Remark on Voronovskaja theorem for q-Bernstein operators

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Abstract. We establish quantitative Voronovskaja type theorems for the q-Bernstein operators introduced by Phillips in 1997. Our estimates are given with the aid of the first order Ditzian-Totik modulus of smoothness.

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1. Introduction

Let $q > 0$ and $n$ be a non-negative integer. Then the q-integers $[n]_q$ and the q-factorials $[n]_q!$ are defined by

$$[n]_q = \begin{cases} 1 + q + \ldots + q^{n-1}, & \text{if } n \geq 1 \\ 0, & \text{if } n = 0 \end{cases}$$

and

$$[n]_q! = \begin{cases} [1]_q[2]_q \ldots [n]_q, & \text{if } n \geq 1 \\ 1, & \text{if } n = 0. \end{cases}$$

For integers $0 \leq k \leq n$, the q-binomial coefficients are defined by

$$\left[ \begin{array}{c} n \\ k \end{array} \right]_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}.$$  

The so-called q-Bernstein operators were introduced by G.M. Phillips [3] and they are defined by $B_{n,q} : C[0,1] \to C[0,1]$,

$$(B_{n,q}f)(x) \equiv B_{n,q}(f, x) = \sum_{k=0}^{n} f \left( \frac{[k]_q}{[n]_q} \right) p_{n,k}(q, x),$$
where
\[ p_{n,k}(q;x) = \left[ \begin{array}{c} n \\ k \end{array} \right] q^k (1-x)(1-qx) \ldots (1-q^{n-k-1}x), \quad x \in [0,1], \]
and an empty product denotes 1. Note that for \( q = 1 \), we recover the classical Bernstein operators. It is well-known that Voronovskaja’s theorem [5] deals with the asymptotic behaviour of Bernstein operators. Then naturally raises the following question: can we state a similar Voronovskaja theorem for the q-Bernstein operators? The positive answer was given in [3] as follows.

**Theorem 1.1.** Let \( q = q_n \) satisfy \( 0 < q_n < 1 \) and let \( q_n \to 1 \) as \( n \to \infty \). If \( f \) is bounded on \([0,1] \), differentiable in some neighborhood of \( x \) and has second derivative \( f''(x) \) for some \( x \in [0,1] \), then the rate of convergence of the sequence \( \{(B_{n,q_n},f)(x)\} \) is governed by
\[
\lim_{n \to \infty} |n| q_n \{(B_{n,q_n},f)(x) - f(x)\} = \frac{1}{2} x (1-x) f''(x). \tag{1.1}
\]

In [4], the convergence (1.1) was given in quantitative form as follows.

**Theorem 1.2.** Let \( q = q_n \) satisfy \( 0 < q_n < 1 \) and let \( q_n \to 1 \) as \( n \to \infty \). Then for any \( f \in C^2[0,1] \) the following inequality holds
\[
|n| q_n \{(B_{n,q_n},f)(x) - f(x)\} - \frac{1}{2} x (1-x) f''(x) \leq c x (1-x) \omega \left(f'', [n] q_n^{-1/2}\right),
\]
where \( c \) is an absolute positive constant, \( x \in [0,1], n = 1, 2, \ldots \) and \( \omega \) is the first order modulus of continuity.

The goal of this note is to obtain new quantitative Voronovskaja type theorems for the q-Bernstein operators. Our results will be formulated with the aid of the first order Ditzian-Totik modulus of smoothness (see [1]), which is given for \( f \in C[0,1] \) by
\[
\omega^1_{\varphi}(f, \delta) = \sup_{0 < h < \delta} \| \Delta_{h\varphi(x)}^1 f(\cdot) \|,
\tag{1.2}
\]
where \( \varphi(x) = \sqrt{x(1-x)}, x \in [0,1], \| \cdot \| \) is the uniform norm and
\[
\Delta_{h\varphi(x)}^1 f(x) = \begin{cases} f \left(x + \frac{1}{2} h \varphi(x)\right) - f \left(x - \frac{1}{2} h \varphi(x)\right), & \text{if } x \pm \frac{1}{2} h \varphi(x) \in [0,1] \\ 0, & \text{otherwise}. \end{cases}
\]
Further, the corresponding \( K \)-functional to (1.2) is defined by
\[
K^1_{1,\varphi}(f, \delta) = \inf\{\| f - g \| + \delta \| \varphi g' \| : g \in W^1(\varphi)\},
\]
where \( W^1(\varphi) \) is the set of all \( g \in C[0,1] \) such that \( g \) is absolutely continuous on every interval \([a,b] \subset [0,1]\) and \( \| \varphi g' \| < +\infty \). Then, in view of [1, p.11], there exists \( C > 0 \) such that
\[
K^1_{1,\varphi}(f, \delta) \leq C \omega^1_{\varphi}(f, \delta). \tag{1.3}
\]
Here we mention that throughout this paper \( C \) denotes a positive constant independent of \( n \) and \( x \), but it is not necessarily the same in different cases.
2. Main result

Our result is the following.

Theorem 2.1. Let \( \{q_n\} \) be a sequence such that \( 0 < q_n < 1 \) and \( q_n \to 1 \) as \( n \to \infty \). Then for every \( f \in C^2[0,1] \) the following inequalities hold

\[
\left| [n]_{q_n} \left( (B_{n,q_n} f)(x) - f(x) \right) - \frac{1}{2} x(1-x) f''(x) \right| \\
\leq C \omega_1^1 \left( f'', \sqrt{[n]_{q_n}^{-1} x(1-x)} \right), \tag{2.1}
\]

\[
\left| [n]_{q_n} \left( (B_{n,q_n} f)(x) - f(x) \right) - \frac{1}{2} x(1-x) f''(x) \right| \\
\leq C \sqrt{x(1-x)} \omega_1^1 \left( f'', \sqrt{[n]_{q_n}^{-1}} \right), \tag{2.2}
\]

where \( x \in [0,1] \) and \( n = 1, 2, \ldots \).

Proof. We recall some properties of the q-Bernstein operators (see [3]):

\[
B_{n,q_n}(1, x) = 1, \quad B_{n,q_n}(t, x) = x, \quad B_{n,q_n}(t^2, x) = x^2 + [n]_{q_n}^{-1} x(1-x) \tag{2.3}
\]

and \( B_{n,q_n} \) are positive.

Let \( f \in C^2[0,1] \) be given and \( t, x \in [0,1] \). Then, by Taylor’s formula,

\[
f(t) = f(x) + f'(x)(t-x) + \int_x^t f''(u)(t-u) \, du.
\]

Hence

\[
f(t) - f(x) - f'(x)(t-x) - \frac{1}{2} f''(x)(t-x)^2 \\
= \int_x^t f''(u)(t-u) \, du - \int_x^t f''(x)(t-u) \, du \\
= \int_x^t [f''(u) - f''(x)](t-u) \, du.
\]

In view of (2.3), we obtain

\[
\left| B_{n,q_n}(f, x) - f(x) - \frac{1}{2} [n]_{q_n}^{-1} x(1-x) f''(x) \right| \\
= \left| B_{n,q_n} \left( \int_x^t [f''(u) - f''(x)](t-u) \, du, x \right) \right| \\
\leq B_{n,q_n} \left( \left| \int_x^t [f''(u) - f''(x)] \, |t-u| \, du \right|, x \right). \tag{2.4}
\]

In what follows we estimate \( \int_x^t |f''(u) - f''(x)| \, |t-u| \, du \). For \( g \in W^1(\varphi) \), we have
where we have used the inequality $|\int_a^b f''(u) - f''(x)| |t-u| \, du |$ \\
\[ \leq \left| \int_a^b f''(u) - g(u) |t-u| \, du \right| + \left| \int_a^b g(u) - g(x) |t-u| \, du \right| \\
\[ + \left| \int_b^c g(x) - f''(x) |t-u| \, du \right| \\
\[ \leq 2\|f'' - g\|(t - x)^2 \]

On the other hand, by [2, p. 440], we have the following property: for any $m = 1, 2, \ldots$ and $0 < q < 1$, there exists a constant $C(m) > 0$ such that \\
\[ |B_{n,q}(t-x)^m| \leq C(m) \frac{\varphi^2(x)}{[n]_q^{(m+1)/2}}, \] 

where we have used the inequality $\frac{|u-v|}{\varphi^2(v)} \leq \frac{|u-x|}{\varphi^2(x)}$, $v$ is between $u$ and $x$ (see [1, p. 141]).

Now combining (2.4), (2.5), (2.6) and the Cauchy-Schwarz inequality, we find that \\
\[ \left| (B_{n,q}f)(x) - f(x) - \frac{1}{2} [n]_{q^n}^{-1} x (1-x) f''(x) \right| \]

where $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$ and $\lfloor a \rfloor$ is the integer part of $a \geq 0$ (see also [4, (4.2) and (5.6)]).
Because $\varphi^2(x) \leq \varphi(x) \leq 1$, $x \in [0, 1]$, we obtain, in view of (2.7),

$$\left| [n]_{q_n} \{ (B_{n,q} f)(x) - f(x) \} - \frac{1}{2} x(1-x) f''(x) \right| \leq C \left\{ \|f'' - g\| + \frac{\varphi(x)}{[n]^{1/2}_{q_n}} \|\varphi g'\| \right\}$$

(2.8)

and

$$\left| [n]_{q_n} \{ (B_{n,q} f)(x) - f(x) \} - \frac{1}{2} x(1-x) f''(x) \right| \leq C \varphi(x) \left\{ \|f'' - g\| + \frac{1}{[n]^{1/2}_{q_n}} \|\varphi g'\| \right\},$$

(2.9)

respectively. Taking the infimum on the right hand side of (2.8) and (2.9) over all $g \in W^1(\varphi)$, we obtain

$$\left| [n]_{q_n} \{ (B_{n,q} f)(x) - f(x) \} - \frac{1}{2} x(1-x) f''(x) \right| \leq \begin{cases} C K_{1,\varphi} (f'', \varphi(x)[n]^{-1/2}_{q_n}) \\ C \varphi(x) K_{1,\varphi} (f'', [n]^{-1/2}_{q_n}) \end{cases}.$$

Hence, by (1.3), we find the estimates (2.1) and (2.2). Thus the theorem is proved. □

References


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