Rigid body time-stepping schemes in a quasi-static setting

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Abstract. We discuss how linear complementary problems (LCPs) can be used to simulate rigid-body systems in a quasi-static setting. LCP-based time-stepping schemes were successfully used in [1] in order to plan and control meso-scale manipulation tasks.

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1. Introduction

In [1] we considered the canonical problem of assembling a peg into a hole. Simulation of this quasi-static system was used in order to select the control parameters. The integration step in the simulator was formulated as a mixed linear complementarity problem (MLCP). MLCPs should be thought of as linear complementarity problems (LCPs) coupled with additional linear equality constraints. A brief description of the linear complementarity problem and results concerning LCPs with copositive matrices are given in the following subsections. For a detailed analysis of these problems we refer the reader to the excellent manuscript [2].

1.1. Linear complementarity problems

In this section we present the definitions for the linear complementarity problem (LCP) and the mixed linear complementarity problem (MLCP).

Definition 1.1. The problem of finding $z \in \mathbb{R}^n$ such that

$$z \geq 0, \quad Mz + b \geq 0, \quad \text{and} \quad z^T(Mz + b) = 0, \quad (1.1)$$

where $b \in \mathbb{R}^n$ and $M \in \mathbb{R}^{n \times n}$ is called a linear complementarity problem.
In the above definition the inequality $z \geq 0$, $z \in \mathbb{R}^n$ is to be understood componentwise, i.e., $z_i \geq 0$, $i = 1, n$. The non-negativity and complementarity conditions (1.1) can be also written in the more compact form:

$$0 \leq z \perp w := Mz + b \geq 0.$$ 

We denote the problem (1.1) by $LCP(b, M)$. If in addition to the complementarity constraints we add some equality constraints we obtain a mixed linear complementarity problem (MLCP). To be more precise, we follow the definition in [2] and consider the matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{m \times m}$, $C \in \mathbb{R}^{n \times m}$ and $D \in \mathbb{R}^{m \times n}$. Let $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$ be given.

**Definition 1.2.** The mixed linear complementarity problem is the problem of finding vectors $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$ such that

$$a + Au + Cv = 0$$

$$b + Du + Bv \geq 0$$

$$v \geq 0$$

$$v^T(b + Du + Bv) = 0 \quad (1.2)$$

We note that if the matrix $A$ in (1.2) is invertible we can write $u$ in terms of $v$ and use this form to reduce the problem to a standard LCP formulation.

1.2. LCPs with copositive matrices

The matrix of the underlying LCP used in the time-stepping schemes such as the one used in [1] is a copositive matrix.

**Definition 1.3.** A matrix $M \in \mathbb{R}^{n \times n}$ is said to be copositive if

$$x^T M x \geq 0 \text{ for all } x \in \mathbb{R}^n, \ x \geq 0.$$ 

In general a linear complementarity problem with a copositive matrix is not guaranteed to possess a solution. Solvability of such LCPs is discussed in the following Theorem.

**Theorem 1.4 ([2], Th. 3.8.6).** Let $M \in \mathbb{R}^{n \times n}$ be a copositive matrix and let $b \in \mathbb{R}^n$ be given. If the implication

$$[v \geq 0, \ Mv \geq 0, \ v^T Mv = 0] \Rightarrow [v^T b \geq 0]$$

holds, then $LCP(b, M)$ has a solution. Lemke’s algorithm with precautions taken against cycling will always find a solution of $LCP(b, M)$.

Lemke’s algorithm is a pivoting method similar to the simplex method of linear programming. Cycling here refers to the possibility of using the same basis twice.
2. The quasi-static model

The continuous-time model under the rigid body assumption is given by the following differential complementarity problem (DCP):

\[ \dot{q}(t) = v(t), \quad (2.1) \]
\[ E v(t) - W_n(q, u, t) \lambda_n(t) - W_t(q, u, t) \lambda_t(t) = 0, \quad (2.2) \]
\[ 0 \leq \Psi_n(q, u, t) \perp \lambda_n(t) \geq 0, \quad (2.3) \]
\[ \dot{s}^+_{tk}(t) - \dot{s}^-_{tk}(t) = (W_{tk}(q, u, t))^T v(t) + \frac{\partial \Psi_{tk}}{\partial t}(q, u, t), \quad k = 1, \ldots, n_c, \quad (2.4) \]
\[ 0 \leq \dot{s}^+_{tk}(t) \perp \mu_k \lambda_{nk}(t) + \lambda_{tk}(t) \geq 0, \quad k = 1, \ldots, n_c, \quad (2.5) \]
\[ 0 \leq \dot{s}^-_{tk}(t) \perp \mu_k \lambda_{nk}(t) - \lambda_{tk}(t) \geq 0, \quad k = 1, \ldots, n_c. \quad (2.6) \]

Here \( q \) denotes the generalized system position and \( v \) the generalized system velocity. The control parameters are encoded in the vector \( u \). The quasi-static assumption is reflected by the equilibrium equation (2.2), where \( E \) is a damping matrix, assumed to be symmetric positive definite. The vectors \( \lambda_n(t) \in \mathbb{R}^{n_c} \) and \( \lambda_t(t) \in \mathbb{R}^{n_c} \) represent all normal and tangential forces, while \( W_n(q, u, t) \) and \( W_t(q, u, t) \) are the normal and tangential wrench matrices. More precisely, \( k \)-th column of \( W_n(q, u, t) \) (\( W_t(q, u, t) \)) is the normal (tangential) wrench vector \( W_{nk}(q, u, t) \) (\( W_{tk}(q, u, t) \)) corresponding to contact \( k \), \( k = 1, \ldots, n_c \), with \( n_c \) denoting the number of active contacts. The vector \( \Psi_n(q, u, t) \) contains the normal displacements for configuration \( q \), controls \( u \) and time \( t \). More precisely, \( \Psi_n(q, u, t) = [\Psi_{n1}(q, u, t), \ldots, \Psi_{nnc}(q, u, t)]^T \), where \( \Psi_{nk}(q, u, t) \) represents the normal displacement function corresponding to contact \( k \). In a similar way, one defines the vector of tangential displacements, \( \Psi_t(q, u, t) = [\Psi_{t1}(q, u, t), \ldots, \Psi_{tnc}(q, u, t)]^T \). Equation (2.3) represents the contact and non-penetration constraints; that is whenever the normal separation at contact \( k \) is strictly positive (\( \Psi_{nk}(q, u, t) > 0 \)), the corresponding normal force is 0 (\( \lambda_{nk} = 0 \)), while whenever contact \( k \) is established (\( \Psi_{nk}(q, u, t) = 0 \)), the corresponding normal force is nonnegative (\( \lambda_{nk} \geq 0 \)).

Equation (2.4) defines the positive, \( \dot{s}^+_{tk}(t) \), and negative, \( \dot{s}^-_{tk}(t) \), sliding velocities at contact \( k \). The right-hand side of (2.4) represents the (overall) sliding velocity \( \dot{s}_{tk}(t) := \dot{\Psi}_{tk}(q, u, t) = (W_{tk}(q, u, t))^T v(t) + \frac{\partial \Psi_{tk}}{\partial t}(q, u, t) \) at contact \( k \). The last two equations, namely (2.5) and (2.6), represent Coulomb’s friction law at contact \( k \), with \( \mu_k \in [0, 1] \) being the friction coefficients.

3. The time-stepping scheme

Let \( t_l \) denote the time at which one has a solution configuration \( q^l \) and let \( t_{l+1} = t_l + h \) denote the time at which one would want an estimate of the solution. We approximate the new configuration \( q^{l+1} \) using a backward Euler formula, as follows

\[ q^{l+1} = q^l + hv^{l+1}, \]
where \( v_{l+1} \) is an estimate for the new velocity and will be found by solving a mixed linear complementarity problem. At each integration step the unknowns \( (h v_{l+1}, h \lambda_{n l+1}, h \lambda_{f l+1}, h \sigma_{l+1}) \) may be obtained as the solution of the following MLCP:

\[
\begin{pmatrix}
0 & E & -W^l_n & -W^l_f & \rho^l_{n+1} \\
(W^l_n)^T & 0 & 0 & 0 & \rho^l_{f+1} \\
(W^l_f)^T & 0 & 0 & E_f & s^{l+1} \\
0 & U_f & -E^T_f & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
h v_{l+1} \\
h \lambda_{n l+1} \\
h \lambda_{f l+1} \\
h \sigma_{l+1} \\
\end{pmatrix} =
\begin{pmatrix}
0 \\
\Psi^l_n + h \frac{\partial \Psi^l_{n l}}{\partial t} \\
\Psi^l_f + h \frac{\partial \Psi^l_{f l}}{\partial t} \\
0 \\
\end{pmatrix}
\]

(3.1)

with \( 0 \leq \begin{bmatrix} \rho^l_{n+1}, \rho^l_{f+1}, s^{l+1} \end{bmatrix} \perp \begin{bmatrix} h \lambda_{n l+1}, h \lambda_{f l+1}, h \sigma_{l+1} \end{bmatrix} \geq 0 \). Here \( U_f \in \mathbb{R}^{n_c \times n_c}, E_f \in \mathbb{R}^{2n_c \times n_c} \) with \( U_f \) a diagonal matrix with elements on its diagonal equal to \( \mu_k, k = 1, ..., n_c \) and \( E_f \) a block diagonal matrix, with diagonal blocks given by the vector \( e \) (\( e \) is a two-dimensional vector of all ones). That is,

\[
U_f = \begin{pmatrix}
\mu_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \mu_{n_c}
\end{pmatrix},
E_f = \begin{pmatrix}
1 & \cdots & 0 \\
1 & \cdots & 0 \\
0 & \cdots & 1 \\
0 & \cdots & 1
\end{pmatrix}.
\]

The superscript \( l \) used in the MLCP (3.1) indicates that all the corresponding quantities are calculated with \( q := q^l \) and \( t := t_l \). For each contact \( k \) we define the \( 3 \times 2 \) matrix \( W_{fk}(q, u, t) \) by joining the column vectors \( W_{tk}(q, u, t) \) and \( -W_{tk}(q, u, t) \). That is,

\[
W_{fk}(q, u, t) = [W_{tk}(q, u, t) \quad -W_{tk}(q, u, t)].
\]

If we put all the active contacts together we obtain the "frictional" wrench matrix \( W_f(q, u, t) \) appearing in formulation (3.1). In a similar way, we get the vector \( \Psi_f(q, u, t) \).

**Solvability and the Friction Cone.** For an active contact \( k \), we define the friction cone corresponding to that contact by

\[
FC_k(q, u, t) = \left\{ z = W_{nk} \lambda_{nk} + W_{fk} \lambda_{fk} \mid \lambda_{nk} \geq 0, \lambda_{fk} \geq 0, e^T \lambda_{fk} \leq \mu_k \lambda_{nk} \right\},
\]

(3.2)

where \( W_{nk} := W_{n,k}(q, u, t) \), \( W_{fk} := W_{fk}(q, u, t) \) and \( e = [1, 1]^T \). The total friction cone, \( FC(q, u, t) \), which accounts for all active contacts is defined by

\[
FC(q, u, t) = \sum_{k=1}^{n_c} FC_k(q, u, t).
\]
Using the fact that the matrix $E$ in the MLCP (3.1) is positive definite, we can eliminate the variables $h_{t_{l+1}}$ and reduce the MLCP to a standard LCP with a copositive matrix. It can be shown that the resulting LCP, is solvable whenever the total friction cone $FC(q^l, u, t_i)$ is pointed. We recall that a cone is pointed if it doesn’t contain any proper subspace. The lack of pointedness for the friction cone results in jammed configurations (see [3]) and therefore this regularity assumption is very realistic and can be successfully used in devising randomized plans (see [1]).

4. Conclusions

We have discussed an LCP-based time-stepping scheme that can be used to simulate rigid body systems in a quasi-static setting. The scheme was introduced and successfully used for a particular case in [1]. Solvability of the integration step is guaranteed by the pointedness of the friction cone, an assumption that is common in dynamic settings as well (see [3] and [4] for example).

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