Asymptotic expansions for Favard operators and their left quasi-interpolants

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Abstract. In 1944 Favard [5, pp. 229, 239] introduced a discretely defined operator which is a discrete analogue of the familiar Gauss-Weierstrass singular convolution integral. In the present paper we consider a slight generalization $F_{n,\sigma_n}$ of the Favard operator and its Durrmeyer variant $\tilde{F}_{n,\sigma_n}$ and study the local rate of convergence when applied to locally smooth functions. The main result consists of the complete asymptotic expansions for the sequences $(F_{n,\sigma_n}f)(x)$ and $(\tilde{F}_{n,\sigma_n}f)(x)$ as $n$ tends to infinity. Furthermore, these asymptotic expansions are valid also with respect to simultaneous approximation. Finally, we define left quasi-interpolants for the Favard operator and its Durrmeyer variant in the sense of Sablonniere.

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1. Introduction

In 1944 J. Favard [5, pp. 229, 239] introduced the operator

$$(F_n f)(x) = \frac{1}{\sqrt{\pi n}} \sum_{\nu=-\infty}^{\infty} f\left(\frac{\nu}{n}\right) \exp\left(-n\left(\frac{\nu}{n} - x\right)^2\right)$$

(1.1)

which is a discrete analogue of the familiar Gauss-Weierstrass singular convolution integral

$$(W_n f)(x) = \frac{n}{\pi} \int_{-\infty}^{\infty} f(t) \exp\left(-n(t-x)^2\right) dt.$$
Basic properties such as saturation in weighted spaces can be found in [3] and [2]. For a sequence of positive reals \( \sigma_n \), the generalization

\[
(F_{n,\sigma_n} f) (x) = \sum_{\nu=-\infty}^{\infty} p_{n,\nu,\sigma_n} (x) f \left( \frac{\nu}{n} \right),
\]

where

\[
p_{n,\nu,\sigma_n} (x) = \frac{1}{\sqrt{2\pi n\sigma_n}} \exp \left( -\frac{1}{2} \frac{\nu^2}{\sigma_n^2} \right),
\]

was introduced and studied by Gawronski and Stadtmüller [7]. The particular case \( \sigma_n^2 = \gamma/(2n) \) with a constant \( \gamma > 0 \) reduces to Favard’s classical operators (1.1). The operators can be applied to functions \( f \) defined on \( \mathbb{R} \) satisfying the growth condition

\[
f (t) = O \left( e^{Kt^2} \right) \quad \text{as} \quad |t| \to \infty,
\]

for a constant \( K > 0 \).

In 2007 Nowak and Sikorska-Nowak [11] considered a Kantorovich variant [11, Eq. (1.5)]

\[
(\hat{F}_{n,\sigma_n} f) (x) = n \sum_{\nu=-\infty}^{\infty} p_{n,\nu,\sigma_n} (x) \int_{\nu/n}^{(\nu+1)/n} p_{n,\nu,\sigma_n} (t) f (t) \, dt
\]

and a Durrmeyer variant [11, Eq. (1.6)]

\[
(\tilde{F}_{n,\sigma_n} f) (x) = n \sum_{\nu=-\infty}^{\infty} p_{n,\nu,\sigma_n} (x) \int_{-\infty}^{\infty} p_{n,\nu,\sigma_n} (t) f (t) \, dt
\]

of Favard operators. Further related papers are [12] and [13].

The main result of this paper consists of the complete asymptotic expansions

\[
F_{n,\sigma_n} f \sim f + \sum_{k=0}^{\infty} c_k (f) \sigma_n^k \quad \text{and} \quad \tilde{F}_{n,\sigma_n} f \sim f + \sum_{k=0}^{\infty} \tilde{c}_k (f) \sigma_n^k \quad (n \to \infty),
\]

for \( f \) sufficiently smooth. The coefficients \( c_k \) and \( \tilde{c}_k \), which depend on \( f \) but are independent of \( n \), are explicitly determined. It turns out that \( c_k (f) = 0 \), for all odd integers \( k > 0 \). Moreover, we deal with simultaneous approximation by the operators (1.2).

Finally, we define left quasi-interpolants for the Favard operator and its Durrmeyer variant in the sense of Sablonniere.

### 2. Complete asymptotic expansions

Throughout the paper, we assume that

\[
\sigma_n > 0, \quad \sigma_n \to 0, \quad \sigma_n^{-1} = O \left( n^{1-\eta} \right) \quad (n \to \infty)
\]

(2.1)
Asymptotic expansions for Favard operators

with (an arbitrarily small) constant $\eta > 0$. Note that the latter condition implies that $n \sigma_n \to \infty$ as $n \to \infty$.

Under these conditions, the operators possess the basic property that $(F_n f)(x)$ converges to $f(x)$ in each continuity point $x$ of $f$. Among other results, Gawronski and Stadtmüller [7, Eq. (0.6)] established the Voronovskaja-type theorem

$$\lim_{n \to \infty} \frac{1}{\sigma_n^2} [(F_n, \sigma_n f)(x) - f(x)] = \frac{1}{2} f''(x)$$

(2.2)

uniformly on proper compact subsets of $[a, b]$, for $f \in C^2[a, b]$ ($a, b \in \mathbb{R}$) and $\sigma_n \to 0$ as $n \to \infty$, provided that certain conditions on the first three moments of $F_n, \sigma_n$ are satisfied. Actually, Eq. (2.2) was proved for a truncated variant of (1.2) which possesses the same asymptotic properties as (1.2) [7, cf. Theorem 1 (iii) and Remark (i), p. 393]. For a Voronovskaja-type theorem in the particular case $\sigma_n^2 = \gamma/(2n)$ cf. [3, Theorem 4.3]. Abel and Butzer extended Formula (2.2) by deriving a complete asymptotic expansion of the form

$$F_{n, \sigma_n} f \sim f + \sum_{k=0}^{\infty} c_k(f) \sigma_n^k \quad (n \to \infty),$$

for $f$ sufficiently smooth. The latter formula means that, for all positive integers $q$, there holds pointwise on $\mathbb{R}$

$$F_{n, \sigma_n} f = f + \sum_{k=1}^{q} c_k(f) \sigma_n^k + o(\sigma_n^q) \quad (n \to \infty).$$

The following theorem presents the main result of this paper, the complete asymptotic expansion for the sequence $(\tilde{F}_{n, \sigma_n})(x)$ as $n \to \infty$. For $r \in \mathbb{N}$ and $x \in \mathbb{R}$ let $W[r; x]$ be the class of functions on $\mathbb{R}$ satisfying growth condition (1.3), which admit a derivative of order $r$ at the point $x$.

**Theorem 2.1.** Let $q \in \mathbb{N}$ and $x \in \mathbb{R}$. Suppose that the real sequence $(\sigma_n)$ satisfies the conditions (2.1). For each function $f \in W[2q; x]$, the Favard-Durrmeyer operators (1.4) possess the complete asymptotic expansions

$$(F_{n, \sigma_n} f)(x) = f(x) + \sum_{k=1}^{q} \frac{f(2k)(x)}{(2k)!!} \sigma_n^{2k} + o(\sigma_n^{2q})$$

(2.3)

and

$$(\tilde{F}_{n, \sigma_n} f)(x) = f(x) + \sum_{k=1}^{q} \frac{f(2k)(x)}{k!} \sigma_n^{2k} + o(\sigma_n^{2q})$$

(2.4)

as $n \to \infty$.

Here $m!!$ denote the double factorial numbers defined by $0!! = 1!! = 1$ and $m!! = m \times (m - 2)!!$ for integers $m \geq 2$. It turns out that the asymptotic expansions contain only terms with even order derivatives of the function $f$.

As an immediate consequence we obtain the following Voronovskaja-type theorems.
Corollary 2.2. Let \( x \in \mathbb{R} \). Suppose that the real sequence \((\sigma_n)\) satisfies the conditions (2.1). For each function \( f \in W[2; x] \), there hold the asymptotic relations

\[
\lim_{n \to \infty} \sigma_n^{-2} ((F_n, \sigma_n) f (x) - f (x)) = \frac{1}{2} f'' (x)
\]

and

\[
\lim_{n \to \infty} \sigma_n^{-2} \left( (\tilde{F}_n, \sigma_n) f (x) - f (x) \right) = f'' (x).
\]

Concerning simultaneous approximation, it turns out that the complete asymptotic expansion (2.3) can be differentiated term-by-term. Indeed, there holds

Theorem 2.3. Let \( \ell \in \mathbb{N}_0 \), \( q \in \mathbb{N} \) and \( x \in \mathbb{R} \). Suppose that the real sequence \((\sigma_n)\) satisfies condition (2.1). For each function \( f \in W[2 (\ell + q) ; x] \), the following complete asymptotic expansions are valid as \( n \to \infty \):

\[
(F_n, \sigma_n) f^{(\ell)} (x) = f (x) + \sum_{k=1}^{q} \frac{f^{(2k+\ell)} (x)}{(2k)!! \sigma_n^2} + o \left( \sigma_n^{2q} \right) \tag{2.5}
\]

and

\[
(\tilde{F}_n, \sigma_n) f^{(\ell)} (x) = f^{(\ell)} (x) + \sum_{k=1}^{q} \frac{f^{(2k+\ell)} (x)}{k! \sigma_n^2} + o \left( \sigma_n^{2q} \right). \tag{2.6}
\]

Remark 2.4. The latter formulas can be written in the equivalent form

\[
\lim_{n \to \infty} \sigma_n^{-2q} \left( (F_n, \sigma_n) f^{(\ell)} (x) - f^{(\ell)} (x) - \sum_{k=1}^{q} \frac{f^{(2k+\ell)} (x)}{(2k)!! \sigma_n^2} \right) = 0,
\]

\[
\lim_{n \to \infty} \sigma_n^{-2q} \left( (\tilde{F}_n, \sigma_n) f^{(\ell)} (x) - f^{(\ell)} (x) - \sum_{k=1}^{q} \frac{f^{(2k+\ell)} (x)}{k! \sigma_n^2} \right) = 0.
\]

Assuming smoothness of \( f \) on intervals \( I = (a, b) \), \( a, b \in \mathbb{R} \), it can be shown that the above expansions hold uniformly on compact subsets of \( I \).

The proofs are based on localization theorems which are interesting in themselves. We quote only the result for the ordinary Favard operator (1.2).

Proposition 2.5. Fix \( x \in \mathbb{R} \) and let \( \delta > 0 \). Assume that the function \( f : \mathbb{R} \to \mathbb{R} \) vanishes in \( (x - \delta, x + \delta) \) and satisfies, for positive constants \( M_x, K_x \), the growth condition

\[
|f (t)| \leq M_x e^{K_x (t-x)^2} \quad (t \in \mathbb{R}). \tag{2.7}
\]

Then, for positive \( \sigma < 1/\sqrt{2K_x} \), there holds the estimate

\[
|F_n, \sigma f (x)| \leq \sqrt{\frac{2}{\pi}} \frac{M_x \sigma / \delta}{1 - 2K_x \sigma^2} \exp \left( - \frac{1 - 2K_x \sigma^2}{2} \left( \frac{\delta}{\sigma} \right)^2 \right).
\]
Consequently, under the general assumption (2.1) a positive constant $A$ (independent of $\delta$) exists such that the sequence $((F_{n,\sigma_n}f)(x))$ can be estimated by

$$(F_{n,\sigma_n}f)(x) = o\left( \exp\left( -A\frac{\delta^2}{\sigma_n^2} \right) \right) \quad (n \to \infty).$$

**Remark 2.6.** Note that a function $f : \mathbb{R} \to \mathbb{R}$ satisfies condition (2.1) if and only if condition (2.7) is valid. The elementary inequality $(t - x)^2 \leq 2(t^2 + x^2)$ implies that

$$M_x e^{K_x(t-x)^2} \leq M e^{Kt^2} \quad (t, x \in \mathbb{R})$$

with constants $M = M_x e^{2Kx^2}$ and $K = 2K_x$.

### 3. Quasi-interpolants

The results of the preceding section show that the optimal degree of approximation cannot be improved in general by higher smoothness properties of the function $f$. In order to obtain much faster convergence quasi-interpolants were considered. Let us shortly recall the definition of the quasi-interpolants in the sense of Sablonniere [14]. For another method to construct quasi-interpolants see [8] and [9].

If the operators $B_n$ let invariant the space of algebraic polynomials $\Pi_j$ of each order $j = 0, 1, 2, \ldots$ (the most approximation operators possess this property), i.e.,

$$B_n(\Pi_j) \subseteq \Pi_j \quad (0 \leq j \leq n),$$

$B_n : \Pi_n \to \Pi_n$ is an isomorphism which can be represented by linear differential operators

$$B_n = \sum_{k=0}^{n} \beta_{n,k} D^k$$

with polynomial coefficients $\beta_{n,k}$ and $Df = f'$, $D^0 = \text{id}$. The inverse operator $B_n^{-1} \equiv A : \Pi_n \to \Pi_n$ satisfies

$$A = \sum_{k=0}^{n} \alpha_{n,k} D^k$$

with polynomial coefficients $\alpha_{n,k}$. Sablonniere defined new families of intermediate operators obtained by composition of $B_n$ and its truncated inverses

$$A_n^{(r)} = \sum_{k=0}^{r} \alpha_{n,k} D^k.$$

In this way he obtained a family of left quasi-interpolants (LQI) defined by

$$B_n^{(r)} = A_n^{(r)} \circ B_n, \quad 0 \leq r \leq n,$$

and a family of right quasi-interpolants (RQI) defined by

$$B_n^{[r]} = B_n \circ A_n^{(r)}, \quad 0 \leq r \leq n.$$
Obviously, there holds $B_n^{(0)} = B_n^{[0]} = B_n$, and $B_n^{(n)} = B_n^{[n]} = I$ when acting on $\Pi_n$. In the following we consider only the family of LQI. The definition reveals that $B_n^{(r)} f$ is a linear combination of derivatives of $B_n f$. Furthermore, $B_n^{(r)} (0 \leq r \leq n)$ has the nice property to preserve polynomials of degree up to $r$, because, for $p \in \Pi_r$, we have

$$B_n^{(r)} p = \left(A_n^{(r)} \circ B_n\right) p = \sum_{k=0}^{r} \alpha_{n,k} D^k B_n p = \sum_{k=0}^{n} \alpha_{n,k} D^k (B_n p)$$

$$= \left(A_n^{-1} \circ B_n\right) p = p.$$ 

In many instances there holds $L_n^{(r)} f - f = O\left(n^{-|r/2+1|}\right)$ as $n \to \infty$.

Unfortunately, the Favard operator as well as its Durrmeyer variant doesn’t let invariant the spaces $\Pi_j$, for $0 \leq j \leq n$. However, under appropriate assumptions on the sequence ($\sigma_n$) they do it asymptotically up to a remainder which decays exponentially fast as $n$ tends to infinity. Writing $\simeq$ for this “asymptotic equality” we obtain, for fixed $n \in \mathbb{N}$,

$$F_{n,\sigma_n} p_k \simeq e_k$$

with

$$p_k = k! \sum_{j=0}^{\left\lfloor k/2 \right\rfloor} (-1)^j \frac{\sigma_n^{2j}}{2j! (k-2j)!} e_{k-2j},$$

where $e_m$ denote the monomials $e_m(t) = t^m$ ($m = 0, 1, 2, \ldots$). Hence, for the inverse,

$$(F_{n,\sigma_n})^{-1} e_k \simeq p_k = \sum_{j=0}^{\left\lfloor k/2 \right\rfloor} (-1)^j \frac{\sigma_n^{2j}}{2j!} D^{2j} e_k$$

Note that $\beta_n,2k+1 = \alpha_n,2k+1 = 0$ ($k = 0, 1, 2, \ldots$) and that neither $\beta_n,k$ nor $\alpha_n,k$ depend on the variable $x$. The analogous results for the Favard-Durrmeyer operators are similar. Proceeding in this way we define the following operators:

**Definition 3.1 (Favard quasi-interpolants).** The left quasi-interpolants $F_n^{(r)}$ and $\tilde{F}_n^{(r)}$ ($r = 0, 1, 2, \ldots$) of the Favard and Favard-Durrmeyer operators, respectively, are given by

$$F_{n,\sigma_n}^{(r)} = \sum_{k=0}^{r} \alpha_{n,k} D^k F_{n,\sigma_n} := \sum_{k=0}^{\left\lfloor r/2 \right\rfloor} (-1)^k \frac{\sigma_n^{2k}}{k!} D^{2k} F_{n,\sigma_n}$$

and

$$\tilde{F}_{n,\sigma_n}^{(r)} = \sum_{k=0}^{r} \tilde{\alpha}_{n,k} D^k \tilde{F}_{n,\sigma_n} := \sum_{k=0}^{\left\lfloor r/2 \right\rfloor} (-1)^k \frac{\sigma_n^{2k}}{k!} D^{2k} \tilde{F}_{n,\sigma_n}.$$ 

**Remark 3.2.** Note that $F_{n,\sigma_n}^{(2r)} = F_{n,\sigma_n}^{(2r+1)}$ and $\tilde{F}_{n,\sigma_n}^{(2r)} = \tilde{F}_{n,\sigma_n}^{(2r+1)}$ ($r = 0, 1, 2, \ldots$).

The local rate of convergence is given by the next theorem.
**Theorem 3.3.** Let $\ell \in \mathbb{N}_0$, $q \in \mathbb{N}$ and $x \in \mathbb{R}$. Suppose that the real sequence $(\sigma_n)$ satisfies condition (2.1). For each function $f \in W[2(\ell + q); x]$, the following complete asymptotic expansions are valid as $n \to \infty$:

\[
\left( F_{n,\sigma_n}^{(2r)} f \right)^{(\ell)} (x) \sim f^{(\ell)} (x) + (-1)^r \sum_{k=r+1}^{\infty} \binom{k-1}{r} \frac{f^{(2k+\ell)} (x)}{(2k)!!} \sigma_n^{2k}
\]

and

\[
\left( \tilde{F}_{n,\sigma_n}^{(2r)} f \right)^{(\ell)} (x) = f^{(\ell)} (x) + (-1)^r \sum_{k=1}^{q} \binom{k-1}{r} \frac{f^{(2k+\ell)} (x)}{k!} \sigma_n^{2k} + o \left( \sigma_n^{2q} \right).
\]

**Remark 3.4.** An immediate consequence are the asymptotic relations

\[
\left( F_{n,\sigma_n}^{(2r)} f \right)^{(\ell)} (x) - f (x) = O \left( \sigma_n^{2(r+1)} \right)
\]

and

\[
\left( \tilde{F}_{n,\sigma_n}^{(2r)} f \right)^{(\ell)} (x) - f (x) = O \left( \sigma_n^{2(r+1)} \right)
\]
as $n \to \infty$.

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