ORDER OF CLOSE-TO-CONVEXITY FOR ANALYTIC FUNCTIONS OF COMPLEX ORDER

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Dedicated to Professor Grigore Ştefan Şalăgean on his 60th birthday

Abstract. The aim of this paper is to find the order of close-to-convexity for certain analytic functions of complex order.

1. Introduction and definitions

Let $A$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $U = \{ z : |z| < 1 \}$. A function $f(z)$ in $A$ is said to be starlike function of complex order $\gamma (\gamma \in \mathbb{C} - \{0\})$, if and only if

$$\Re \left\{ 1 + \frac{1}{\gamma} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right\} > 0, \quad (z \in U).$$

We denote by $S(\gamma)$ the class of all such functions. Also, a function $f(z)$ in $A$ is said to be convex function of complex order $\gamma (\gamma \in \mathbb{C} - \{0\})$, that is, $f \in C(\gamma)$, if and only if

$$\Re \left\{ 1 + \frac{1}{\gamma} \left( \frac{zf''(z)}{f'(z)} \right) \right\} > 0, \quad (z \in U).$$

The class $S(\gamma)$ was introduced by Nasr and Aouf [7] and the class $C(\gamma)$ was introduced by Wiatrowski [15] and considered in [6] (see also [5], [10], [13] and [2]).
We note that \( f(z) \in \mathcal{C}(\gamma) \Leftrightarrow zf'(z) \in \mathcal{S}(\gamma) \) and \( \mathcal{S}(1-\alpha) = \mathcal{S}^*(\alpha), \mathcal{C}(1-\alpha) = \mathcal{C}(\alpha) \) where \( \mathcal{S}^*(\alpha) \) and \( \mathcal{C}(\alpha) \) denote, respectively, the familiar classes of starlike and convex functions of a real order \( \alpha (0 \leq \alpha < 1) \) in \( \mathcal{U} \) (see, for example, [14]).

A function \( f(z) \) in \( \mathcal{A} \) is said to be close-to-convex of complex order \( \gamma (\gamma \in \mathbb{C} - \{0\}) \) and type \( \delta \) if there exists a function \( g(z) \) belonging to \( \mathcal{S}(\gamma) \) such that

\[
\text{Re} \left\{ 1 + \frac{1}{\gamma} \left( \frac{zf'(z)}{g(z)} - 1 \right) \right\} > \delta, \quad (z \in \mathcal{U}).
\] (1.4)

We denote by \( \mathcal{K}(\gamma, \delta) \) the subclass of \( \mathcal{A} \) consisting of functions which are close-to-convex of complex order \( \gamma \) and type \( \beta \) in \( \mathcal{U} \). We note that the class \( \mathcal{K}(1,0) \) is the class of close-to-convex functions introduced by Kaplan [4] and Ozaki [11].

Pflatzgraff et al. [12] have proved that if \( f(z) \) in \( \mathcal{A} \) satisfies the condition

\[
\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \quad (\frac{1}{2} \leq \alpha < 1),
\] (1.5)

then \( f(z) \) in the class \( \mathcal{S} \) (and convex in at least one direction in \( \mathcal{U} \)). Furthermore, Cerebiez-Tarabicka et al. [1] have shown that if \( f(z) \) in \( \mathcal{A} \) satisfies the condition

\[
\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > -\frac{1}{2} \quad (\frac{1}{2} \leq \alpha < 1),
\] (1.6)

then

\[
\text{Re} \left( \frac{zf'(z)}{g(z)} \right) > 0, \quad (z \in \mathcal{U}).
\] (1.7)

Recently, Owa [9] proved that if \( f(z) \) in \( \mathcal{A} \) satisfies the condition

\[
\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} - \beta \right) > 0 \quad (z \in \mathcal{U})
\] (1.8)

then

\[
\text{Re} \left( \frac{zf'(z)}{g(z)} \right) > \frac{3}{5} \quad (z \in \mathcal{U})
\] (1.9)

where \( g(z) \in \mathcal{S}^*(\alpha/(\alpha + 1)), \alpha \geq 0 \).

Also, Frasin and Oros [3] proved that if the function \( f(z) \) in \( \mathcal{A} \) satisfies the condition

\[
\text{Re} \left( \frac{zf''(z)}{f'(z)} - \beta \right) > 0 \quad (z \in \mathcal{U})
\] (1.10)

then

\[
\text{Re} \left( \frac{zf'(z)}{g(z)} \right) > \frac{1}{2\beta - 1} \quad (z \in \mathcal{U})
\] (1.11)
where \( g(z) \in \mathcal{S}^* \) and \( 1 \leq \beta \leq 3/2 \).

In order to show our results, we shall need the following lemma due to Obradović et al. [8].

**Lemma 1.1.** Let \( f \in \mathcal{S}(b) \), \( b \in \mathbb{C} \setminus \{0\} \), and let \( a \in \mathbb{C} \setminus \{0\} \) with \( 0 \leq 2ab \leq 1 \). Then

\[
\Re \left\{ \left( \frac{f(z)}{z} \right)^a \right\} > 2^{-2ab} \quad (z \in \mathcal{U}).
\]

(1.12)

2. Main results

With the aid of Lemma 1.1, we can prove the following result.

**Theorem 2.1.** If the functions \( f(z) \) and \( g(z) \) are in \( \mathcal{A} \) and satisfies the conditions

\[
\Re \left\{ 1 + \frac{1}{b} \left( \frac{zf''(z)}{f'(z)} \right) \right\} > 0 \quad (z \in \mathcal{U}),
\]

(2.1)

with \( 0 \leq 2a\gamma \leq 1 \), \( \gamma = b/(a+1) \); \( a, b \in \mathbb{C} \setminus \{0\} \); \( a \neq -1 \), and

\[
\Im \left( \frac{a+1}{b} \right) \leq 0 \text{ or } \Im \left( \frac{zf'(z)}{g(z)} \right) \leq 0,
\]

(2.2)

then \( f(z) \) belongs to the class \( \mathcal{K}(\gamma, \delta) \), where

\[
\delta = 1 + \left( \frac{2-2ab}{a+1} - 1 \right) \Re \left( \frac{a+1}{b} \right).
\]

Proof. If we define \( g(z) \) by

\[
1 + \frac{a+1}{b} \left( \frac{zf'(z)}{g(z)} - 1 \right) = 1 + \frac{1}{b} \left( \frac{zf''(z)}{f'(z)} \right),
\]

(2.3)

then from the condition (2.1) and (2.3), we have \( g(z) \in \mathcal{S}(\gamma) \), with \( \gamma = b/(a+1) \). It is easy to see that (2.3) implies

\[
f'(z) = \left( \frac{g(z)}{z} \right)^{a+1}
\]

(2.4)
or

\[
\frac{zf'(z)}{g(z)} = \left( \frac{g(z)}{z} \right)^a
\]

(2.5)
Applying Lemma 1.1 to \( g(z) \), we obtain
\[
\Re \left\{ 1 + \frac{a + 1}{b} \frac{zf'(z)}{g(z)} - 1 \right\} = \Re \left\{ 1 + \frac{a + 1}{b} \left( \frac{g(z)}{z} \right)^a - 1 \right\} \\
= 1 + \Re \left( \frac{a + 1}{b} \right) \Re \left\{ \left( \frac{g(z)}{z} \right)^a - 1 \right\} \\
- \Im \left( \frac{a + 1}{b} \right) \Im \left\{ \left( \frac{g(z)}{z} \right)^a - 1 \right\} \\
\geq 1 + \Re \left( \frac{a + 1}{b} \right) \Re \left\{ \left( \frac{g(z)}{z} \right)^a - 1 \right\} \\
> 1 + (2^{-2a\gamma} - 1) \Re \left( \frac{a + 1}{b} \right) \\
= 1 + \left( 2^{-2a\gamma} - 1 \right) \Re \left( \frac{a + 1}{b} \right).
\]

This completes the proof of Theorem 2.1. □

Letting \( a = 1 \) in Theorem 2.1, we have

**Corollary 2.2.** If the function \( f \in C(b) \) with \( 0 < b \leq 2 \), then \( f \in K(b/2, \delta) \), where
\[
\delta = 1 + \frac{2^{1-b} - 2}{b}.
\]

Letting \( b = 1 \) in Theorem 2.1, we have

**Corollary 2.3.** If the functions \( f(z) \) and \( g(z) \) are in \( A \) and satisfies the conditions
\[
\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0 \quad (z \in U), \tag{2.6}
\]
with \( 0 < 2a\gamma \leq 1 \), \( \gamma = 1/(a + 1) \); \( a \in \mathbb{C} \setminus \{0\} \); \( a \neq -1 \), and
\[
\Im (a + 1) \leq 0 \text{ or } \Im \left( \frac{zf'(z)}{g(z)} \right) \leq 0, \tag{2.7}
\]
then \( f(z) \) belongs to the class \( K(\gamma, \delta) \), where
\[
\delta = 1 + \left( 2^{\frac{2}{2a\gamma}} - 1 \right) \Re (a + 1).
\]

Letting \( b = 1 \) in Corollary 2.2 or \( a = 1 \) in Corollary 2.3, we have
Corollary 2.4. Let the functions $f(z)$ and $g(z)$ be in $A$. If

$$\text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0 \quad (z \in U), \quad (2.8)$$

then

$$\text{Re} \left\{ \frac{z^2 f'(z)}{g(z)} \right\} > \frac{1}{2} \quad (z \in U), \quad (2.9)$$

Therefore, if $f(z)$ is convex in $U$ then $f(z)$ is close-to-convex of order $1/2$ in $U$.

Letting $b = a + 1$ in in Theorem 2.1, we have

Corollary 2.5. Let the functions $f(z)$ and $g(z)$ be in $A$. If

$$\text{Re} \left\{ 1 + \frac{1}{a+1} \left( \frac{zf''(z)}{f'(z)} \right) \right\} > 0 \quad (z \in U), \quad (2.10)$$

where $0 < a \leq 1/2$, then

$$\text{Re} \left\{ \frac{zf'(z)}{g(z)} \right\} > \frac{1}{4a} \quad (z \in U). \quad (2.11)$$

Letting $a = 1/2$ in Corollary 2.5, we have

Corollary 2.6. Let the functions $f(z)$ and $g(z)$ be in $A$. If

$$\text{Re} \left\{ 1 + \frac{2}{3} \left( \frac{zf''(z)}{f'(z)} \right) \right\} > 0 \quad (z \in U), \quad (2.12)$$

then

$$\text{Re} \left\{ \frac{zf'(z)}{g(z)} \right\} > \frac{1}{2} \quad (z \in U), \quad (2.13)$$

That is, $f(z)$ is close-to-convex of order $1/2$ in $U$.

References


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