup \zeta\) of \(\xi\) and \(\zeta\) is defined by \((\xi \cup \zeta)(x) = \max \{ \xi(x), \zeta(x) \}\) for all \(x \in X\).
2. The intersection, \(\xi \cap \zeta\) of \(\xi\) and \(\zeta\) is defined by \((\xi \cap \zeta)(x) = \min \{ \xi(x), \zeta(x) \}\) for all \(x \in X\).

Obviously, \((X, \xi \cup \zeta)\) and \((X, \xi \cap \zeta)\) are \(\mathcal{N}\)-structures which are called the union and the intersection of \((X, \xi)\) and \((X, \zeta)\), respectively.

**Proposition 3.12.** If \((X, \xi)\) and \((X, \zeta)\) are left (resp. right) \(\mathcal{N}\)-associative \(\mathcal{I}\)-ideals of \(X\), then the union \((X, \xi \cup \zeta)\) is a left (resp. right) \(\mathcal{N}\)-associative \(\mathcal{I}\)-ideal of \(X\).

Now, we give an example to show that the intersection of two \(\mathcal{N}\)-\(\mathcal{I}\)-ideals may not be an \(\mathcal{N}\)-\(\mathcal{I}\)-ideal.

**Example 3.13.** Consider the two \(\mathcal{N}\)-\(\mathcal{I}\)-ideals \((X, \xi)\) and \((X, \zeta)\) given in Example 3.3. The intersection \(\xi \cap \zeta\) is given by

\[\xi \cap \zeta = \begin{pmatrix} 0 & a & b & c \\ t_0 & t_0 & t_0 & t_1 \end{pmatrix}, \text{ where } t_0 < t_1 \text{ in } [-1,0].\]

\(\xi \cap \zeta\) is not an \(\mathcal{N}\)-\(\mathcal{I}\)-ideal of \(X\), since \((\xi \cap \zeta)(c) = t_1 \nless \max \{ (\xi \cap \zeta)(c \ast b), (\xi \cap \zeta)(b) \}\) = \(t_0\).

For any \(\mathcal{N}\)-function \(\xi\) on \(X\) and \(t \in [-1,0]\), define the set \(C(\xi, t)\) as

\[C(\xi, t) = \{ x \in X \mid \xi(x) \leq t \}.\]

**Theorem 3.14.** An \(\mathcal{N}\)-structure \((X, \xi)\) is a left (resp. right) \(\mathcal{N}\)-associative \(\mathcal{I}\)-ideal of \(X\) if and only if every non-empty set \(C(\xi, t)\) is a left (resp. right) associative \(\mathcal{I}\)-ideal of \(X\) for all \(t \in [-1,0]\).

**Proof.** Assume that \((X, \xi)\) is a left (resp. right) \(\mathcal{N}\)-associative \(\mathcal{I}\)-ideal of \(X\) and let \(t \in [-1,0]\) be such that \(C(\xi, t) \neq \emptyset\). Let \(x \in X\) and \(\alpha \in C(\xi, t)\). Then, \(\xi(\alpha) \leq t\). It follows from (C1) that \(\xi(x \cdot \alpha) \leq \xi(\alpha) \leq t\) (resp. \(\xi(\alpha \cdot x) \leq \xi(\alpha) \leq t\)). Hence, \(x \cdot \alpha \in C(\xi, t)\) (resp. \(\alpha \cdot x \in C(\xi, t)\)). Now, let \((x \ast y) \ast z \in C(\xi, t)\) and \((y \ast z) \in C(\xi, t)\). Then, \(\xi((x \ast y) \ast z) \leq t\).
and \( \xi(y \ast z) \leq t \). Using (C3) we obtain, \( \xi(x) \leq \max \{ \xi((x \ast y) \ast z), \xi(y \ast z) \} \leq t \). Thus \( x \in C(\xi, t) \). Therefore, \( C(\xi, t) \) is a left (resp. right) associative \( I \)-ideal of \( X \) for all \( t \in [-1, 0) \).

Conversely, suppose that every non-empty set \( C(\xi, t) \) is a left (resp. right) associative \( I \)-ideal of \( X \) for all \( t \in [-1, 0) \). If there are \( a, b \in X \) such that \( \xi(a \cdot b) > \xi(b) \) (resp. \( \xi(a \cdot b) > \xi(a) \) then, \( \xi(a \cdot b) > t_0 \geq \xi(b) \) (resp. \( \xi(a \cdot b) > t_0 \geq \xi(a) \)) for some \( t_0 \in [-1, 0) \). Hence, \( b \in C(\xi, t_0) \) (resp. \( a \in C(\xi, t_0) \)) and \( a \cdot b \notin C(\xi, t_0) \). This is a contradiction.

Thus, \( \xi(x \cdot y) \leq \xi(y) \) (resp. \( \xi(x \cdot y) \leq \xi(x) \)) for all \( x, y \in X \). Now, assume that there exist \( a, b, c \in X \) such that \( \xi(a) > \max \{ \xi((a \ast b) \ast c), \xi(b \ast c) \} \). Then, \( \xi(a) > t_1 \geq \max \{ \xi((a \ast b) \ast c), \xi(b \ast c) \} \) for some \( t_1 \in [-1, 0) \). Hence, \( (a \ast b) \ast c, b \ast c \in C(\xi, t_1) \) and \( a \notin C(\xi, t_1) \), which is a contradiction. Therefore, \( (X, \xi) \) is a left (resp. right) \( N \)-associative \( I \)-ideal of \( X \). □

**Theorem 3.15.** Let \( A \) be a left (resp. right) associative \( I \)-ideal of \( X \) and let \( (X, \xi) \) be an \( N \)-structure in \( X \) defined by

\[
\xi(x) = \begin{cases} 
t_0 & \text{if } x \in A \\
t_1 & \text{otherwise}
\end{cases}
\]

where \( t_0 < t_1 \in [-1, 0) \). Then, the \( N \)-structure \( (X, \xi) \) is a left (resp. right) \( N \)-associative \( I \)-ideal of \( X \).

**Proof.** It follows directly from Theorem 3.14. □

For any \( N \)-structure \( (X, \xi) \) and any element \( w \in X \), consider the set

\[
D_w := \{ x \in X \mid \xi(x) \leq \xi(w) \}.
\]

Then, \( D_w \) is non-empty subset of \( X \).

**Theorem 3.16.** If an \( N \)-structure \( (X, \xi) \) is a left (resp. right) \( N \)-associative \( I \)-ideal of \( X \), then \( D_w \) is a left (resp. right) associative \( I \)-ideal of \( X \) for all \( w \in X \).

**Proof.** Let \( a \in D_w \) and \( x \in X \). Then, \( \xi(a) \leq \xi(w) \). By (C1) it follows that \( \xi(x \cdot a) \leq \xi(a) \leq \xi(w) \) (resp. \( \xi(a \cdot x) \leq \xi(a) \leq \xi(w) \)). Hence \( x \cdot a \in D_w \) (resp. \( a \cdot x \in D_w \)). Now, let \( x, y, z \in X \) be such that \( (x \ast y) \ast z \in D_w \) and \( y \ast z \in D_w \). Then, \( \xi((x \ast y) \ast z) \leq \xi(w) \) and \( \xi(y \ast z) \leq \xi(w) \). By (C3) it follows that \( \xi(x) \leq \max \{ \xi((x \ast y) \ast z), \xi(y \ast z) \} \leq \xi(w) \). Hence, \( x \in D_w \). Therefore, \( D_w \) is a left (resp. right) associative \( I \)-ideal of \( X \) for all \( w \in X \). □

**References**


$\mathcal{N}$-structures applied to associative-$\mathcal{I}$-ideals in IS-algebras


Ali H. Handam
Department of Mathematics
Al al-Bayt University
P.O. Box: 130095, Al Mafraq, Jordan
e-mail: ali.handam@windowslive.com