Some results on the solutions of a functional-integral equation

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Abstract. In this paper we give existence, uniqueness, data dependence and comparison theorems for the solutions of a functional-integral equation of the same type as that considered by L. Olszowy [6]. We apply some results from Picard and weakly Picard operators’ theory (see I.A. Rus, [7]).

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1. Introduction

The fixed point theory has a lot of applications in the field of functional-differential equations (see for example [1]-[6], [8]). In the paper [6] has been given theorems on the existence and asymptotic characterization of the solutions of the following problem:

\[ y'(t) = f(t, y(H(t)), y'(h(t))), t \in [0, \infty) \]  (1.1)
\[ y(0) = 0. \]  (1.2)

Technique linking measures of noncompactness with the Tichonov’ fixed point principle in suitable Fréchet space was used.

As it was shown in [6], the problem (1.1)+(1.2) is equivalent with the following functional-integral equation:

\[ x(t) = f(t, \int_0^{H(t)} x(s)ds, x(h(t))), t \in [0, \infty) \]  (1.3)

The aim of this paper is to give existence, uniqueness, data dependence and comparison theorems for the solutions of a functional-integral equation
of the same type as that considered in [6]. We apply some results from Picard and weakly Picard operators’ theory (see [7] and [8]).

2. Weakly Picard operators

Here, first we present some notions and results from the weakly Picard operators’ theory.

Let \((X, d)\) be a metric space and \(A : X \to X\) an operator.

We denote by \(A^0 := 1_X, A^1 := A, ..., A^{n+1} := A \circ A^n, n \in \mathbb{N}\), the iterate operators of the operator \(A\). Also:

\[
P(X) := \{Y \subset X \mid Y \neq \emptyset\},
I(A) := \{Y \in P(X) \mid A(Y) \subset Y\},
\]

the family of all nonempty invariant subsets of \(A\),

\[
F_A = \{x \in X \mid A(x) = x\},
\]

the fixed point set of the operator \(A\).

Following Rus I.A. [7] and [8], we have:

**Definition 2.1.** The operator \(A\) is a Picard operator if there exists \(x^* \in X\) such that

1) \(F_A = \{x^*\} \);

2) the successive approximation sequence \((A^n(x_0))_{n \in \mathbb{N}}\) converges to \(x^*\),

for all \(x_0 \in X\).

**Definition 2.2.** \(A\) is a weakly Picard operator if the sequence \((A^n(x_0))_{n \in \mathbb{N}}\) converges for all \(x_0 \in X\) and the limit (which generally depends on \(x_0\)) is a fixed point of \(A\).

**Definition 2.3.** For an weakly Picard operator \(A : X \to X\) we define the operator \(A^{\infty}\) as follows:

\[
A^{\infty} : X \to X, \quad A^{\infty}(x) := \lim_{n \to \infty} A^n(x), \text{ for all } x \in X.
\]

**Remark 2.4.** \(A^{\infty}(X) = F_A\).

We have

**Theorem 2.5. (Data dependence theorem)** Let \((X, d)\) be a complete metric space and \(A, B : X \to X\) two operators. We suppose that:

(i) \(A\) is an \(\alpha\)-contraction and let \(F_A = \{x^*_A\}\);
(ii) \(F_B \neq \emptyset\) and let \(x^*_B \in F_B\);
(iii) there exists \(\delta > 0\), such that \(d(A(x), B(x)) \leq \delta\), for all \(x \in X\).

Then

\[
d(x^*_A, x^*_B) \leq \frac{\delta}{1 - \alpha}.
\]
Theorem 2.6. (Characterization theorem) Let \((X, d)\) be a metric space and \(A : X \to X\) an operator. The operator \(A\) is a weakly Picard operator if and only if there exists a partition of \(X\), \(X = \bigcup_{\lambda \in \Lambda} X_\lambda\), such that:

(i) \(X_\lambda \in I(A)\);
(ii) \(A|_{X_\lambda} : X_\lambda \to X_\lambda\) is a Picard operator, for all \(\lambda \in \Lambda\).

Lemma 2.7. Let \((X, \leq)\) be an ordered metric space and \(A : X \to X\) an operator. We suppose that:

(i) \(A\) is a weakly Picard operator;
(ii) \(A\) is increasing.
Then the operator \(A^\infty\) is increasing.

Lemma 2.8. (Abstract Gronwall lemma) Let \((X, \leq)\) be an ordered metric space and \(A : X \to X\) an operator. We suppose that:

(i) \(A\) is a Picard operator;
(ii) \(A\) is increasing.
If we denote by \(x^*_A\) the unique fixed point of \(A\), then:

(a) \(x \leq A(x)\) implies \(x \leq x^*_A\);
(b) \(x \geq A(x)\) implies \(x \geq x^*_A\).

Lemma 2.9. (Abstract comparison lemma) Let \((X, \leq)\) be an ordered metric space and the operators \(A, B, C : X \to X\) be such that:

(i) \(A \leq B \leq C\);
(ii) the operators \(A, B, C\) are weakly Picard operators;
(iii) the operator \(B\) is increasing.
Then \(x \leq y \leq z\) implies \(A^\infty(x) \leq B^\infty(y) \leq C^\infty(z)\).

3. Existence, uniqueness and data dependence results

Let us consider the following functional-integral equation:

\[ x(t) = \alpha + f(t, \int_0^{g(t)} x(s)ds, x(h(t)))) , t \in [0, T] \] (3.1)

under the following assumptions:

\((A_1)\) \(f \in C([0, T] \times \mathbb{R}^2)\);

\((A_2)\) \(g, h \in C([0, T], [0, T])\) and \(g(t) \leq t, h(t) \leq t\), for all \(t \in [0, T]\);

\((A_3)\) \(\alpha \in \mathbb{R}\) and \(f(0, 0, \alpha) = 0\);

\((A_4)\) there exists \(k_1 > 0\) and \(0 < k_2 < 1\), such that

\[ |f(t, u_1, v_1) - f(t, u_2, v_2)| \leq k_1 |u_1 - u_2| + k_2 |v_1 - v_2| , \]

for all \(t \in [0, T]\) and all \(u_i, v_i \in \mathbb{R}, i = 1, 2\).

We have

Theorem 3.1. If all the conditions \((A_1) - (A_4)\) are satisfied, then the equation (3.1) has in \(C[0, T]\) a unique solution.
Proof. On $C[0,T]$, we consider a Bielecki norm $\| \cdot \|_\tau$, defined by

$$
\|x\|_\tau = \max_{t \in [0,T]} |x(t)| e^{-\tau t},
$$

where $\tau > 0$, and the operator

$$
A : (C[0,T], \| \cdot \|_\tau) \to (C[0,T], \| \cdot \|_\tau),
$$

defined by

$$
A(x)(t) := \alpha + f(t, \int_0^t x(s)ds, x(h(t))), t \in [0,T].
$$

So, we have a fixed point equation:

$$
x = A(x).
$$

Let $x, z \in C[0,T]$ be. We obtain

$$
|A(x)(t) - A(z)(t)| =
$$

$$
= |f(t, \int_0^t x(s)ds, x(h(t))) - f(t, \int_0^t z(s)ds, z(h(t)))| \leq
$$

$$
\leq k_1 \int_0^t |x(s) - z(s)| e^{-\tau s} e^{\tau t} ds + k_2 |x(h(t)) - z(h(t))| e^{-\tau h(t)} e^{\tau h(t)} \leq
$$

$$
\leq (k_1 \int_0^t e^{\tau t} ds + k_2 e^{\tau h(t)} t) \|x - z\|_\tau \leq
$$

$$
\leq (k_1 \int_0^t e^{\tau t} ds + k_2 e^{\tau h(t)}) \|x - z\|_\tau \leq
$$

$$
\leq \left( \frac{k_1}{\tau} + k_2 \right) e^{\tau t} \|x - z\|_\tau,
$$

for all $t \in [0,T]$.

So,

$$
|A(x)(t) - A(z)(t)| e^{-\tau t} \leq \left( \frac{k_1}{\tau} + k_2 \right) \|x - z\|_\tau,
$$

for all $t \in [0,T]$.

It follows that

$$
\|A(x) - A(z)\|_\tau \leq \left( \frac{k_1}{\tau} + k_2 \right) \|x - z\|_\tau,
$$

for all $x, z \in C[0,T]$.

We choose $\tau$ large enough, such that $\frac{k_1}{\tau} + k_2 < 1$. By applying Contraction mapping principle, we obtain that $A$ is a Picard operator. \qed
Now, together with (3.1), we consider the following equation:

$$x(t) = \alpha + F(t, \int_0^{g(t)} x(s)ds, x(h(t))), t \in [0, T],$$  \hfill (3.2)

where $F \in C([0, T] \times \mathbb{R}^2)$ and $\alpha, g, h$ are the same as in (3.1).

We have

**Theorem 3.2.** We suppose that:

(i) the conditions $(A_1)-(A_4)$ are satisfied and $x^* \in C[0, T]$ is the unique solution of the equation (3.1);

(ii) the equation (3.2) has solutions in $C[0, T]$ and $z^* \in C[0, T]$ is a solution of (3.2);

(iii) there exists $\eta > 0$ such that

$$|f(t, u, v) - F(t, u, v)| \leq \eta, \text{ for all } t \in [0, T] \text{ and all } u, v \in \mathbb{R}.$$

Then

$$||x^* - z^*||_{\tau} \leq \frac{\eta}{1 - (k_1 + k_2)},$$

where $\tau$ is large enough such that $k_1 + k_2 < 1$.

**Proof.** Consider

$$A_F : (C[0, T], || \cdot ||_{\tau}) \to (C[0, T], || \cdot ||_{\tau}),$$

$$A_F(x)(t) := \alpha + F(t, \int_0^{g(t)} x(s)ds, x(h(t))), t \in [0, T],$$

the corresponding operator of (3.2).

We have

$$|A(x)(t) - A_F(x)(t)| \leq \eta,$$

for all $t \in [0, T]$, and consequently

$$||A(x) - A_F(x)||_{\tau} \leq \eta,$$

for all $x \in C[0, T]$. \hfill $\square$

Now, we apply Data dependence theorem (Theorem 2.5).

**Theorem 3.3.** We suppose that:

(i) the conditions $(A_1)-(A_4)$ are satisfied and $x^* \in C[0, T]$ is the unique solution of the equation (3.1);

(ii) $u_i, v_i \in \mathbb{R}, i = 1, 2$ and $u_1 \leq u_2, v_1 \leq v_2$ implies $f(t, u_1, v_1) \leq f(t, u_2, v_2)$, for all $t \in [0, T]$.

Then

$$x \leq A(x) \text{ implies } x \leq x^*$$

and

$$x \geq A(x) \text{ implies } x \geq x^*.$$

**Proof.** The operator $A$ is a Picard operator and $A$ is increasing. So, we apply Abstract Gronwall lemma (Lemma 2.8). \hfill $\square$
4. Comparison results

Consider the following functional-integral equation:
\[
x(t) = x(0) + f(t, \int_0^{g(t)} x(s, x(h(t))) \, ds), \quad t \in [0, T].
\]
(4.1)

The corresponding operator,
\[
A_f : (C[0, T], \| \cdot \|_\tau) \to (C[0, T], \| \cdot \|_\tau),
\]
\[
A_f (x)(t) := x(0) + f(t, \int_0^{g(t)} x(s) \, ds, x(h(t))), \quad t \in [0, T],
\]
is a continuous operator but it isn’t a contraction.

We denote
\[
S_f = \{ \alpha \in \mathbb{R} / f(0, 0, \alpha) = 0 \} \quad \text{and} \quad X_\alpha := \{ x \in C[0, T] / x(0) = \alpha \}.
\]
Then
\[
\bigcup_{\alpha \in S_f} X_\alpha \quad \text{is a partition of} \quad C[0, T]
\]
and \( X_\alpha \) is an invariant subset of \( A_f \) if and only if \( \alpha \in S_f \).

We have

**Theorem 4.1.** We suppose that:

(i) the conditions \((A_1) - (A_4)\) are satisfied for (4.1);
(ii) \( S_f \neq \emptyset \).

Then
\[
A_f |_{\bigcup_{\alpha \in S_f} X_\alpha} : \bigcup_{\alpha \in S_f} X_\alpha \to \bigcup_{\alpha \in S_f} X_\alpha
\]
is a weakly Picard operator and \( \text{card } F_{A_f} = \text{card } S_f \).

**Proof.** By using the result of Theorem 3.1, we have that
\[
A_f |_{X_\alpha} : X_\alpha \to X_\alpha \quad \text{is a Picard operator, for all} \quad \alpha \in S_f.
\]

So, we apply Characterization theorem of the weakly Picard operators (Theorem 2.6). \( \square \)

**Remark 4.2.** If the conditions \((A_1) - (A_4)\) are satisfied and \( S_f = \{ \alpha^* \} \), then the equation (4.1) has in \( C[0, T] \) a unique solution.

We have

**Theorem 4.3.** We suppose that:

(i) all the conditions of Theorem 4.1 are satisfied;
(ii) \( u_i, v_i \in \mathbb{R}, i = 1, 2 \) and \( u_1 \leq u_2, v_1 \leq v_2 \) implies \( f(t, u_1, v_1) \leq f(t, u_2, v_2) \), for all \( t \in [0, T] \).

Let \( x^* \) be a solution of the equation (4.1) and \( x^{**} \) a solution of the following inequality:
\[
x(t) \leq x(0) + f(t, \int_0^{g(t)} x(s) \, ds, x(h(t))), \quad t \in [0, T].
\]

Then
\[
x^{**}(0) \leq x^*(0) \quad \text{implies} \quad x^{**} \leq x^*.
\]
Proof. We remark that

\[ x^* = A_f(x^*) \quad \text{and} \quad x^{**} \leq A_f(x^{**}). \]

From Lemma 2.7 and the condition \((ii)\) we have that the operator \(A_f^\infty\) is increasing. If \(\beta \in \mathbb{R}\) then we consider \(\tilde{\beta} \in \mathcal{C}[0,T]\) defined by \(\tilde{\beta}(t) = \beta\), for all \(t \in [0,T]\). By using the previous considerations and because the operator \(A_f^\infty\) is increasing, we obtain:

\[ x^{**} \leq A_f^\infty(x^{**}(0)) = A_f^\infty(x^{**}(0)) \leq A_f^\infty(x^{*}(0)) = x^*. \]

\[ \square \]

Now, we consider the following functional-integral equations:

\[ x(t) = x(0) + f_i(t, \int_0^{\sigma(t)} x(s, x(h(t)))) \quad t \in [0,T], \quad (4.2) \]

where \(g, h\) are the same in all three equations.

We have

**Theorem 4.4.** We suppose that:

(i) the corresponding conditions of Theorem 4.1 are satisfied for all equations \((4.2)\);

(ii) \(f_2(t, \cdot, \cdot) : \mathbb{R}^2 \to \mathbb{R}\) is increasing for all \(t \in [0,T]\);

(iii) \(f_1 \leq f_2 \leq f_3\).

Let \(x_i^*\) be a solution of the corresponding equation \((4.2)\), \(i = \overline{1,3}\). Then

\[ x_1^*(0) \leq x_2^*(0) \leq x_3^*(0) \quad \text{implies} \quad x_1^* \leq x_2^* \leq x_3^*. \]

**Proof.** First we remark that the operators \(A_{f_i}, i = \overline{1,3}\) are weakly Picard operators (Theorem 4.1). From \((ii)\) we have that the operator \(A_{f_2}\) is increasing. From the condition \((iii)\) we have that \(A_{f_1} \leq A_{f_2} \leq A_{f_3}\). On the other hand, \(x_i^* = A_{f_i}^\infty(x_i^*(0)), i = \overline{1,3}\). Now, the proof follows from Abstract comparison lemma (Lemma 2.9).

\[ \square \]

**References**


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