Almost greedy uniformly bounded orthonormal bases in rearrangement invariant Banach function spaces

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Abstract. We construct uniformly bounded orthogonal almost greedy bases in rearrangement invariant Banach spaces.

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1. Introduction

Let \( \{x_n\}_{n \in \mathbb{N}} \) be a semi-normalized basis in a Banach space \( X \). This means that \( \{x_n\}_{n \in \mathbb{N}} \) is a Schauder basis and is semi-normalized i.e. \( 0 < \inf_{n \in \mathbb{N}} \|x_n\| \leq \sup_{n \in \mathbb{N}} \|x_n\| < \infty \). For an element \( x \in X \) we define the error of the best \( m \)-term approximation as follows

\[
\sigma_m(x) = \inf \{ \|x - \sum_{n \in A} \alpha_n x_n\| \},
\]

where the inf is taken over all subsets \( A \subset \mathbb{N} \) of cardinality at most \( m \) and all possible scalars \( \alpha_n \). The main question in approximation theory concerns the construction of efficient algorithms for \( m \)-term approximation. A computationally efficient method to produce \( m \)-term approximations, which has been widely investigated in recent years, is the so called greedy algorithm. We define the greedy approximation of \( x = \sum_n a_n x_n \in X \) as

\[
G_m(x) = \sum_{n \in A} a_n x_n,
\]

where \( A \subset \mathbb{N} \) is any set of the cardinality \( m \) in such a way that \( |a_n| \geq |a_l| \) whenever \( n \in A \) and \( l \not\in A \). We say that a semi-normalized basis \( \{x_n\}_{n \in \mathbb{N}} \) is

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greedy if there exists a constant $C$ such that for all $m = 1, 2, \ldots$ and all $x \in X$ we have

$$\|x - G_m(x)\| \leq C \sigma_m(x).$$

This notion evolved in theory of non-linear approximation (see e.g. [1], [2]). A result of Konyagin and Temlyakov [3] characterizes greedy bases in a Banach spaces $X$ as those which are unconditional and democratic, the latter meaning that for some constant $C > 0$

$$\left\| \sum_{\alpha \in A} \frac{x_\alpha}{\|x_\alpha\|} \right\| \leq C \left\| \sum_{\alpha \in A'} \frac{x_\alpha}{\|x_\alpha\|} \right\|$$

holds for all finite sets of indices $A, A' \subset \mathbb{N}$ with the same cardinality.

Wavelet systems are well known examples of greedy bases for many function and distribution spaces. Indeed, Temlyakov showed in [1] that the Haar system is greedy in the Lebesgue spaces $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. When wavelets have sufficient smoothness and decay, they are also greedy bases for the more general Sobolev and Tribel-Lizorkin classes (see e.g. [4-5]).

A bounded Schauder basis for a Banach space $X$ is called quasi-greedy if there exists a constant $C$ such that for $x \in X \|x - G_m(x)\| \leq C \|x\|$ for $m \geq 1$.

Wojtaszczyk [2] proved the following result which gives a more intuitive interpretation of quasi-greedy bases.

**Theorem 1.1.** A bounded Schauder basis for a Banach space $X$ is quasi-greedy if and only if $\lim_{m \to \infty} \|x - G_m(x)\|_X = 0$ for every element $x \in X$.

A bounded Schauder basis for a Banach space $X$ is almost greedy if there exists a constant $C$ such that $x \in X \|x - G_m(x)\| \leq C \inf \{\|x - \sum_{n \in A} \chi_{[0,1]}(t/s) \| : A \subset \mathbb{N}, \ |A| = m \}$. It was proved in [6] that a basis is almost greedy if and only if it is quasi-greedy and democratic.

A Banach function space on $[0, 1]$ is said to be a rearrangement invariant (r.i) space provided $f^*(t) \leq g^*(t)$ for every $t \in [0, 1]$ and $g \in X$ imply $f \in X$ and $\|f\|_X \leq \|g\|_X$, where $f^*(t)$ denotes the decreasing rearrangement of $|f|$. An r.i. space $X$ with a norm $\|\cdot\|_X$ has the Fatou property if for any increasing positive sequence $f_n$ in $X$ with $\sup_n \|f_n\|_X < \infty$ we have that $\sup_n f_n \in X$ and $\|\sup_n f_n\|_X = \sup_n \|f_n\|_X$. We will assume that the r.i. space $X$ has the Fatou property.

Given $s > 0$, the dilation operator $\sigma_s$ given by

$$\sigma_s f(t) = f(t/s) \chi_{[0,1]}(t/s), \ t \in [0, 1]$$

($\chi_A$ denotes the characteristic function of a measurable set $A \subset [0, 1]$) is well defined in every r.i. space $X$. The classical Boyd indices of $X$ are defined by

$$p_X = \lim_{s \to \infty} \frac{\ln s}{\ln \|\sigma_s\|_X}, \ q_X = \lim_{s \to 0^+} \frac{\ln s}{\ln \|\sigma_s\|_X}.$$ 

In general, $1 \leq p_X \leq q_X \leq \infty$. 


Any r.i. function space $X$ on $[0,1]$ satisfies $L^\infty([0,1]) \subset X \subset L^1([0,1])$. If we have information on the Boyd indices of $X$ then a stronger assertion is valid. Indeed for every $1 \leq p < p_X$ and $q_X < q < \infty$, we have

$$L^q([0,1]) \subset X \subset L^p([0,1]) \quad (1.1)$$

with the inclusion maps being continuous. Let $X'$ denote the associate Banach function space of $X$. Then $X'$ is a r.i. Banach function space whose Boyd indices are defined as $1/p_X + 1/q_X' = 1$ and $1/q_X + 1/p_X' = 1$ (see [7]).

M. Nielsen in [8] proved that there exists a uniformly bounded orthonormal almost greedy basis in $L^p([0,1]), \ 1 < p < \infty$, that shows that it is not possible to extend Orlicz's theorem, stating that there are no uniformly bounded orthonormal unconditional bases for $L^p([0,1]), \ p \neq 2$, to the class of almost greedy bases.

The purpose of this paper is to study these problems in the r.i. function spaces. Namely, the following theorem is obtained.

**Theorem 1.2.** Let $X$ be a separable r.i. Banach function space on $[0,1]$ and $1 < p_X \leq q_X < 2$ or $2 < p_X \leq q_X < \infty$. Then there exists a uniformly bounded orthogonal almost greedy basis in $X$.

**2. Proof of theorem**

Let us construct some system in the following way. For $k = 1, 2, \ldots$, we define the $2^k \times 2^k$ Olevskii matrix $A^k = (a_{ij}^{(k)})_{i,j=1}^{2^k}$ by the following formulas

$$a_{i1}^k = 2^{-\frac{k}{2}}, \text{ for } i = 1, 2, \ldots, 2^k,$$

and for $j = 2^s + \nu$, with $1 \leq \nu \leq 2^s$ and $s = 0, 1, \ldots, k - 1$, we let

$$a_{ij}^{(k)} = \begin{cases} 2^{\frac{s-k}{2}} & \text{for } (\nu - 1)2^{k-s} < i \leq (2\nu - 1)2^{k-s-1} \\ -2^{\frac{s-k}{2}} & \text{for } (2\nu - 1)2^{k-s-1} < i \leq \nu2^{k-s} \\ 0 & \text{otherwise.} \end{cases}$$

It is known [16] that $A^k$ are orthogonal matrices and there exists a finite constant $C$ such that for all $i, k$ we have

$$\sum_{j=1}^{2^k} |a_{ij}^{(k)}| \leq C.$$
The block $B_k := \{\phi_k, r_{F_k-1+1}, ..., r_{F_k}\}$ has length $N_k$ and we apply $A^{10^k}$ to $B_k$ to obtain a new orthonormal system $\{\psi_i^{(k)}\}_{i=1}^{N_k}$ given by

$$\psi_i^{(k)} = \frac{\phi_k}{\sqrt{N_k}} + \sum_{j=2}^{N_k} a^{10^k}_{ij} r_{F_k-1+j-1}.$$ 

The system ordered $\psi^{(1)}_1, ..., \psi^{(1)}_{N_1}, \psi^{(2)}_1, ..., \psi^{(2)}_{N_2}, ...$ will be denoted by $B = \{\psi_k\}_{k=1}^\infty$. It is easy to verify that $B$ is an orthonormal basis for $L_2$ since each matrix $A^{10^k}$ is orthogonal and it is uniformly bounded also.

**Lemma 2.1.** Let $X$ be a r.i. Banach function space on $[0,1]$ and $1 < p_X \leq q_X < \infty$. The system $B = \{\psi_k\}_{k=1}^\infty$ is democratic in $X$ with

$$\| \sum_{k \in A} \psi_k \|_X \asymp |A|^{\frac{1}{2}}.$$ 

**Proof.** Taking into account that fact that $B\| \cdot \|_{p_X} \leq \| \cdot \|_X \leq C\| \cdot \|_{q_X}$ and the estimate (see [8])

$$\| \sum_{k \in A} \psi_k \|_p \asymp |A|^{\frac{1}{2}}$$

for any $1 < p < \infty$ we obtain our result. \hfill \Box

**Lemma 2.2.** (Khintchine’s inequality )Suppose that $X$ is a r.i. Banach function space on $[0,1]$, $1 < p_X \leq q_X < \infty$, and $r_k(t), k \geq 1$, are the Rademacher functions. Then there exist $A, B$ such that for any sequence $\{a_k\}_{k \geq 1}$,

$$A(\sum_k |a_k|^2)^{\frac{1}{2}} \leq \| \sum_k a_k r_k(t) \|_X \leq B(\sum_k |a_k|^2)^{\frac{1}{2}}.$$ 

**Proof.** It is known that (see [10]) for $1 \leq p < \infty$ there exist $A_p, B_p$ such that for any sequence $\{a_k\}_{k \geq 1}$,

$$A_p(\sum_k |a_k|^2)^{\frac{1}{2}} \leq \| \sum_k a_k r_k(t) \|_p \leq B_p(\sum_k |a_k|^2)^{\frac{1}{2}}.$$ 

Taking into account that fact that $B\| \cdot \|_{q_X} \leq \| \cdot \|_X \leq C\| \cdot \|_{p_X}$ and the above inequality we obtain Lemma 2.2. \hfill \Box

**Lemma 2.3.** Suppose that $X$ is a r.i. Banach function space on $[0,1]$, $1 < p_X \leq q_X < \infty$, and $r_k(t), k \geq 1$, are the Rademacher functions. Then for $f \in X$ we have

$$\left(\sum_{k=1}^\infty | < f, r_k > |^2 \right)^{\frac{1}{2}} \leq C\|f\|_X.$$ 

**Proof.** For any $n \geq 1$ by the Hölder inequality and Khintchine’s inequality we obtain

$$\sum_{k=1}^n | < f, r_k > |^2 = \int_0^1 f(x)(\sum_{k=1}^n r_k(x) < f, r_k >)dx \leq 2\| \sum_{k=1}^n < f, r_k > r_k \|_X \|f\|_X \leq C(\sum_{k=1}^n | < f, r_k > |^2)^{1/2}\|f\|_X.$$ 

This implies
\[
\left( \sum_{k=1}^{n} | < f, r_k > |^2 \right)^{\frac{1}{2}} \leq B\|f\|_X.
\]

Now taking the limit when \( n \to \infty \) we obtain our result. \( \square \)

**Lemma 2.4.** Let \( X \) be a separable r.i. Banach function space on \([0, 1]\) and \( 1 < p_X \leq q_X < \infty \). Then the system \( \mathcal{B} = \{ \psi_k \}_{k=1}^{\infty} \) is a Schauder basis for \( X \).

**Proof.** Notice that \( \text{span}(\mathcal{B}) = \text{span}(\mathcal{W}) \) by construction, so \( \text{span}(\mathcal{B}) \) is dense in \( X \), since \( \mathcal{W} \) is a Schauder basis for \( X \) (see [11]).

Let \( S_n(f) = \sum_{k=1}^{n} < f, \psi_k > \psi_k \) be the partial sum operator. We need to prove that the family of operators \( \{ S_n \}_{n=1}^{\infty} \) is uniformly bounded on \( X \).

Let \( f \in L^\infty([0, 1]) \subset L^2([0, 1]) \). For \( n \in \mathbb{N} \) we can find \( L \geq 1 \) and \( 1 \leq m \leq N_L \) such that
\[
S_n(f) = \sum_{k=1}^{n} < f, \psi_k > \psi_k = \sum_{k=1}^{L-1} \sum_{j=1}^{N_k} < f, \psi_j^{(k)} > \psi_j^{(k)} + \sum_{k=1}^{m} < f, \psi_k^{(L)} > \psi_k^{(L)} := T_1 + T_2.
\]

Let us estimate \( T_1 \). If \( L = 1 \) then \( T_1 = 0 \), so we may assume \( L > 1 \). The construction of \( \mathcal{B} \) shows that \( T_1 \) is the orthogonal projection of \( f \) onto \( \text{span } \cup_{k=1}^{L-1} \cup_{j=1}^{N_k} \psi_k^{(j)} \) = \( \text{span } \{ W_0, W_1, ..., W_{L-2} \} \cup \{ r_{l_0}, r_{l_0+1}, ..., r_{F_{L-1}} \} \), with \( l_0 = \lfloor \log_2(L) \rfloor \). It follows that we can rewrite \( T_1 \) as
\[
T_1 = \sum_{k=0}^{L-2} < f, W_k > W_k + P_R(f),
\]
where \( P_R(f) \) is the orthogonal projection of \( f \) onto \( \text{span } \{ r_{l_0}, r_{l_0+1}, ..., r_{F_{L-1}} \} \).

Thus, using the fact that \( \mathcal{W} \) is a Schauder basis for \( X \), Khintchine’s inequality and Lemma 2.3, we will have
\[
\|T_1\|_X \leq C\|f\|_X.
\]

Let us now estimate \( T_2 \).
\[
T_2 = \sum_{k=1}^{m} < f, \psi_k^{(L)} > \psi_k^{(L)}
\]
\[
= \sum_{k=1}^{m} < f, \frac{\phi_L}{\sqrt{N_L}} > + \sum_{j=2}^{N_L} a_j^{(10^L)} r_{F_{L-1}+j-1} \geq (\frac{\phi_L}{\sqrt{N_L}}) f + \sum_{t=2}^{N_L} a_t^{(10^L)} r_{F_{L-1}+t-1}
\]
\[
= \frac{m}{N_L} < f, \phi_L > + \frac{\phi_L}{\sqrt{N_L}} \sum_{j=2}^{N_L} (\sum_{k=1}^{m} a_k^{(10^L)}) < f, r_{F_{L-1}+j-1} > + < f, \frac{\phi_L}{\sqrt{N_L}} > \sum_{j=2}^{N_L} (\sum_{k=1}^{m} a_k^{(10^L)}) r_{F_{L-1}+j-1}
\]
\[ + \sum_{k=1}^{m} \left( \sum_{j=2}^{N_L} a_{kj}^{(10_L)} \right) \langle f, r_{F_{L-1}+j-1} \rangle \sum_{t=2}^{N_L} a_{kt}^{(10_L)} r_{F_{L-1}+t-1} \]  

\[ = G_1 + G_2 + G_3 + G_4. \]

Using that fact that \( 1 \leq m \leq N_L \) and Hölder inequality we obtain \( \| G_1 \|_X \leq C \| f \|_X \). Using the Hölder and Khintchine’s inequality, the fact that matrices \( A^k \) are orthonormal, and Lemma 2.3 we obtain \( \| G_i \|_X \leq C \| f \|_X \) for \( i = 2, 3, 4 \) for some constant \( C \) independent of \( f \in L^\infty([0, 1]) \). Consequently for some constant \( C \) independent of \( f \in L^\infty([0, 1]) \) we have \( \| S_n f \|_X \leq C \| f \|_X \).

**Theorem 2.5.** Let \( X \) be a separable r.i. Banach function space on \([0, 1]\) and \( 1 < p_X \leq q_X < \infty \). Then there exists a uniformly bounded orthonormal democratic basis in \( X \).

**Lemma 2.6.** Let \( X \) be a separable r.i. Banach function space on \([0, 1]\) and \( 1 < p_X \leq q_X \leq 2 \) or \( 2 < p_X \leq q_X < \infty \). Then the system \( \mathcal{B} = \{ \psi_k \}_{k=1}^\infty \) is a quasi-greedy basis for \( X \).

**Proof.** First we consider \( 2 < p_X \leq q_X < \infty \) case. Let \( f \in X \subset L_2 \). We have

\[ f = \sum_{i=1}^{\infty} \langle f, \psi_i \rangle > \psi_i, \]

with \( \| \{ \langle f, \psi_i \rangle \} \|_2 \leq \| f \|_2 \leq C \| f \|_X \). We must prove that \( G_m(f) \) is convergent in \( X \).

Let us formally write

\[ f = \sum_{k=1}^{\infty} \sum_{j=1}^{N_k} \langle f, \psi_j^{(k)} \rangle > \psi_j^{(k)} = \sum_{k=1}^{\infty} \sum_{j=1}^{N_k} \langle f, \psi_j^{(k)} \rangle > \frac{\phi_k}{\sqrt{N_k}} + \sum_{k=1}^{\infty} \sum_{i=1}^{N_k} \langle f, \psi_i^{(k)} \rangle > \sum_{j=2}^{N_k} a_{ij}^{(10_k)} r_{F_{k-1}+j-1} \]

\[ = S_1 + S_2. \]

Consider \( \varepsilon_i^k \in \{0, 1\} \). By Khintchine’s inequality and the fact that each \( A^{10_k} \) is orthogonal we conclude that \( S_2 \) converges unconditionally in \( X \). Indeed

\[ \left\| \sum_{k=1}^{\infty} \sum_{j=2}^{N_k} \left( \sum_{i=1}^{N_k} \varepsilon_i^k \langle f, \psi_i^{(k)} \rangle > a_{ij}^{(10_k)} \right) r_{F_{k-1}+j-1} \right\|_X \]

\[ \leq C \left( \sum_{k=1}^{N_k} \sum_{i=1}^{N_k} \varepsilon_i^k | \langle f, \psi_i^{(k)} \rangle > |^2 \right)^{1/2}. \]
The series defining $S_2$ converges unconditionally, so it suffices to prove that the series defining $S_1$ converges in $X$ when the coefficients $\langle f, \psi \rangle$ are arranged in decreasing order. Let us consider the sets

$$\Lambda_k^1 = \left\{ j : \frac{1}{N_k} < \left| \langle f, \psi_j^{(k)} \rangle \right| < \frac{1}{N_k^{1/10}} \right\}$$

$$\Lambda_k^2 = \left\{ j : \left| \langle f, \psi_j^{(k)} \rangle \right| \leq \frac{1}{N_k} \right\}$$

$$\Lambda_k^3 = \left\{ j : \left| \langle f, \psi_j^{(k)} \rangle \right| \geq \frac{1}{N_k^{1/10}} \right\}.$$  

Then

$$S_1 = \sum_{k=1}^{\infty} \sum_{j \in \Lambda_k^1} <f, \psi_j^{(k)}> \frac{\phi_k}{\sqrt{N_k}} + \sum_{k=1}^{\infty} \sum_{j \in \Lambda_k^2} <f, \psi_j^{(k)}> \frac{\phi_k}{\sqrt{N_k}} + \sum_{k=1}^{\infty} \sum_{j \in \Lambda_k^3} <f, \psi_j^{(k)}> \frac{\phi_k}{\sqrt{N_k}} = T_1 + T_2 + T_3.$$

By the construction of sets $\Lambda_k^i$, we can conclude that the series defining $T_2$ and $T_3$ converges absolutely in $X$.

From the definition of $\Lambda_k^1$ we get

$$\left| \langle f, \psi_i^{(k)} \rangle \right| > \frac{1}{N_k} \geq \frac{1}{N_k^{1/10}} \geq \left| \langle f, \psi_j^{(k+1)} \rangle \right|,$$

$i \in \Lambda_k^1$, $j \in \Lambda_{k+1}^1$, $k = 1, 2, ...$ so when we arrange $T_1$ by decreasing order the rearrangement can only take place inside the blocks. From the estimate

$$\sum_{j \in \Lambda_k^1} \left\| <f, \psi_j^{(k)}> \frac{\phi_k}{\sqrt{N_k}} \right\|_X \leq \left( \sum_{j \in \Lambda_k^1} \left| <f, \psi_j^{(k)}> \right|^2 \right)^{1/2} \frac{|\Lambda_k^1|^{1/2}}{\sqrt{N_k}}, \ k \geq 1$$

we obtain that the rearrangements inside blocks are well-behaved, and

$$\sum_{j \in \Lambda_k^1} \left\| <f, \psi_j^{(k)}> \frac{\phi_k}{\sqrt{N_k}} \right\|_X \to 0, \ k \to \infty.$$

We can conclude that $G_m(f)$ is convergent in $X$.

Using Theorem 1.1 we conclude that $B$ is a quasi-greedy basis and consequently almost greedy in $X$.

Let $1 < p_X \leq q_X < 2$. By the results proved above it follows that the system $B$ is almost greedy in $X$. From [6, Theorem 5.4] we conclude that $B$ is quasi-greedy basis and consequently almost greedy in $X$.

This completes the proof. □
References


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