A class of uniformly convex functions involving a differential operator

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Abstract. The main purpose of this paper is to introduce a new class \( \mathcal{UH}(\alpha, \beta, \gamma, \lambda, k) \), of functions which are analytic in the open disc \( \Delta = \{ z \in \mathbb{C} : |z| < 1 \} \). We obtain various results including characterization, coefficients estimates, distortion and covering theorems, radii of close-to-convexity, starlikeness and convexity for functions belonging to the class \( \mathcal{UH}(\alpha, \beta, \gamma, \lambda, k) \).

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1. Introduction and motivations

Let \( \mathcal{A} \) denote the class of functions of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n,
\]

that are analytic in the open unit disc \( \Delta := \{ z \in \mathbb{C} : |z| < 1 \} \). Let \( \mathcal{S} \) be a subclass of \( \mathcal{A} \) consisting of univalent functions in \( \Delta \). By \( \mathcal{K}(\beta) \), and \( \mathcal{S}^*(\beta) \) respectively, we mean the classes of analytic functions that satisfy the analytic conditions

\[
\text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \beta \quad \text{and} \quad \text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \beta, \quad z \in \Delta
\]

for \( 0 \leq \beta < 1 \). In particular, \( \mathcal{K} = \mathcal{K}(0) \) and \( \mathcal{S}^* = \mathcal{S}^*(0) \) respectively, are the well-known standard class of convex and starlike functions.

The function \( f \in \mathcal{A} \) is said to be close-to-convex of order \( \beta, \beta \geq 0 \), with respect to a starlike function \( g \) and \( \phi \in \mathbb{R} \) if

\[
\left| \arg e^{i\phi} \frac{f(z)}{g(z)} \right| \leq \beta \frac{\pi}{2}, \quad z \in \Delta.
\]

Let \( \mathcal{CC}(\beta) \) denote the union of all such close-to-convex functions of order \( \beta \).
Let $T$ denote the subclass of $S$ of functions of the form
\[ f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0, \] (1.1)
that are analytic in the open unit disk $\Delta$. This class was introduced and studied in [9]. Analogous to the subclasses $S^*(\beta)$ and $K(\beta)$ of $S$ respectively, the subclasses of $T$ denoted by $T^*(\beta)$ and $C(\beta)$, $0 \leq \beta < 1$, were also investigated in [9].

The main class which we investigate in this present paper uses the operator known as the Cho-Srivastava operator. In fact, one important concept that is useful in discussing this operator is the convolution or Hadamard product. Here by convolution we mean the following: For $f, g$ analytic with $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ and $g(z) = b_0 + b_1 z + b_2 z^2 + \cdots$, the (Hadamard) convolution of $f$ and $g$ is defined by $(f * g)(z) = a_0 b_0 + a_1 b_1 z + a_2 b_2 z^2 + \cdots$. It is natural to use the notation $f(z) * g(z)$ for $(f * g)(z)$ and vice versa frequently.

For functions $f \in A$, we recall the multiplier transformation $I(\lambda, k)$ introduced by Cho and Srivastava [3] defined as
\[ I(\lambda, k)f(z) = z + \sum_{n=2}^{\infty} \Psi_n a_n z^n \quad (\lambda \geq 0; \ k \in \mathbb{Z}) \] (1.2)
where
\[ \Psi_n := \left( \frac{n + \lambda}{1 + \lambda} \right)^k \] (1.3)
so that, obviously,
\[ I(\lambda, k)(I(\lambda, m)f(z)) = I(\lambda, k + m)f(z) \quad (k, m \in \mathbb{Z}). \] (1.4)

For $\lambda = 1$, the operators $I(\lambda, k)$ were studied by Uralegaddi and Somanatha [12]. The operators $I(\lambda, k)$ are closely related to the multiplier transformations studied by Flett [4] and also to the differential and integral operators investigated by Sălăgean [7]. For a detailed analysis of various convolution operators, which are related to the multiplier transformations of Flett [4], refer the work of Li and Srivastava [5] (as well as the references cited by them). Now we define an unified class of analytic function based on this operator.

**Definition 1.1.** For $0 \leq \gamma \leq 1$, $0 \leq \beta < 1$, $\alpha \geq 0$, and for all $z \in \Delta$, we let the class $UH(\alpha, \beta, \gamma, \lambda, k)$, consists of functions $f \in T$ is said to be in the class satisfying the condition
\[ \Re \left\{ \frac{zF'(z)}{F(z)} \right\} > \alpha \left| \frac{zF'(z)}{F(z)} - 1 \right| + \beta, \] (1.5)
with,
\[ F(z) := \gamma(1 + \lambda)I(\lambda, k + 1)f(z) + (1 - \gamma(1 + \lambda))I(\lambda, k)f(z), \] (1.6)
where $I(\lambda, k)f(z)$ is the Cho-Srivastava operator as defined by (1.2)
The family $UH(\alpha, \beta, \gamma, \lambda, k)$, unifies various well known classes of analytic univalent functions. We list a few of them. The class $UH(2, 1, \lambda, \beta, 0)$ studied in [1]. Many classes including $UH(2, 1, 0, \beta, 0)$ and $UH(2, 1, 1, \beta, 0)$ given in [11], are particular cases of this class. Further that, the class $UH(2, 1, \lambda, 0, \beta, k)$ is the class of $k-$uniformly convex of order $\beta$, was introduced and studied in [10] (also see [2]).

In this present paper, we obtain a characterization, coefficients estimates, distortion theorem and covering theorem, extreme points and radii of close-to-convexity, starlikeness and convexity for functions belonging to the class $UH(\alpha, \beta, \gamma, \lambda, k)$.

2. Characterization and coefficient estimates

Theorem 2.1. Let $f \in T$. Then $f \in UH(\alpha, \beta, \gamma, \lambda, k)$, $0 \leq \gamma \leq 1$, $0 \leq \beta < 1$ and $\alpha \geq 0$,

$$\sum_{n=2}^{\infty} [n(\alpha + 1) - (\alpha + \beta)] (\gamma(n - 1) + 1) \Psi_n |a_n| \leq 1 - \beta.$$  \hspace{1cm} (2.1)

This result is sharp for the function

$$f(z) = z - \frac{1 - \beta}{n(\alpha + 1) - (\alpha + \beta)][\gamma(n - 1) + 1]\Psi_n} z^n n \geq 2. \hspace{1cm} (2.2)$$

Proof. We employ the technique adopted by [2]. We have

$$f \in UH(\alpha, \beta, \gamma, \lambda, k),$$

if and only if the condition (1.5) is satisfied, which is equivalent to

$$\text{Re} \left\{ \frac{zF'(z)(1 + ke^{i\theta}) - F(z)ke^{i\theta}}{F(z)} \right\} > \beta, \hspace{1cm} -\pi \leq \theta < \pi.$$  \hspace{1cm} (2.3)

Now, letting $G(z) = zF'(z)(1 + ke^{i\theta}) - F(z)ke^{i\theta}$, equation (2.3) is equivalent to

$$|G(z) + (1 - \beta)F(z)| > |G(z) - (1 + \beta)F(z)|, \hspace{1cm} 0 \leq \beta < 1.$$  

where $F(z)$ is as defined in (1.6). Now a simple computation gives

$$|G(z) + (1 - \beta)F(z)| \geq (2 - \beta)|z| - \sum_{n=2}^{\infty} \left( n(\alpha + 1) - (\alpha + \beta) + 1 \right) (\gamma(n - 1) + 1) \Psi_n |a_n| |z|^n$$

and similarly,

$$|G(z) - (1 + \beta)F(z)| \leq \beta |z| + \sum_{n=2}^{\infty} \left( n(\alpha + 1) - (\alpha + \beta) - 1 \right) (\gamma(n - 1) + 1) \Psi_n |a_n| |z|^n.$$  

Therefore,

$$|G(z) + (1 - \beta)F(z)| - |G(z) - (1 + \beta)F(z)|$$
\[ \geq 2(1 - \beta)|z| - 2 \sum_{n=2}^{\infty} \left( (n(\alpha + 1) - (\alpha + \beta)) \right) (\gamma(n - 1) + 1) \Psi_n a_n |z|^n \geq 0, \]

which is equivalent to the result (2.1).

On the other hand, for all \(-\pi \leq \theta < \pi\), we must have

\[ \text{Re} \left\{ \frac{zF'(z)}{F(z)} (1 + ke^{i\theta}) - ke^{i\theta} \right\} > \beta. \]

Now, choosing the values of \(z\) on the positive real axis, where \(0 \leq |z| = r < 1\), and using \(\text{Re} \{-e^{i\theta}\} \geq -|e^{i\theta}| = -1\), the above inequality can be written as

\[ \text{Re} \left\{ (1 - \beta) - \sum_{n=2}^{\infty} \left( (n(\alpha + 1) - (\alpha + \beta)) \right) \left( (\gamma(n - 1) + 1) \Psi_n a_n r^{n-1} \right) \right\} \geq 0. \]

Setting \(r \to 1^-\), we get the desired result. \(\square\)

Many known results can be obtained as particular cases of Theorem 2.1. For details, we refer to [6, 8].

By taking \(\alpha = 0, \gamma = 1, \lambda = 0\) and \(k = 1\) in Theorem 2.1, we get the following interesting result given in [9].

**Corollary 2.2.** [9] If \(f \in T\), then \(f \in C(\beta)\) if and only if

\[ \sum_{n=2}^{\infty} n(n - \beta)a_n \leq 1 - \beta. \]

Indeed, since \(f \in UH(\alpha, \beta, \gamma, \lambda, k)\), (2.1), we have

\[ \sum_{n=2}^{\infty} \left( (n(\alpha + 1) - (\alpha + \beta)) \left( (\gamma(n - 1) + 1) \Psi_n a_n \right) \right. \]

Hence for all \(n \geq 2\), we have

\[ a_n \leq \frac{1 - \beta}{\left( (n(\alpha + 1) - (\alpha + \beta)) \left( (\gamma(n - 1) + 1) \Psi_n \right) \right.}, \]

whenever \(0 \leq \gamma \leq 1, 0 \leq \beta < 1\) and \(\alpha \geq 0\). Hence we state this important observation as a separate theorem.

**Theorem 2.3.** If \(f \in UH(q, s, \lambda, \beta, k)\), then

\[ a_n \leq \frac{1 - \beta}{\left( (n(\alpha + 1) - (\alpha + \beta)) \left( (\gamma(n - 1) + 1) \Psi_n \right) \right.}, \quad n \geq 2, \tag{2.4} \]

where \(0 \leq \gamma \leq 1, 0 \leq \beta < 1\) and \(\alpha \geq 0\). Equality in (2.4) holds for the function

\[ f(z) = z - \frac{1 - \beta}{\left( (n(\alpha + 1) - (\alpha + \beta)) \left( (\gamma(n - 1) + 1) \Psi_n \right) \right.}. \tag{2.5} \]

This theorem also contains many known results for the special values of the parameters. For example, see [6, 8].
3. Distortion and covering theorems

**Theorem 3.1.** If \( f \in \mathcal{UH}(\alpha, \beta, \gamma, \lambda, k) \), then \( f \in \mathcal{T}^*(\delta) \), where

\[
\delta = 1 - \frac{1 - \beta}{(2(\alpha + 1) - (\alpha + \beta))(\gamma + 1)\Psi_2 - (1 - \beta)}.
\]

This result is sharp with the extremal function being

\[
f(z) = z - \frac{1 - \beta}{(2(\alpha + 1) - (\alpha + \beta))(\gamma + 1)\Psi_2} z^2.
\]

**Proof.** It is sufficient to show that (2.1) implies \( \sum_{n=2}^{\infty} (n - \delta) a_n \leq 1 - \delta \) [9], that is,

\[
\frac{n - \delta}{1 - \delta} \leq \frac{(n(\alpha + 1) - (\alpha + \beta))(\gamma(n - 1) + 1)\Psi_n}{1 - \beta} \quad n \geq 2. \tag{3.1}
\]

Since, for \( n \geq 2 \), (3.1) is equivalent to

\[
\delta \leq 1 - \frac{(n-1)(1-\beta)}{(n(\alpha + 1) - (\alpha + \beta))(\gamma(n - 1) + 1)\Psi_n - (1 - \beta)} = \Phi(n),
\]

and \( \Phi(n) \leq \Phi(2) \), (3.1) holds true for any \( 0 \leq \gamma \leq 1, \quad 0 \leq \beta < 1 \) and \( \alpha \geq 0 \). This completes the proof of the Theorem 3.1. \( \square \)

As in the previous cases we note this result has many special cases. If we take \( \alpha_1 = 1, \alpha_2 = 1, \beta_1 = 1, q = 2, s = 1, \lambda = 1 \) and \( k = 0 \) in Theorem 3.1, then we have the following result of [9].

**Corollary 3.2.** [9] If \( f \in C(\beta) \), then \( f \in \mathcal{T}^*\left(\frac{2}{3-\beta}\right) \). The result is sharp for the extremal function

\[
f(z) = z - \frac{1 - \beta}{2(2 - \beta)} z^2.
\]

**Remark.** Since distortion theorem and covering theorem are available for the class \( \mathcal{T}^*(\beta) \) [9], we can also obtain the corresponding results for the class \( \mathcal{UH}(\alpha, \beta, \gamma, \lambda, k) \), from the respective results of \( \mathcal{T}^*(\beta) \) by using Theorem 3.1, and we state them without proof.

**Theorem 3.3.** Let \( \Psi_n \) be defined as in (1.3). Then, for \( f \in \mathcal{UH}(\alpha, \beta, \gamma, \lambda, k) \), with \( z = re^{it} \in \Delta \), we have

\[
r - B(\alpha, \beta, \gamma, \lambda)r^2 \leq |f(z)| \leq r + B(\alpha, \beta, \gamma, \lambda)r^2, \tag{3.2}
\]

where,

\[
B(\alpha, \beta, \gamma, \lambda) := \frac{1 - \beta}{2(\alpha + 1) - (\alpha + \beta)(\gamma + 1)\Psi_2}.
\]

**Theorem 3.4.** If \( f \in \mathcal{UH}(\alpha, \beta, \gamma, \lambda, k) \), then for \( |z| = r < 1 \)

\[
1 - B(\alpha, \beta, \gamma, \lambda)r \leq |f'(z)| \leq 1 + B(\alpha, \beta, \gamma, \lambda)r, \tag{3.3}
\]

where \( B(\alpha, \beta, \gamma, \lambda) \) as in Theorem 3.3.
Note that in Theorem 3.3 and Theorem 3.4 equality holds for the function
\[ f(z) = z - \frac{1 - \beta}{(2(\alpha + 1) - (\alpha + \beta))(\gamma + 1)} \Psi_2 z^2. \]

4. Extreme points of the class \( \mathcal{UH}(\alpha, \beta, \gamma, \lambda, k) \),

**Theorem 4.1.** Let \( f_1(z) = z \) and
\[ f_n(z) = z - \frac{1 - \beta}{(n(\alpha + 1) - (\alpha + \beta))(\gamma(n - 1) + 1)} \Psi_n z^n, \quad n \geq 2 \]
and \( \Psi_n \) be as defined in (1.3). Then \( f \in \mathcal{UH}(\alpha, \beta, \gamma, \lambda, k) \), if and only if it can be represented in the form
\[ f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z), \quad \mu_n \geq 0, \quad \sum_{n=1}^{\infty} \mu_n = 1. \tag{4.1} \]

**Proof.** Suppose \( f(z) \) can be written as in (4.1). Then
\[ f(z) = z - \sum_{n=2}^{\infty} \mu_n \left\{ \frac{1 - \beta}{(n(\alpha + 1) - (\alpha + \beta))(\gamma(n - 1) + 1)} \Psi_n z^n \right\} = \sum_{n=1}^{\infty} \mu_n f_n(z), \quad \mu_n \geq 0, \quad \sum_{n=1}^{\infty} \mu_n = 1. \]

Now,
\[ \sum_{n=2}^{\infty} \mu_n \frac{(n(\alpha + 1) - (\alpha + \beta))(\gamma(n - 1) + 1)}{(1 - \beta)(n(\alpha + 1) - (\alpha + \beta))(\gamma(n - 1) + 1)} \Psi_n = \sum_{n=2}^{\infty} \mu_n = 1 - \mu_1 \leq 1. \]

Thus \( f \in \mathcal{UH}(\alpha, \beta, \gamma, \lambda, k) \). Conversely, let us have \( f \in \mathcal{UH}(\alpha, \beta, \gamma, \lambda, k) \). Then by using (2.4), we may write
\[ \mu_n = \frac{(n(\alpha + 1) - (\alpha + \beta))(\gamma(n - 1) + 1)}{1 - \beta} \Psi_n a_n, \quad n \geq 2, \]
and \( \mu_1 = 1 - \sum_{n=2}^{\infty} \mu_n \). Then \( f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z) \), with \( f_n(z) \) is as in the Theorem. \( \square \)

**Corollary 4.2.** The extreme points of \( f \in \mathcal{UH}(\alpha, \beta, \gamma, \lambda, k) \), are the functions \( f_1(z) = z \) and
\[ f_n(z) = z - \frac{1 - \beta}{(n(\alpha + 1) - (\alpha + \beta))(\gamma(n - 1) + 1)} \Psi_n z^n, \quad n \geq 2. \]

**Remark.** As in earlier theorems, we can deduce known results for various other classes and we omit details.

**Theorem 4.3.** The class \( \mathcal{UH}(\alpha, \beta, \gamma, \lambda, k) \) is a convex set.
Proof. Let the function
\[ f_j(z) = \sum_{n=2}^{\infty} a_{n,j} z^n, \quad a_{n,j} \geq 0, \quad j = 1, 2, \] (4.2)
be the class \( UH(\alpha, \beta, \gamma, \lambda, k) \). It suffices to show that the function \( g(z) \) defined by
\[ g(z) = \mu f_1(z) + (1 - \mu) f_2(z), \quad 0 \leq \mu \leq 1, \]
is in the class \( UH(\alpha, \beta, \gamma, \lambda, k) \). Since
\[ g(z) = z - \sum_{n=2}^{\infty} [\mu a_{n,1} + (1 - \mu) a_{n,2}] z^n, \]
an easy computation with the aid of Theorem 2.1 gives,
\[
\sum_{n=2}^{\infty} \left( n(\alpha + 1) - (\alpha + \beta) \right) \left( \gamma(n - 1) + 1 \right) \Psi_n [\mu a_{n,1} + (1 - \mu) a_{n,2}]
\]
\[ + (1 - \mu) \sum_{n=2}^{\infty} \left( n(\alpha + 1) - (\alpha + \beta) \right) \left( \gamma(n - 1) + 1 \right) \Psi_n \]
\[ \leq \mu (1 - \beta) + (1 - \mu) (1 - \beta) \leq 1 - \beta, \]
which implies that \( g \in UH(\alpha, \beta, \gamma, \lambda, k) \). Hence \( UH(\alpha, \beta, \gamma, \lambda, k) \) is convex. \[ \square \]

5. Modified Hadamard products

For functions of the form (4.2), we define the modified Hadamard product as
\[ (f_1 \ast f_2)(z) = z - \sum_{n=2}^{\infty} a_{n,1} a_{n,2} z^n. \] (5.1)

Theorem 5.1. If \( f_j(z) \in UH(q, s, \lambda, \beta, k), j = 1, 2 \), then
\[ (f_1 \ast f_2)(z) \in UH(q, s, \lambda, \beta, k, \xi), \]
where
\[ \xi = \frac{(2 - \beta) \left( 2(\alpha + 1) - (\alpha + \beta) \right) (\gamma + 1) \Psi_2 - 2(1 - \beta)^2}{(2 - \beta) \left( 2(\alpha + 1) - (\alpha + \beta) \right) (\gamma + 1) \Psi_2 - (1 - \beta)^2}, \]
with \( \Psi_n \) be defined as in (1.3).

Proof. Since \( f_j(z) \in UH(q, s, \lambda, \beta, k), j = 1, 2 \), we have
\[
\sum_{n=2}^{\infty} \left( n(\alpha + 1) - (\alpha + \beta) \right) \left( \gamma(n - 1) + 1 \right) \Psi_n a_{n,j} \leq 1 - \beta, \quad j = 1, 2. \]
The Cauchy-Schwartz inequality leads to
\[
\sum_{n=2}^{\infty} \left( n(\alpha + 1) - (\alpha + \beta) \right) \left( \gamma(n - 1) + 1 \right) \Psi_n a_{n,j} \leq 1 - \beta \]
\[ \leq \sqrt{a_{n,1} a_{n,2}} \leq 1. \] (5.2)
Note that we need to find the largest $\xi$ such that
\[
\sum_{n=2}^{\infty} \frac{(n(k + 1) - (k + \xi))(\gamma(n - 1) + 1)}{1 - \xi} \Psi_n a_{n,1} a_{n,2} \leq 1. \tag{5.3}
\]
Therefore, in view of (5.2) and (5.3), whenever
\[
\frac{n - \xi}{1 - \xi} \sqrt{a_{n,1} a_{n,2}} \leq \frac{n - \beta}{1 - \beta}, \quad n \geq 2
\]
holds, then (5.3) is satisfied. We have, from (5.2),
\[
\sqrt{a_{n,1} a_{n,2}} \leq \frac{1 - \beta}{(n(\alpha + 1) - (\alpha + \beta))(\gamma(n - 1) + 1)\Psi_n}, \quad n \geq 2. \tag{5.4}
\]
Thus, if
\[
\left(\frac{n - \xi}{1 - \xi}\right) \left[\frac{1 - \beta}{(n(\alpha + 1) - (\alpha + \beta))(\gamma(n - 1) + 1)\Psi_n}\right] \leq \frac{n - \beta}{1 - \beta}, \quad n \geq 2,
\]
or, if
\[
\xi \leq \frac{(n - \beta)(n(\alpha + 1) - (\alpha + \beta))(\gamma(n - 1) + 1)\Psi_n - n(1 - \beta)^2}{(n - \beta)(n(\alpha + 1) - (\alpha + \beta))(\gamma(n - 1) + 1)\Psi_n - (1 - \beta)^2}, \quad n \geq 2,
\]
then (5.2) is satisfied. Note that the right hand side of the above expression is an increasing function on $n$. Hence, setting $n = 2$ in the above inequality gives the required result. Finally, by taking the function
\[
f(z) = z - \frac{1 - \beta}{(2 - \beta)(2(\alpha + 1) - (\alpha + \beta))(\gamma + 1)\Psi_n}z^2,
\]
we see that the result is sharp. \qed

6. Radii of close-to-convexity, starlikeness and convexity

**Theorem 6.1.** Let the function $f \in T$ be in the class $UH(q,s,\lambda,\beta,k)$. Then $f(z)$ is close-to-convex of order $\rho$, $0 \leq \rho < 1$ in $|z| < r_1(\alpha, \beta, \gamma, \rho)$, where
\[
r_1(\alpha, \beta, \gamma, \rho) = \inf_n \left[\frac{(1 - \rho)(n(\alpha + 1) - (\alpha + \beta))(\gamma(n - 1) + 1)\Psi_n}{n(1 - \beta)}\right]^{\frac{1}{n - 1}}, \quad n \geq 2, \text{ with } \Psi_n \text{ be defined as in (1.3). This result is sharp for the function } f(z) \text{ given by (2.2).}
\]

**Proof.** It is sufficient to show that $|f'(z) - 1| \leq 1 - \rho$, $0 \leq \rho < 1$, for $|z| < r_1(\alpha, \beta, \gamma, \rho)$, or equivalently
\[
\sum_{n=2}^{\infty} \left(\frac{n}{1 - \rho}\right) a_n |z|^{n-1} \leq 1. \tag{6.1}
\]
By Theorem 2.1, (6.1) will be true if
\[
\left(\frac{n}{1 - \rho}\right) |z|^{n-1} \leq \frac{(n(\alpha + 1) - (\alpha + \beta))(\gamma(n - 1) + 1)\Psi_n}{1 - \beta}
\]
or, if
\[ |z| \leq \left[ \frac{(1 - \rho) \left( n(\alpha + 1) - (\alpha + \beta) \right) \left( \gamma(n - 1) + 1 \right) \Psi_n}{n(1 - \beta)} \right]^{\frac{1}{n-1}}. \]  
(6.2)

The theorem follows easily from (6.2).

**Theorem 6.2.** Let the function \( f(z) \) defined by (1.1) be in the class \( UH(\alpha, \beta, \gamma, \lambda, k) \). Then \( f(z) \) is starlike of order \( \rho \), \( 0 \leq \rho < 1 \) in \( |z| < r_2(\alpha, \beta, \gamma, \rho) \), where
\[
r_2(\alpha, \beta, \gamma, \rho) = \inf_n \left[ \frac{(1 - \rho) \left( n(\alpha + 1) - (\alpha + \beta) \right) \left( \gamma(n - 1) + 1 \right) \Psi_n}{n(1 - \beta)} \right]^{\frac{1}{n-1}},
\]

\( n \geq 2 \), with \( \Psi_n \) be defined as in (1.3). This result is sharp for the function \( f(z) \) given by (2.2).

**Proof.** It is sufficient to show that
\[
\left| \frac{zf''(z)}{f'(z)} - 1 \right| \leq 1 - \rho, \text{ or equivalently } \sum_{n=2}^{\infty} \left( \frac{n - \rho}{1 - \rho} \right) a_n |z|^{n-1} \leq 1,
\]
(6.3)
for \( 0 \leq \rho < 1 \) and \( |z| < r_2(\alpha, \beta, \gamma, \rho) \). Proceeding as in Theorem 6.1, we get the required result.

**Theorem 6.3.** Let the function \( f(z) \) defined by (1.1) be in the class \( UH(\alpha, \beta, \gamma, \lambda, k) \). Then \( f(z) \) is convex of order \( \rho \), \( 0 \leq \rho < 1 \) in \( |z| < r_3(\alpha, \beta, \gamma, \rho) \), where
\[
r_3(\alpha, \beta, \gamma, \rho) = \inf_n \left[ \frac{(1 - \rho) \left( n(\alpha + 1) - (\alpha + \beta) \right) \left( \gamma(n - 1) + 1 \right) \Psi_n}{n(1 - \beta)} \right]^{\frac{1}{n-1}},
\]

\( n \geq 2 \), with \( \Psi_n \) be defined as in (1.3). This result is sharp for the function \( f(z) \) given by (2.2).

**Proof.** It is sufficient to show that
\[
\left| \frac{zf''(z)}{f'(z)} \right| \leq 1 - \rho \text{ or equivalently } \sum_{n=2}^{\infty} \left( \frac{\left( n(n - \rho) \right)}{1 - \rho} \right) a_n |z|^{n-1} \leq 1,
\]
(6.4)
for \( 0 \leq \rho < 1 \) and \( |z| < r_3(\alpha, \beta, \gamma, \rho) \). Proceeding as in Theorem 6.1, we get the required result.

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80 Srikandan Sivasubramanian and Chellakutti Ramachandran


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