On the rate of convergence of a new $q$-Szász-Mirakjan operator

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Abstract. In the present paper we introduce a new $q$-generalization of Szász-Mirakjan operators and we investigate their approximation properties. By using a weighted modulus of smoothness, we give local and global estimations for the error of approximation.

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1. Introduction

The aim of this paper is to study the approximation properties of a new Szász-Mirakjan type operator constructed by using $q$-Calculus. Firstly, we recall some basic definitions and notations used in quantum calculus, see, e.g., [6, pp. 7-13].

Let $q > 0$. For any $n \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$ the $q$-integer $[n]_q$ is defined by

$$[n]_q := 1 + q + \ldots + q^{n-1} \quad (n \in \mathbb{N}), \quad [0]_q := 0,$$

and the $q$-factorial $[n]_q!$ by

$$[n]_q! := [1]_q[2]_q \ldots [n]_q \quad (n \in \mathbb{N}), \quad [0]_q! := 1.$$

Also, the $q$-binomial coefficients are denoted by $\left[ \begin{array}{c} n \\ k \end{array} \right]_q$ and are defined by

$$\left[ \begin{array}{c} n \\ k \end{array} \right]_q = \frac{[n]_q!}{[k]_q![n-k]_q!}, \quad k = 0, 1, \ldots, n.$$

The $q$-derivative of a function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad x \neq 0, \quad D_q f(0) := \lim_{x \to 0} D_q f(x),$$

and the high $q$-derivatives $D_q^0 f := f, \quad D_q^n f := D_q(D_q^{n-1} f), \quad n \in \mathbb{N}.$
The product rule is
\[ D_q (f(x)g(x)) = D_q (f(x)) g(x) + f(qx)D_q (g(x)). \] (1.1)

We recall the $q$-Taylor theorem as it is given in [4, p. 103].

**Theorem 1.1.** If the function $g(x)$ is capable of expansion as a convergent power series and $q$ is not a root of unity, then
\[ g(x) = \sum_{r=0}^{\infty} \frac{(x-a)^r}{[r]_q!} D_q^r g(a), \]
where
\[ (x-a)_q = \prod_{s=0}^{r-1} (x - qa) = \sum_{k=0}^{r} \binom{r}{k} q^{\binom{k(k-1)}{2}} x^{r-k} (-a)^k. \]

### 2. Auxiliary results

Throughout the paper we consider $q \in (0, 1)$.

We define a suitable $q$-difference operator as follows
\[ \Delta^0_q f_{k,s} = f_{k,s}, \] (2.1)
\[ \Delta^{r+1}_q f_{k,s} = q^r \Delta^r_q f_{k+1,s} - \Delta^r_q f_{k,s-1}, \quad r \in \mathbb{N}_0, \] (2.2)
where $f_{k,s} = f \left( \frac{[k]_q}{q^{[n]_q}} \right)$, $k \in \mathbb{N}_0$, $s \in \mathbb{Z}$.

The following lemma gives an expression for the $r$-th $q$-differences $\Delta^r_q f_{k,s}$ as a sum of multiplies of values of $f$.

**Lemma 2.1.** The $q$-difference operator $\Delta^r_q$ defined by (2.1)-(2.2) satisfies
\[ \Delta^r_q f_{k,s} = \sum_{j=0}^{r} (-1)^{r-j} q^{j(j-1)/2} \binom{r}{j} f_{k+j,j+s-r} \quad \text{for} \quad r, k \in \mathbb{N}_0, \ s \in \mathbb{Z}. \] (2.3)

Taking into account the relations (2.1)-(2.2) and the formula
\[ \binom{r+1}{j+1}_q = q^{r-j} \binom{r}{j}_q + \binom{r}{j+1}_q, \]
the identity (2.3) can be easily obtained by induction over $r \in \mathbb{N}_0$.

In what follows, the monomial of $m$ degree is denoted by $e_m$, $m \in \mathbb{N}_0$.

Let us denote by $[x_0, x_1, \ldots, x_n; f]$ the divided difference of the function $f$ with respect to the points $x_0, x_1, \ldots, x_n$.

**Lemma 2.2.** For all $k, r \in \mathbb{N}_0$, $s \in \mathbb{Z}$, we have
\[ [x_{k,s-1}, \ldots, x_{k+r,s+r-1}; f] = \frac{q^{(r+2s-1)/2} [r]_q^r}{[r]_q!} \Delta^r_q f_{k+r,s-1}, \] (2.4)
where $x_{k,s-1} = \frac{[k]_q}{q^{s-1}[n]_q}$. 
Proof. We use the mathematical induction with respect to $r$. For $r = 0$ the equality (2.4) follows immediately from (2.1). Let us assume that (2.4) holds true for some $r \geq 0$ and all $k \in \mathbb{N}_0$, $s \in \mathbb{Z}$.

We have

$$\left[ x_{k,r}, \ldots, x_{k+r+1,s+r}; f \right] = \frac{\left[ x_{k+1,r}, \ldots, x_{k+r+1,s+r}; f \right] - \left[ x_{k,s}, \ldots, x_{k+r,s+r+1}; f \right]}{x_{k+r+1,s+r} - x_{k,s}}.$$ 

Since $x_{k+r+1,s+r} - x_{k,s} = \frac{[r+1]q\Delta^r f_{k+r,s}}{q^{r+1}[n]_q}$, by using (2.2) we get

$$\left[ x_{k,s}, \ldots, x_{k+r+1,s+r}; f \right] = q^{(r+1)(r+2s)/2[n]_q^r+1} \frac{\Delta^r f_{k,r+s} - \Delta^r f_{k,r+s-1}}{[r+1]q!} = q^{(r+1)(r+2s)/2[n]_q^r+1} \Delta^r f_{k,r+s}.$$

\[\square\]

3. Construction of the operators

In 1987 A. Lupaș [9] introduced the first $q$-analogue of Bernstein operator and investigated its approximating and shape-preserving properties. Another $q$-generalization of the classical Bernstein polynomials is due to G. Phillips [13]. More properties of these two $q$-extensions were obtained over time in several papers such as [3], [10], [11], [1]. We mention that the comprehensive survey [12] due to S. Ostrovska gives a good perspective of the most important achievements during a decade relative to these operators.

Two of the known expansions in $q$-calculus of the exponential function are given as follows (see, e.g., [6, p. 31])

$$E_q(x) = \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{x^k}{[k]_q!}, \quad x \in \mathbb{R}, \ |q| < 1,$$

$$e_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!}, \quad |x| < \frac{1}{1-q}, \ |q| < 1.$$ 

It is obvious that $\lim_{q \rightarrow 1^{-}} E_q(x) = \lim_{q \rightarrow 1^{-}} e_q(x) = e^x$.

For $q \in (0,1)$, in [2] A. Aral introduced the first $q$-analogue of the classical Szász-Mirakjan operators given by

$$S^q_n(f; x) = E_q \left( -[n]_q x \frac{b_n}{b_{n+1}} \sum_{k=0}^{\infty} f \left( \frac{[k]_q}{[n]_q} \frac{x}{b_n} \right) \frac{[n]_q x^k}{[k]_q ![b_n]_q^k} \right),$$

where $0 \leq x < \frac{b_n}{1-q^n}$, $(b_n)_n$ is a sequence of positive numbers such that $\lim_n b_n = \infty$. 

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The operator $S_n^q$ reproduces linear functions and

$$S_n^q(e^x; x) = qx^2 + \frac{b_n}{[n]^q}x, \quad 0 \leq x < \frac{b_n}{1 - q^n}.$$  

Motivated by this work, for $q \in (0, 1)$ we give another $q$-analogue of the same class of operators as follows

$$S_{n,q}^q(f; x) = \sum_{k=0}^{\infty} \binom{[n]^q x}{[k]^q} q^{k(k-1)} E_q \left( -[n]^q q^k x \right) f \left( \frac{[k]^q}{[n]^q q^{k-1}} \right), \quad x \geq 0,$$

where $f \in \mathcal{F}(\mathbb{R}_+) := \{ f : \mathbb{R} \to \mathbb{R}, \ \text{the series in (3.1) is convergent} \}.$

Since $E_q(x)$ is convergent for every $x \in \mathbb{R}$, by using Theorem 1.1 and the property $D^r q E_q(ax) = a E_q(q^r x)$ we obtain

$$\sum_{r=0}^{\infty} \binom{-x}{r} q^{r(r-1)} E_q(q^r x) = E_q(0) = 1, \quad x \in \mathbb{R},$$

which yields that the operator $S_{n,q}$ is well defined.

For $q \to 1^-$, the above operators reduce to the classical Szász-Mirakjan operators. In this case, the approximation function $S_{n,q} f$ is defined on $\mathbb{R}_+$ for each $n \in \mathbb{N}$.

**Theorem 3.1.** Let $q \in (0, 1)$ and $S_{n,q}, \ n \in \mathbb{N}$, be defined by (3.1). For any $f \in \mathcal{F}(\mathbb{R}_+)$ we have

$$S_{n,q}(f; x) = \sum_{r=0}^{\infty} \binom{[n]^q x}{[r]^q} q^{r(r-1)} \Delta_q^r f_{0,r-1}, \quad x \geq 0. \quad (3.2)$$

**Proof.** Let $f \in \mathcal{F}(\mathbb{R}_+)$. By using (2.1), the operator $S_{n,q}$ can be expressed as follows

$$S_{n,q}(f; x) = \sum_{k=0}^{\infty} \binom{[n]^q x}{[k]^q} q^{k(k-1)} E_q \left( -[n]^q q^k x \right) \Delta_q^0 f_{k,k-1}.$$  

Applying $q$-derivative operator to $S_{n,q} f$ and taking into account the product rule (1.1) and the property $D_q E_q(ax) = a E_q(aqx)$, (see e.g. [6, pp. 29-32]), we have

$$D_q S_{n,q}(f; x)$$

$$= [n]^q \sum_{k=0}^{\infty} \binom{[n]^q x}{[k]^q} q^{k(k+1)} E_q \left( -[n]^q q^{k+1} x \right) \left( \Delta_q^0 f_{k+1,k} - \Delta_q^0 f_{k,k-1} \right)$$

$$= [n]^q \sum_{k=0}^{\infty} \binom{[n]^q x}{[k]^q} q^{k(k+1)} E_q \left( -[n]^q q^{k+1} x \right) \Delta_q^1 f_{k,k}.$$
For \( n \in \mathbb{N} \) and \( x \in \mathbb{R}_+ \), by induction with respect to \( r \in \mathbb{N} \), we can prove
\[
D^r_q S_{n,q}(f; x) = [n]^r_q q^{\frac{r(r-1)}{2}} \sum_{k=0}^{\infty} \left[ \frac{[n]_q x}{[k]_q} \right]^k q^{k(2r+k-1)} E_q \left(-[n]_q q^{k+r} x\right) \Delta^r_q f_{k,k+r-1}.
\]
Choosing \( x = 0 \), we deduce \( D^r_q S_{n,q}(f; 0) = [n]^r_q q^{\frac{r(r-1)}{2}} \Delta^r_q f_{0,r-1} \).

Choosing \( a = 0 \) in Theorem 1.1, we obtain
\[
S_{n,q}(f; x) = \sum_{r=0}^{\infty} \frac{([n]_q x)^r}{[r]_q} q^{\frac{r(r-1)}{2}} \Delta^r_q f_{0,r-1},
\]
which completes the proof. \( \square \)

**Corollary 3.2.** Let \( q \in (0,1) \) and \( S_{n,q} \), \( n \in \mathbb{N} \), be defined by (3.1). For any \( f \in \mathcal{F}(\mathbb{R}_+) \) we have
\[
S_{n,q}(f; x) = \sum_{r=0}^{\infty} x^r \left[ 0, \frac{1}{[n]_q}, \frac{[2]_q}{q[n]_q}, \ldots, \frac{[r]_q}{q^{r-1}[n]_q}; f \right], \quad x \geq 0.
\]

**Proof.** The identity (3.3) is obtained from the above theorem and (2.4) by choosing \( k = s = 0 \). \( \square \)

**Corollary 3.3.** For all \( n \in \mathbb{N} \), \( x \in \mathbb{R}_+ \) and \( 0 < q < 1 \), we have
\[
S_{n,q}(e_0; x) = 1, \quad S_{n,q}(e_1; x) = x, \quad S_{n,q}(e_2; x) = x^2 + \frac{1}{[n]_q} x.
\]

Moreover, for \( m \in \mathbb{N}_0 \) and \( 0 < q < 1 \), the operator \( S_{n,q} \) defined by (3.1) can be expressed as
\[
S_{n,q}(e_m; x) = \sum_{r=0}^{m} x^r \left[ 0, \frac{1}{[n]_q}, \frac{[2]_q}{q[n]_q}, \ldots, \frac{[r]_q}{q^{r-1}[n]_q}; e_m \right], \quad x \geq 0.
\]

**Proof.** Since for any distinct points \( x_0, \ldots, x_r \), the divided difference
\[
[x_0, \ldots, x_r; e_m] = \begin{cases} 
0 & \text{if } m < r, \\
1 & \text{if } m = r, \\
x_0 + \ldots + x_r & \text{if } m = r + 1,
\end{cases}
\]
(see e.g. [5, p.63]), the identities (3.4)-(3.7) are obvious. \( \square \)

**Lemma 3.4.** For \( m \in \mathbb{N}_0 \) and \( q \in (0,1) \) we have
\[
S_{n,q}(e_m; x) \leq A_{m,q}(1 + x^m), \quad x \geq 0, \quad n \in \mathbb{N},
\]
where \( A_{m,q} \) is a positive constant depending only on \( q \) and \( m \).
Proof. Let \( m \in \mathbb{N} \). From (3.7) we get
\[
S_{n,q}(e_m; x) \leq (1 + x^m) \sum_{r=1}^{m} \left[ \frac{[r]}{[n]_q}, \ldots, \frac{[r]}{q^{r-1}[n]_q}; e_m \right].
\]
Applying the well known Lagrange’s Mean Value Theorem, we can write
\[
S_{n,q}(e_m; x) \leq (1 + x^m) \sum_{r=1}^{m} \left( \frac{m}{r} \right) (\xi_r)^m - r,
\]
where \( 0 < \xi_r < \frac{[r]}{q^{r-1}[n]_q}, 0 < r \leq m \).
Consequently, we have
\[
S_{n,q}(e_m; x) \leq (1 + x^m) \sum_{r=1}^{m-1} \left( \frac{m}{r} \right) \frac{[r]}{q^{(r-1)(m-r)}[n]_q^{m-r}} \]
\[
\leq A_{m,q}(1 + x^m),
\]
where
\[
A_{m,q} := \frac{[m]_q^{m-1}}{m} \left( 1 + \frac{1}{q^m} \right)^m, \quad m \geq 1. \tag{3.9}
\]
For \( m = 0 \) we can take \( A_{0,q} = \frac{1}{2} \).

Examining relation (3.6) it is clear that the sequence of the operators \( (S_{n,q})_n \) does not satisfies the conditions of Bohman-Korovkin theorem.
Further on, we consider a sequence \((q_n)_n\), \( q_n \in (0, 1) \), such that
\[
\lim_{n \to \infty} q_n = 1. \tag{3.10}
\]
The condition (3.10) guarantees that \([n]_{q_n} \to \infty\) for \( n \to \infty \).

**Theorem 3.5.** Let \((q_n)_n\) be a sequence satisfying (3.10) and let the operators \( S_{n,q_n}, \ n \in \mathbb{N}, \) be defined by (3.1). For any compact \( J \subset \mathbb{R}_+ \) and for each \( f \in C(\mathbb{R}_+) \) we have
\[
\lim_{n \to \infty} S_{n,q_n}(f; x) = f(x), \text{ uniformly in } x \in J.
\]

*Proof.* Replacing \( q \) by a sequence \((q_n)_n\) with the given conditions, the result follows from (3.4)-(3.6) and the well-known Bohman-Korovkin theorem (see [7], pp. 8-9).

\[\square\]

### 4. Error of approximation

Let \( \alpha \in \mathbb{N} \). We denote by \( B_\alpha(\mathbb{R}_+) \) the weighted space of real-valued functions \( f \) defined on \( \mathbb{R}_+ \) with the property \( |f(x)| \leq M_f(1 + x^\alpha) \) for all \( x \in \mathbb{R}_+ \), where \( M_f \) is a constant depending on the function \( f \). We also consider the weighted subspace \( C_\alpha(\mathbb{R}_+) \) of \( B_\alpha(\mathbb{R}_+) \) given by
\[
C_\alpha(\mathbb{R}_+) := \{ f \in B_\alpha(\mathbb{R}_+) : \text{f continuous on } \mathbb{R}_+ \}.
\]
Endowed with the norm $\| \cdot \|_\alpha$, where $\| f \|_\alpha := \sup_{x \in \mathbb{R}_+} |f(x)|^{1/\alpha}$, both $B_\alpha(\mathbb{R}_+)$ and $C_\alpha(\mathbb{R}_+)$ are Banach spaces.

We can give estimates of the error $|S_{n,q}(f; \cdot) - f|$, $n \in \mathbb{N}$, for unbounded functions by using a weighted modulus of smoothness associated to the space $B_\alpha(\mathbb{R}_+)$. We consider

$$\Omega_\alpha(f; \delta) := \sup_{x \geq 0, 0 < h \leq \delta} \frac{|f(x + h) - f(x)|}{1 + (x + h)^\alpha}, \delta > 0, \alpha \in \mathbb{N}. \quad (4.1)$$

It is evident that for each $f \in B_\alpha(\mathbb{R}_+)$, $\Omega_\alpha(f; \cdot)$ is well defined and

$$\Omega_\alpha(f; \delta) \leq 2 \| f \|_\alpha, \delta > 0, f \in B_\alpha(\mathbb{R}_+), \alpha \in \mathbb{N}. \quad (4.2)$$

The weighted modulus of smoothness $\Omega_\alpha(f; \cdot)$ possesses the following properties ([8]):

$$\begin{align*}
\Omega_\alpha(f; \lambda \delta) &\leq (\lambda + 1)\Omega_\alpha(f; \delta), \quad \delta > 0, \lambda > 0, \\
\Omega_\alpha(f; n \delta) &\leq n\Omega_\alpha(f; \delta), \quad \delta > 0, n \in \mathbb{N}, \\
\lim_{\delta \to 0^+} \Omega_\alpha(f; \delta) &= 0.
\end{align*} \quad (4.3)$$

**Theorem 4.1.** Let $(q_n)_n$ be a sequence satisfying (3.10). Let $q_0 = \inf_{n \in \mathbb{N}} q_n$ and $\alpha \in \mathbb{N}$. For each $n \in \mathbb{N}$ and every $f \in B_\alpha(\mathbb{R}_+)$ one has

$$|S_{n,q_n}(f; x) - f(x)| \leq C_{\alpha,q_0}(1 + x^{\alpha+1})\Omega_\alpha(f; \sqrt{1/[n] q_n}), \quad x \geq 0, \quad (4.4)$$

where $C_{\alpha,q_0}$ is a positive constant independent of $f$ and $n$.

**Proof.** Let $n \in \mathbb{N}$, $f \in B_\alpha(\mathbb{R}_+)$ and $x \geq 0$ be fixed. Setting $\mu_{x,\alpha}(t) := 1 + (x + |t - x|)^\alpha$ and $\psi_x(t) := |t - x|$, $t \geq 0$, relations (4.1) and (4.2) imply

$$|f(t) - f(x)| \leq (1 + (x + |t - x|)^\alpha) \left(1 + \frac{1}{\delta} |t - x|\right) \Omega_\alpha(f; \delta)$$

$$= \mu_{x,\alpha}(t) \left(1 + \frac{1}{\delta} \psi_x(t)\right) \Omega_\alpha(f; \delta), \quad t \geq 0.$$

By using the Cauchy inequality for linear positive operators which preserve the constants, we obtain

$$|S_{n,q_n}(f; x) - f(x)| \leq S_{n,q_n}(|f - f(x)|; x) \leq \left(S_{n,q_n}(\mu_{x,\alpha}; x) + \frac{1}{\delta} S_{n,q_n}(\mu_{x,\alpha,\psi}; x)\right) \Omega_\alpha(f; \delta) \leq \sqrt{S_{n,q_n}(\mu_{x,\alpha,\psi}; x)} \left(1 + \frac{1}{\delta} \sqrt{S_{n,q_n}(\psi_x^2; x)}\right) \Omega_\alpha(f; \delta).$$

Since

$$\mu_{x,\alpha}^2(t) = (1 + (x + |t - x|)^\alpha)^2 \leq 2 \left(1 + (2x + t)^{2\alpha}\right) \leq 2 \left(1 + 2^{2\alpha} ((2x + t)^{2\alpha} + t^{2\alpha})\right),$$

we have

$$|S_{n,q_n}(f; x) - f(x)| \leq S_{n,q_n}(|f - f(x)|; x) \leq C_{\alpha,q_0}(1 + x^{\alpha+1})\Omega_\alpha(f; \sqrt{1/[n] q_n}), \quad x \geq 0,$$
and taking into account (3.4) and (3.8) we get
\[ S_{n,q_n}(\mu_x^2; x) \leq B_{\alpha,q_n}^2 (1 + x^{2\alpha}), \tag{4.5} \]
where \( B_{\alpha,q_n}^2 = 2^{\alpha+1} (2^{2\alpha} + A_{2\alpha,q_n}). \)

According to (3.4)-(3.6) we have
\[ S_{n,q_n}(\psi_x^2; x) = 1. \]

By choosing \( \delta := \sqrt{\frac{1}{[n]_{q_n}}} \) in (4.3), from (4.5) follows
\[ |S_{n,q_n}(f; x) - f(x)| \leq B_{\alpha,q_n} \theta \sqrt{1 + x^{2\alpha}} (1 + \sqrt{x}) \Omega_\alpha \left( f; \sqrt{\frac{1}{[n]_{q_n}}} \right). \]

Finally, since \( 1 + \sqrt{x} \leq \sqrt{2} \sqrt{1 + x} \) and \( (1 + x^{2\alpha})(1 + x) \leq 4(1 + x^{\alpha+1}) \) for \( x \geq 0 \) and \( \alpha \in \mathbb{N} \), we obtain
\[ |S_{n,q_n}(f; x) - f(x)| \leq C_{\alpha,q_0} (1 + x^{\alpha+1}) \Omega_\alpha \left( f; \sqrt{1/[n]_{q_n}} \right), \quad x \geq 0, \]
where \( q_0 := \inf_{n \in \mathbb{N}} q_n \) and \( C_{\alpha,q_0} := 2\sqrt{2} B_{\alpha,q_0}. \)

On the basis of Theorem 4.1 we give the following global estimate.

**Corollary 4.2.** Let \((q_n)_n\) be a sequence satisfying (3.10) and \( \alpha \in \mathbb{N} \). For each \( n \in \mathbb{N} \) and every \( f \in B_\alpha (\mathbb{R}_+) \) one has
\[ \|S_{n,q_n}(f; \cdot) - f\|_{\alpha+1} \leq C_{\alpha,q_0} \Omega_\alpha \left( f; \sqrt{1/[n]_{q_n}} \right), \]
where \( C_{\alpha,q_0} \) is a positive constant independent of \( f \) and \( n \).

**Remark 4.3.** For any function \( f \in B_\alpha (\mathbb{R}_+) \), \( \alpha \in \mathbb{N} \), the rate of convergence of the operators \( S_{n,q_n}(f; \cdot) \) to \( f \) in weighted norm is \( \sqrt{\frac{1}{[n]_{q_n}}} \) which is faster than \( \sqrt{\frac{b_n}{[n]_{q_n}}} \) obtained in [2].

**References**


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