ON THE $\delta(\varepsilon)$-STABLE OF COMPOSED RANDOM VARIABLES

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Abstract. Let $\xi$ be a random variable (r.v.) with the characteristic function $\varphi(t)$ and $\nu$ be a r.v. with the generating function $a(z)$, $\nu$ is independent of $\xi$. It is known (see [1]) that the composed r.v. $\eta$ of $\xi$ and $\nu$ (denote by $\eta = <\nu, \xi>$) is the r.v. having the characteristic function $\psi(t) = a[\varphi(t)]$. The r.v. $\nu$ is called to be the first component of $\eta$ and $\xi$ is called to be the second component of $\eta$. In this paper, we shall investigate the changes of the distribution function of the composed r.v. $\eta$ if we have the small changes of the distribution function of the first component $\nu$ or the second component $\xi$ of $\eta$.

1. Introduction

Let $\xi$ be a random variable (r.v.) with the characteristic function $\varphi(t)$ and the distribution function $F(x)$. Let $\nu$ be a r.v. independent of $\xi$ and has the generating function $a(z)$ with the distribution function $A(x)$. It is known (see [1]) that the composed r.v. of $\nu$ and $\xi$ is denote by

$$\eta = <\nu, \xi>$$

and has the characteristic function

$$\psi(t) = a[\varphi(t)].$$

The r.v. $\nu$ is called to be the first component and the r.v. $\xi$ is called to be the second component of the r.v. $\eta$.
Example 1.1. Let us consider the integer valued nonnegative r.v.:

$$\eta = \sum_{k=1}^{\nu} \xi_k$$  \hspace{1cm} (1.3)

where $\xi_1, \xi_2, ...$ are i.i.d random variables have the same the distribution function with r.v. $\xi$, $\nu$ is a positive value r.v., independent of all $\xi_k$ $(k = 1, 2, ...)$, $\eta$ is composed r.v. of $\nu$ and $\xi$ and $\eta = <\nu, \xi>$.

In many practical problems, we always meet this composed random variable (special in queuing theory - see [7]) where $\nu$ is assumed having Poisson law and $\xi$ has the Exponential law. But, in practice, we also know best that $\nu$ has only a distribution function which arrives at Poisson law or $\xi$ has a distribution function which arrives at Exponential law. Our question is the following: If we have the small changes of the distribution function of $\nu$ or $\xi$, whether the distribution function of $\eta = <\nu, \xi>$ shall has the small changes or not?

The composed r.v. $\eta$ is called to be stable if the small changes in the distribution function of $\nu$ or $\xi$ lead to the small changes in the distribution function of $\eta$.

More detail we have the following definitions:

**Definition 1.1.** Suppose that $\Psi(x)$ and $\psi(t)$ are the distribution function and characteristic function of $\eta$, $A_\varepsilon(x)$ and $a_\varepsilon(z)$ are the distribution function and the generating function of $\nu_\varepsilon$ such that

$$\rho(A; A_\varepsilon) = \sup_{x \in \mathbb{R}} |A(x) - A_\varepsilon(x)| < \varepsilon$$

(for some sufficiently small positive number $\varepsilon$).

Put $\Psi^1_\varepsilon(x)$ be the distribution of the composed r.v. $<\nu_\varepsilon; \xi>$. The composed r.v. $\eta$ is called to be $\delta_1(\varepsilon)$-stable on the first component with metric $\rho(\cdot, \cdot)$ if and only if

$$\rho(\Psi; \Psi^1_\varepsilon) \leq \delta_1(\varepsilon) \hspace{1cm} (\delta_1(\varepsilon) \to 0 \hspace{0.5cm} when \hspace{0.5cm} \varepsilon \to 0).$$

**Definition 1.2.** Suppose that $F_\varepsilon(x)$ and $\varphi_\varepsilon(t)$ are the distribution function and the characteristic function of $\xi_\varepsilon$ such that $\rho(F_\varepsilon; F) < \varepsilon$ (for some sufficiently small
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A positive number $\varepsilon$) and $\Psi_\varepsilon^2(x)$ is distribution function with the characteristic function $\psi_\varepsilon^2(t)$ of the composed r.v. $< \nu; \xi >$.

The composed r.v. $\eta$ is called to be $\delta_2(\varepsilon)$-stable on the second component with metric $\rho(\cdot, \cdot)$ if and only if $\rho(\Psi; \Psi_\varepsilon^2) \leq \delta_2(\varepsilon)$ ($\delta_2(\varepsilon) \to 0$ when $\varepsilon \to 0$).

**Remark 1.1.** In some following stability theorems, metric $\rho(\cdot, \cdot)$ may be changed by metric $\lambda_0(\cdot, \cdot)$ (See [6])

$$\lambda_0(\Psi; \Psi_\varepsilon^2) = \sup_{t \in \mathbb{R}} |\psi(t) - \psi_\varepsilon^2(t)|.$$ 

2. Stability Theorems

**Theorem 2.1.** If the first component of the composed r.v. $\eta$ has the generating function $a(z)$ which satisfies the following condition:

$$|a(z_1) - a(z_2)| \leq K|z_2 - z_1|,$$  

(2.1)

for all complex numbers $z_1, z_2, |z_1| \leq 1, |z_2| \leq 1$ and $K$ is a constant, then $\eta$ shall be $K\varepsilon$-stable on the second component with metric $\lambda_0(\cdot, \cdot)$.

**Proof.** According to the hypothesis $\lambda_0(F, F_\varepsilon) < \varepsilon$,

$$|\varphi(t) - \varphi_\varepsilon(t)| < \varepsilon, \quad \forall t$$

so that

$$|\psi(t) - \psi_\varepsilon^2(t)| = |a[\varphi(t)] - a[\varphi_\varepsilon(t)]| \leq K|\varphi(t) - \varphi_\varepsilon(t)| \leq K\varepsilon$$

for all $t$. That means

$$\lambda_0(\Psi; \Psi_\varepsilon^2) \leq K\varepsilon.$$  

(2.2)

**Example 2.1.** If $\nu$ is the r.v. having the Poisson law with parameter $\lambda > 0$ and $\varphi_1(t)$ is the characteristic function of the r.v. $\xi$ having exponential law with parameter $\theta > 0$ then the composed r.v. $\eta = < \nu; \xi >$ shall be $e^{4\lambda\varepsilon}$-stable on the second component with metric $\lambda_0(\cdot, \cdot)$ (where $e^{4\lambda}$ is a constant).

**Example 2.2.** If $\nu$ is r.v. having the binomial distribution function with the parameters $p, n$ and $\xi$ has the exponential distribution function with parameter $\theta > 0$ then $\eta = < \nu; \xi >$ shall be $np(1 + 2p)^n^{-1}\varepsilon$-stable on the second component with metric $\lambda_0(\cdot, \cdot)$ (where $np(1 + 2p)^n^{-1}\varepsilon$ is a constant).
Example 2.3. If \( \nu \) is r.v. having the geometric distribution function with the parameters \( p \) \( (p = 1 - q) \) and \( \xi \) has the exponential distribution function then \( \eta = \nu; \xi > \) shall be \( \frac{q}{p} \varepsilon \)-stable on the second component with metric \( \lambda_0(\ldots) \) (where \( \frac{q}{p} \varepsilon \) is a constant).

All above examples are immediate from Theorem 2.1 since the corresponding generating functions clearly satisfy the condition (2.1). Indeed, for instance, to show Example 2.3, let \( a_3(z) \) be the generating function of geometric law, i.e.:

\[
a_3(z) = p[1 - qz]^{-1}.
\]

For any complex numbers \( z_1, z_2 \) satisfying \( |z_1| \leq 1, |z_2| \leq 1 \); we have the following estimation:

\[
|a_3(z_1) - a_3(z_2)| = \left| \frac{p}{1 - qz_1} - \frac{p}{1 - qz_2} \right| \leq \frac{pq|z_1 - z_2|}{|1 - qz_1||1 - qz_2|}.
\]

Notice that

\[
|1 - qz_1| \geq |1 - q|z_1|| \geq 1 - q, \quad \text{for all } |z_1| \leq 1
\]

\[
|1 - qz_2| \geq |1 - q|z_2|| \geq 1 - q, \quad \text{for all } |z_2| \leq 1
\]

It follows that

\[
|a_3(z_1) - a_3(z_2)| \leq \frac{pq|z_1 - z_2|}{(1 - q)^2}.
\]

Thus \( a_3(z) \) satisfies the condition (2.1) with the constant \( K = \frac{pq}{(1 - q)^2} \).

Theorem 2.2. (See [2]) Suppose \( \eta = \nu; \xi >, \nu \) has the distribution function \( A(x) \) such that

\[
\mu_1^\alpha = \int_0^{+\infty} z^\alpha A(z) < +\infty, \quad \forall \alpha : 0 < \alpha < 1
\]

and \( \xi \) has the stable law with the characteristic function:

\[
\varphi(t) = \exp\{i\mu t - c|t|^\alpha(1 + i\beta \frac{t}{|t|} \omega(|t|; \alpha))\}, \tag{2.3}
\]

where \( c, \mu, \alpha, \beta \) are real numbers, \( c \geq 0, |\beta| \leq 1 \) and

\[
1 < \alpha_1 \leq \alpha \leq 2; \quad \omega(|t|; \alpha) = tg \frac{\alpha \pi}{2}. \tag{2.4}
\]
For every \( \varepsilon \) sufficiently small positive number is given, such that
\[
\varepsilon < \left( \frac{\pi}{3c_2} \right)^3, c_1 = (c + |\beta||\tan \frac{\alpha_1}{2}| + |\mu|) \quad (2.5)
\]
\( \eta \) shall be \( K_1 \varepsilon^{1/6} \)-stable on the first component with metric \( \rho(.,.) \).

**Theorem 2.3.** Assume that \( \nu \) has any distribution function \( A(z) \) which has moment \( \mu_A = \int_0^\infty zdA(z) < +\infty \), \( \xi \) has the stable law with the characteristic function satisfying condition (2.3), (2.4). Then, the composed r.v. \( \eta = \langle \nu, \xi \rangle \) shall be \( K_2(\varepsilon)^{1/8} \)-stable on the second component with metric \( \rho(.,.) \) for some \( \varepsilon \) is sufficiently small number satisfying condition (2.5).

**Lemma 2.1.** Let \( a \) be a complex number, \( a = \rho e^{i\theta} \), such that
\[
|\theta| \leq \frac{\pi}{3}, 0 \leq \rho \leq 1. \quad (2.6)
\]
Then we always have following estimation
\[
|a^t - 1| \leq \frac{\sqrt{4t}|a - 1|}{(1 - |a - 1|)} \quad \text{for every} \quad t > 0, \ t \in \mathbb{R}. \quad (2.7)
\]

**Proof.** Since \( a = \rho(\cos \theta + i\sin \theta) \), it follows that
\[
|a^t - 1|^2 = (\rho' \cos \theta - 1)^2 + (\rho' \sin \theta)^2. \quad (2.8)
\]
We also have \( (\rho' \cos \theta - 1) = (\rho' - 1) \cos \theta + (\cos \theta - 1) \).

Notice that \( |1 - \cos x| \leq |x| \) for all \( x \in \mathbb{R} \), thus
\[
|\rho' \cos \theta - 1| \leq |\rho' - 1| + |\theta|.
\]
On the other hand, since \( |\sin u| \leq |u| \) for all \( u \in \mathbb{R} \), from (2.8) we shall have
\[
|a^t - 1|^2 \leq 2|\rho' - 1|^2 + 2t^2 \theta^2 + \rho^{2t}(t \theta)^2. \quad (2.9)
\]
We can see \( |a - 1|^2 = (\rho \cos \theta - 1)^2 + \rho^2 \sin^2 \theta \). It follows that
\[
|ho \sin \theta| \leq |a - 1|. \quad (2.10)
\]
Further more
\[
||a| - 1| \leq |a - 1| \Rightarrow |\rho - 1| \leq |a - 1|.
\]
Hence
\[ |\rho - 1| \geq -|a - 1| \Rightarrow \rho \geq 1 - |a - 1|. \]  
(2.11)

From (2.10) we obtain
\[ |\sin \theta| \leq \frac{|a - 1|}{\rho} \leq \frac{|a - 1|}{1 - |a - 1|}. \]

For every \( \theta, |\theta| \leq \frac{\pi}{3} \), we always have inequality: 
\[ |\sin \theta| \geq \frac{|\theta|}{2}. \]

So, from (2.)
\[ |\theta| \leq \frac{2|a - 1|}{1 - |a - 1|}. \]

From (2.9) and (2.11)
\[ |a^t - 1|^2 \leq 2|\rho^t - 1|^2 + \frac{8t^2|a - 1|^2}{(1 - |a - 1|)^2} + 4 \rho^2 t^2 |a - 1|^2 \]
\[ (1 - |a - 1|)^2. \]

(2.12)

For all \( t \geq 0 \), the following inequality holds
\[ 1 - \rho^t \leq \frac{t(1 - \rho)}{\rho}. \]

Notice \( |1 - \rho| = |1 - |a|| \leq |a - 1| \). We have
\[ |1 - \rho^t| \leq \frac{t|a - 1|}{\rho}. \]

(2.13)

Hence by (2.12) and (2.13), we shall get: 
\[ |a^t - 1|^2 \leq \frac{14t^2|a - 1|^2}{(1 - |a - 1|)^2}. \]

**Proof of theorem 2.3.** At first, we shall estimate \( |\psi(t) - \psi_\varepsilon(t)| \) for all \( t, |t| \leq T(\varepsilon) \) where \( T(\varepsilon) \to \infty \) when \( \varepsilon \to 0 \). At last, using Esseen’s inequality (see [4]) we shall have the conclusion. Throughout the proof, we shall denote by \( c_1, c_2, \ldots, c_{14}, c_{15} \) are constants independent of \( \varepsilon \). At first, we have:
\[ |\psi(t) - \psi_\varepsilon(t)| = |a[\varphi(t)] - a[\varphi_\varepsilon(t)]| = \int_1^{+\infty} [\varphi^\varepsilon(t) - \varphi^\varepsilon_\varepsilon(t)]dA(z) \]
\[ \leq \int_1^{+\infty} [\varphi^\varepsilon(t) - \varphi^\varepsilon_\varepsilon(t)]dA(z) + \int_0^1 [\varphi^\varepsilon(t) - \varphi^\varepsilon_\varepsilon(t)]dA(z) = J_1 + J_2. \]

(2.14)

Consider \( J_1 \): Using the Lagrange-formula of the function \( [\varphi(t)]^\varepsilon \) (for \( |z| \geq 1 \)), we get
\[ |\varphi^\varepsilon(t) - \varphi^\varepsilon_\varepsilon(t)| = z|\tilde{\varphi}(t)|^{\varepsilon - 1}|\varphi(t) - \varphi_\varepsilon(t)|, \]

(2.15)

where \( \tilde{\varphi}(t) \) is a complex number satisfying the condition \( |\tilde{\varphi}(t)| \leq \max\{|\varphi(t)|, |\varphi_\varepsilon(t)|\} \).

Notice that:
\[ |\tilde{\varphi}(t)|^{\varepsilon - 1} \leq |\tilde{\varphi}(t)| \leq 1 \quad \text{for all} \quad z : 2 \leq z < +\infty \]
Thus, $\varepsilon^z \leq |\varepsilon|^0 = 1$ for all $z : 1 \leq z < 2$.

i.e., $|\tilde{\varphi}(t)|^{-1} \leq 1$ for all $z : 1 \leq z < +\infty$. \hfill (2.16)

We shall have

$$|\varphi(t) - \varphi_\varepsilon(t)| = \left| \int_{-\infty}^{+\infty} e^{itx} d[F(x) - F_\varepsilon(x)] \right|.$$

For some $N = N(\varepsilon)$ (it also be chosen later), we also have the following estimation:

$$|\varphi(t) - \varphi_\varepsilon(t)| = \left| \int_{-N}^{N} e^{itx} d[F(x) - F_\varepsilon(x)] + 2 \int_{N}^{+\infty} d[F(x) + F_\varepsilon(x)] \right| \leq |[F(x) - F_\varepsilon(x)]|_{N}^{N} + \left| \int_{-N}^{N} [F(x) - F_\varepsilon(x)] d(e^{itx}) \right| + 2 \int_{N}^{+\infty} \frac{\mu_F + \mu_{F_\varepsilon}}{N(\varepsilon)} dx.$$

(where $\mu_F = \int_{-\infty}^{+\infty} |x| dF(x) < +\infty$ and $\mu_{F_\varepsilon} = \int_{-\infty}^{+\infty} |x| dF_\varepsilon(x) < +\infty$). Now, for all $t$, $|t| \leq T(\varepsilon)$ (where $T(\varepsilon) \to \infty$ when $\varepsilon \to 0$, $T(\varepsilon)$ will be chosen later) we always have

$$|\varphi(t) - \varphi_\varepsilon(t)| \leq 2\varepsilon + 2N(\varepsilon)T(\varepsilon)\varepsilon + 2 \frac{\mu_F + \mu_{F_\varepsilon}}{N(\varepsilon)} \cdot \hfill (2.17)$$

Now, consider $J_2$. Using the Lagrange-formula of the function $|\varphi(t)|^z$ for all $z, 0 \leq z \leq 1$ at $\varphi_\varepsilon(t)$, we get

$$|\varphi^z(t) - \varphi_\varepsilon^z(t)| = \frac{z}{|\varphi(t)|^{1-z}} |\varphi(t) - \varphi_\varepsilon(t)|. \hfill (2.18)$$

For every $\varepsilon$- satisfying condition (2.5) we shall choose $T(\varepsilon)$ such that

$$\min\{|\varphi(t)|; |\varphi_\varepsilon(t)| \} \geq c_4 \varepsilon^{1/2} \geq |\varphi(t) - \varphi_\varepsilon(t)| \text{ for all } t, |t| \leq T(\varepsilon),$$

(where $c_4$ is a constant independent of $\varepsilon$).

Because $\varphi(t)$ is the characteristic function of stable law satisfying condition (2.3), we so have the following estimations:

$$|\ln \varphi(t)| \leq |\mu||t| + |t|^\alpha (c + |c\beta|\theta \frac{\alpha \pi}{2}) \leq |\mu||t| + c_2 |t|^\alpha \leq T^\alpha (\varepsilon).$$

Thus,

$$|\varphi(t)| = |e^{\ln \varphi(t)}| \geq e^{-|\ln \varphi(t)|} \geq e^{-c_2 T^\alpha (\varepsilon)}.$$
If we choose:
\[ T(\varepsilon) = \frac{1}{\varepsilon^2} \ln \frac{1}{\varepsilon^{1/8}} \left( T(\varepsilon) \to \infty \text{ when } \varepsilon \to 0 \right). \] (2.19)

Then \( c_2 T^{\alpha}(\varepsilon) \leq \ln \frac{1}{\varepsilon^{1/8}} \) (for all \( \alpha > 1 \)) and \( |\varphi(t)| \geq e^{-c_2 T^{\alpha}(\varepsilon)} \geq \varepsilon^{1/8} \). Now we shall choose \( N(\varepsilon) = \varepsilon^{-1/2} \) (\( N(\varepsilon) \to +\infty \) when \( \varepsilon \to 0 \)). Thus
\[ 2\varepsilon T(\varepsilon) N(\varepsilon) \leq \frac{2}{c_2} \ln \frac{1}{\varepsilon^{1/8}} \varepsilon^{1/2} \leq c_3 \varepsilon^{3/8}. \] (2.20)

Put
\[ c_0(\varepsilon) = 2\varepsilon + 2\varepsilon T(\varepsilon) N(\varepsilon) + 2(\mu_F + \mu_{F_E}) \varepsilon^{1/2} \leq c_4 \varepsilon^{1/2}. \] (2.21)

That means, the condition:
\[ c_4 \varepsilon^{1/2} \geq |\varphi(t) - \varphi_{\varepsilon}(t)| \] (2.22)
shall be satisfied for every \( t, |t| \leq T(\varepsilon) \).

Notice that, from \( |\varphi(t) - \varphi_{\varepsilon}(t)| \leq c_4 \varepsilon^{1/2} \) we always have
\[ ||\varphi(t)| - |\varphi_{\varepsilon}(t)|| \leq |\varphi(t) - \varphi_{\varepsilon}(t)| \] (See[5]).

So
\[ |\varphi(t)| - |\varphi_{\varepsilon}(t)| \leq |\varphi(t) - \varphi_{\varepsilon}(t)| \leq c_4 \varepsilon^{1/2} \]
and
\[ |\varphi_{\varepsilon}(t)| \geq |\varphi(t)| - c_4 \varepsilon^{1/2} \geq \varepsilon^{1/8} - c_4 \varepsilon^{1/2} \geq c_5 \varepsilon^{1/8}. \]

That also means, the estimation \( \min \{|\varphi(t)|; |\varphi_{\varepsilon}(t)|\} \geq c_4 \varepsilon^{1/2} \) shall be satisfied.

On the other hand, for every complex number \( z_3 \) which belong to the interval joining \( z_1 \) and \( z_2 \) we have only two cases:
1) \( |z_3| \geq \min\{|z_1|; |z_2|\} \)
2) \( |z_3| \geq \sqrt{\max\{|z_1|^2; |z_2|^2\} - \frac{|z_1 - z_2|^2}{2}}. \)

Therefore
\[ \tilde{\varphi}(t) \geq \min\{|\varphi(t)|; |\varphi_{\varepsilon}(t)|\} \geq c_5 \varepsilon^{1/8} \]
or
\[ |\tilde{\varphi}(t)| \geq \sqrt{c_2^2/2} - c_4 \varepsilon^{2/4}/2 \geq c_0 \varepsilon^{1/8} \]
i.e., \( |\tilde{\varphi}(t)| \geq c_0 \varepsilon^{1/8} \) in both above cases. Besides that, we always have,
\[ |\tilde{\varphi}(t)|^{1-z} \geq |\tilde{\varphi}(t)| \quad \text{for all complex number } z, 0 \leq |z| \leq 1. \tag{2.23} \]
Taking into account (2.18), (2.20), (2.23) we shall get
\[ J_2 = | \int_{0}^{1} |\varphi^z(t) - \varphi^z_\varepsilon(t)| dA(z) | \leq \int_{0}^{1} |z| \frac{|\varphi(t) - \varphi_\varepsilon(t)|}{|\tilde{\varphi}(t)|} dA(z) \leq c_7 \varepsilon^{3/8}. \tag{2.24} \]
Combine (2.14), (2.16), (2.17), (2.24) we can see that
\[ J_1 + J_2 \leq \mu_A c_0 (\varepsilon) + c_7 \varepsilon^{3/8} \leq c_8 \varepsilon^{3/8}. \tag{2.25} \]
Thus, for all \( t, |t| \leq T(\varepsilon) \) (which is chosen from (2.19)) we always have the estimation
\[ |\psi(t) - \psi_\varepsilon(t)| \leq c_8 \varepsilon^{3/8}. \tag{2.26} \]
Now we shall choose \( \delta = \delta(\varepsilon) \) to be a positive number \( (\delta(\varepsilon) \to 0 \text{ when } \varepsilon \to 0) \) such that
\[ \max\{|\arg[\varphi(t)]|; |\arg[\varphi_\varepsilon(t)]|\} \leq \frac{\pi}{3}, \quad \forall t, |t| \leq \delta(\varepsilon). \tag{2.27} \]
We always have
\[ \int_{-T(\varepsilon)}^{T(\varepsilon)} \frac{\psi(t) - \psi_\varepsilon(t)}{t} dt \leq \int_{-\delta(\varepsilon)}^{\delta(\varepsilon)} \frac{\psi(t) - \psi_\varepsilon(t)}{t} dt + \int_{\delta(\varepsilon) \leq |t| \leq T(\varepsilon)} \frac{\psi(t) - \psi_\varepsilon(t)}{t} dt. \]
Consider \( |\psi(t) - \psi_\varepsilon(t)| \) on \( |t| \leq \delta(\varepsilon) \), we have
\[ |\psi(t) - \psi_\varepsilon(t)| \leq \int_{0}^{+\infty} |\varphi^z(t) - 1| dA(z) + \int_{0}^{+\infty} |\varphi^z_\varepsilon(t) - 1| dA(z). \tag{2.28} \]
In \( |t| \leq \delta(\varepsilon) \), with \( \delta(\varepsilon) \) is chosen from the condition (2.27), the condition (2.6) of Lemma 2.1 shall be satisfied (with \( a = \varphi(t) \)), we shall use Lemma 2.1 and we have the following estimations
\[ |\varphi^z(t) - 1| \leq \frac{\sqrt{14 \varepsilon}}{1 - |\varphi(t) - 1|} |\varphi(t) - 1| \]
and,
\[ |\varphi^z_\varepsilon(t) - 1| \leq \frac{\sqrt{14 \varepsilon}}{1 - |\varphi_\varepsilon(t) - 1|} |\varphi_\varepsilon(t) - 1| \tag{2.29} \]
for all complex numbers \( z \).
Notice that, for all $t$:

$$|e^{tx} - 1| = |(\cos tx - 1)^2 + \sin^2 tx| = 2 \sin \frac{tx}{2} \leq |t||x|. \quad (2.30)$$

In $|t| \leq \delta(\varepsilon)$ with $\delta(\varepsilon) \to 0$ when $\varepsilon \to 0$, so we always have

$$|\varphi(t) - 1| \leq \frac{1}{2}, \quad |\varphi_\varepsilon(t) - 1| \leq \frac{1}{2},$$

and therefore, from (2.7)

$$\int_0^{+\infty} |\varphi_\varepsilon^2(t) - 1|dA(z) \leq \int_0^{+\infty} \sqrt{14|\varphi(t) - 1|dA(z) \leq c_9|t|.}$$

Similarly,

$$\int_0^{+\infty} |\varphi_\varepsilon^2(t) - 1|dA(z) \leq c_{10}|t|.\quad (2.31)$$

Now, if we choose

$$\delta(\varepsilon) = \frac{1}{c_2} \varepsilon^{1/4} \ln \frac{1}{\varepsilon^{1/8}} \quad (\delta(\varepsilon) \to 0 \text{ when } \varepsilon \to 0) \quad (2.32)$$

with $\varepsilon$ satisfying (2.5), then we have

$$\int_{-\delta(\varepsilon)}^{\delta(\varepsilon)} \frac{\psi(t) - \psi_\varepsilon(t)}{t} dt \leq c_{11}\delta(\varepsilon). \quad (2.33)$$

On the other hand, from (2.19) and (2.26)

$$\int_{\delta(\varepsilon)}^{T(\varepsilon)} \frac{\psi(t) - \psi_\varepsilon(t)}{t} dt \leq c_8\varepsilon^{3/8} \int_{\delta(\varepsilon)}^{T(\varepsilon)} \frac{1}{t} dt = c_8\varepsilon^{3/8} \ln \frac{T(\varepsilon)}{\delta(\varepsilon)}. \quad (2.34)$$

and notice that

$$c_8\varepsilon^{3/8} \ln \frac{T(\varepsilon)}{\delta(\varepsilon)} \leq c_8\varepsilon^{3/8} \ln \frac{T^\alpha(\varepsilon)}{\delta(\varepsilon)} \leq c_8\varepsilon^{3/8} \frac{1}{\varepsilon^{1/4}} = c_8\varepsilon^{1/8}. \quad (2.35)$$

With $T(\varepsilon)$ and $\delta(\varepsilon)$ chosen from conditions (2.19) and (2.32), we shall have:

$$\int_{\delta(\varepsilon)}^{T(\varepsilon)} \frac{\psi(t) - \psi_\varepsilon(t)}{t} dt \leq c_8\varepsilon^{3/8} \ln \frac{T(\varepsilon)}{\delta(\varepsilon)} \leq c_8\varepsilon^{1/8}. \quad (2.36)$$
By using Esseen’s inequality (see [4]) and combine (2.33) with (2.36) we can conclude that

$$\rho(\Psi; \Psi_\varepsilon) \leq c_1 \varepsilon^{1/8} + c_8 \varepsilon^{1/8} \leq K_2 \varepsilon^{1/8},$$

where $K_2$ is a constant independent of $\varepsilon$. This completes the proof of Theorem 2.3.

References


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