SOME APPLICATIONS OF SALAGEAN INTEGRAL OPERATOR

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Abstract. In this paper we introduce and study some new subclasses of starlike, convex, close-to-convex and quasi-convex functions defined by Salagean integral operator. Inclusion relations are established and integral operator \( L_c(f)(c \in N = \{1, 2, \ldots\}) \) is also discussed for these subclasses.

1. Introduction

Let \( A \) denote the class of functions of the form:
\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k
\]
which are analytic in the unit disc \( U = \{z : |z| < 1\} \). Also let \( S \) denote the subclass of \( A \) consisting of univalent functions in \( U \). A function \( f(z) \in S \) is called starlike of order \( \gamma \), \( 0 \leq \gamma < 1 \), if and only if
\[
\text{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} > \gamma \quad (z \in U) .
\]
We denote by \( S^*(\gamma) \) the class of all functions in \( S \) which are starlike of order \( \gamma \) in \( U \).

A function \( f(z) \in S \) is called convex of order \( \gamma \), \( 0 \leq \gamma < 1 \), in \( U \) if and only if
\[
\text{Re}\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \gamma \quad (z \in U) .
\]
We denote by \( C(\gamma) \) the class of all functions in \( S \) which are convex of order \( \gamma \) in \( U \).

It follows from (1.2) and (1.3) that:
\[
f(z) \in C(\gamma) \quad \text{if and only if} \quad z f'(z) \in S^*(\gamma) .
\]
The classes $S^*(\gamma)$ and $C(\gamma)$ was introduced by Robertson [12].

Let $f(z) \in A$, and $g(z) \in S^*(\gamma)$. Then $f(z) \in K(\beta, \gamma)$ if and only if
\[
\text{Re} \left\{ \frac{zf'(z)}{g(z)} \right\} > \beta \quad (z \in U),
\]  
(1.5)
where $0 \leq \beta < 1$ and $0 \leq \gamma < 1$. Such functions are called close-to-convex functions of order $\beta$ and type $\gamma$. The class $K(\beta, \gamma)$ was introduced by Libera [4].

A function $f(z) \in A$ is called quasi-convex of order $\beta$ and type $\gamma$ if there exists a function $g(z) \in C(\gamma)$ such that
\[
\text{Re} \left\{ \frac{(zf'(z))'}{g(z)} \right\} > \beta \quad (z \in U),
\]  
(1.6)
where $0 \leq \beta < 1$ and $0 \leq \gamma < 1$. We denote this class by $K^*(\beta, \gamma)$. The class $K^*(\beta, \gamma)$ was introduced by Noor [10].

It follows from (1.5) and (1.6) that:
\[
f(z) \in K^*(\beta, \gamma) \quad \text{if and only if} \quad z f'(z) \in K(\beta, \gamma).
\]  
(1.7)

For a function $f(z) \in A$, we define the integral operator $I^n f(z), n \in N_0 = N \cup \{0\}$, where $N = \{1, 2, \ldots\}$, by
\[
I^0 f(z) = f(z),
\]  
(1.8)
\[
I^1 f(z) = I f(z) = \int_0^z f(t) t^{-1} dt,
\]  
(1.9)
and
\[
I^n f(z) = I(I^{n-1} f(z)).
\]  
(1.10)

It is easy to see that:
\[
I^n f(z) = z + \sum_{k=2}^{\infty} \frac{a_k}{k^n} z^k \quad (n \in N_0),
\]  
(1.11)
and
\[
z(I^n f(z))' = I^{n-1} f(z).
\]  
(1.12)

The integral operator $I^n f(z) \ (f \in A)$ was introduced by Salagean [13] and studied by Aouf et al. [1]. We call the operator $I^n$ by Salagean integral operator.
Using the operator $I^n$, we now introduce the following classes:

\[ S_n^*(\gamma) = \{ f \in A : I^n f \in S^*(\gamma) \} , \]

\[ C_n(\gamma) = \{ f \in A : I^n f \in C(\gamma) \} , \]

\[ K_n(\beta, \gamma) = \{ f \in A : I^n f \in K(\beta, \gamma) \} , \]

and

\[ K_n^*(\beta, \gamma) = \{ f \in A : I^n f \in K^*(\beta, \gamma) \} . \]

In this paper, we shall establish inclusion relation for these classes and integral operator $L_c(f)(c \in \mathbb{N})$ is also discussed for these classes. In [11], Noor introduced and studied some classes defined by Ruscheweyh derivatives and in [6] Liu studied some classes defined by the one-parameter family of integral operator $I^\sigma f(z)(\sigma > 0, f \in A)$. 

2. Inclusion relations

We shall need the following lemma.

**Lemma 2.1.** [8], [9] Let $\varphi(u, v)$ be a complex function, $\phi : D \to C, D \subset C \times C$, and let $u = u_1 + iu_2, v = v_1 + iv_2$. Suppose that $\varphi(u, v)$ satisfies the following conditions:

(i) $\varphi(u, v)$ is continuous in $D$;

(ii) $(1, 0) \in D$ and $\text{Re}\{\varphi(1, 0)\} > 0$;

(iii) $\text{Re}\{\varphi(iu_2, v_1)\} \leq 0$ for all $(iu_2, v_1) \in D$ and such that $v_1 \leq -\frac{1}{2}(1 + u_2^2)$.

Let $h(z) = 1 + c_1z + c_2z^2 + \ldots$ be analytic in $U$, such that $(h(z), zh'(z)) \in D$ for all $z \in U$. If $\text{Re}\{\varphi(h(z), zh'(z))\} > 0 (z \in U)$, then $\text{Re}\{h(z)\} > 0$ for $z \in U$.

**Theorem 2.1.** $S_n^*(\gamma) \subset S_{n+1}^*(\gamma)(0 \leq \gamma < 1, n \in \mathbb{N}_0)$.

**Proof.** Let $f(z) \in S_n^*(\gamma)$ and set

\[ \frac{z(I^{n+1}f(z))'}{I^{n+1}f(z)} = \gamma + (1 - \gamma)h(z) , \tag{2.1} \]

where $h(z) = 1 + h_1z + h_2z^2 + \ldots$. Using the identity (1.12), we have

\[ \frac{I^n f(z)}{I^{n+1}f(z)} = \gamma + (1 - \gamma)h(z) . \tag{2.2} \]
Differentiating (2.2) with respect to \( z \) logarithmically, we obtain
\[
\frac{z(I^n f(z))'}{I^n f(z)} = \frac{z(I^{n+1} f(z))'}{I^{n+1} f(z)} + \frac{(1 - \gamma)zh'(z)}{\gamma + (1 - \gamma)h(z)}
\]
\[
= \gamma + (1 - \gamma)h(z) + \frac{(1 - \gamma)zh'(z)}{\gamma + (1 - \gamma)h(z)} ,
\]
or
\[
\frac{z(I^n f(z))'}{I^n f(z)} - \gamma = (1 - \gamma)h(z) + \frac{(1 - \gamma)zh'(z)}{\gamma + (1 - \gamma)h(z)} . \quad (2.3)
\]
Taking \( h(z) = u = u_1 + iu_2 \) and \( zh'(z) = v = v_1 + iv_2 \), we define the function \( \varphi(u, v) \) by:
\[
\varphi(u, v) = (1 - \gamma)u + \frac{(1 - \gamma)v}{\gamma + (1 - \gamma)u} . \quad (2.4)
\]
Then it follows from (2.4) that
(i) \( \varphi(u, v) \) is continuous in \( D = (C - \{ \frac{\gamma}{1 - \gamma} \}) \times C; \)
(ii) \( (1, 0) \in D \) and \( \text{Re} \{ \varphi(1, 0) \} = 1 - \gamma > 0; \)
(iii) for all \( (iu_2, v_1) \in D \) such that \( v_1 \leq -\frac{1}{2}(1 + u_2^2) \),
\[
\text{Re} \{ \varphi(iu_2, v_1) \} = \text{Re} \left\{ \frac{(1 - \gamma)v_1}{\gamma + (1 - \gamma)iu_2} \right\}
\]
\[
= \frac{\gamma(1 - \gamma)v_1}{\gamma^2 + (1 - \gamma)^2u_2^2}
\]
\[
\leq -\frac{\gamma(1 - \gamma)(1 + u_2^2)}{2[\gamma^2 + (1 - \gamma)^2u_2^2]} < 0 ,
\]
for \( 0 \leq \gamma < 1 \). Therefore, the function \( \varphi(u, v) \) satisfies the conditions in Lemma. It follows from the fact that if \( \text{Re} \{ \varphi(h(z), zh'(z)) \} > 0, z \in U \), then \( \text{Re} \{ h(z) \} > 0 \) for \( z \in U \), that is, if \( f(z) \in S^*_n(\gamma) \) then \( f(z) \in S^*_{n+1}(\gamma) \). This completes the proof of Theorem 2.1. \( \square \)

We next prove:

**Theorem 2.2.** \( C_n(\gamma) \subset C_{n+1}(\gamma) (0 \leq \gamma < 1, n \in N_0) \).

**Proof.** \( f \in C_n(\gamma) \Leftrightarrow I^n f \in C(\gamma) \Leftrightarrow z(I^n f)' \in S^*(\gamma) \Leftrightarrow I^n(zf') \in S^*(\gamma) \Leftrightarrow zf' \in S^*_n(\gamma) \Rightarrow zf' \in S^*_{n+1}(\gamma) \Leftrightarrow I^{n+1}(zf') \in S^*(\gamma) \Leftrightarrow z(I^{n+1}f)' \in S^*(\gamma) \Leftrightarrow I^{n+1}f \in C(\gamma) \Leftrightarrow f \in C_{n+1}(\gamma) \).

This completes the proof of Theorem 2.2. \( \square \)
Theorem 2.3. \( K_n(\beta, \gamma) \subset K_{n+1}(\beta, \gamma) \)(0 \leq \gamma < 1, 0 \leq \beta < 1, n \in N_0).

Proof. Let \( f(z) \in K_n(\beta, \gamma) \). Then there exists a function \( k(z) \in S^*(\gamma) \) such that

\[
\text{Re} \left\{ \frac{z(I^n f(z))'}{k(z)} \right\} > \beta \ (z \in U).
\]

Taking the function \( g(z) \) which satisfies \( I^ng(z) = k(z) \), we have \( g(z) \in S^*_n(\gamma) \) and

\[
\text{Re} \left\{ \frac{z(I^n f(z))'}{I^ng(z)} \right\} > \beta \ (z \in U). \tag{2.5}
\]

Now put

\[
\frac{z(I^{n+1} f(z))'}{I^{n+1} g(z)} - \beta = (1 - \beta)h(z), \tag{2.6}
\]

where \( h(z) = 1 + c_1z + c_2z^2 + \ldots \). Using (1.12) we have

\[
\frac{z(I^n f(z))'}{I^ng(z)} = \frac{I^n(zf'(z))}{I^ng(z)} = \frac{z(I^{n+1}(zf'(z)))'}{z(I^{n+1}g(z))'} = \frac{z(I^{n+1}g(z))}{z(I^{n+1}g(z))'}.
\]  \tag{2.7}

Since \( g(z) \in S^*_n(\gamma) \) and \( S^*_n(\gamma) \subset S^*_{n+1}(\gamma) \), we let \( \frac{z(I^{n+1}g(z))'}{I^{n+1}g(z)} = \gamma + (1 - \gamma)H(z) \), where \( \text{Re} \ H(z) > 0 \ (z \in U) \). Thus (2.7) can be written as

\[
\frac{z(I^n f(z))'}{I^ng(z)} = \frac{z(I^n f(z))'}{I^ng(z)} = \frac{z(I^{n+1}(zf'(z)))'}{I^{n+1}g(z)} = \frac{z(I^{n+1}g(z))}{z(I^{n+1}g(z))'} = \gamma + (1 - \gamma)H(z), \tag{2.8}
\]

Consider

\[
z(I^{n+1} f(z))' = I^{n+1} g(z)[\beta + (1 - \beta)h(z)]. \tag{2.9}
\]

Differentiating both sides of (2.9), we have

\[
\frac{z(I^{n+1}(zf'(z)))'}{I^{n+1}g(z)} = (1 - \beta)zh'(z) + [\beta + (1 - \beta)h(z)] \cdot [\gamma + (1 - \gamma)H(z)]. \tag{2.10}
\]

Using (2.10) and (2.8), we have

\[
\frac{z(I^n f(z))'}{I^ng(z)} - \beta = (1 - \beta)h(z) + \frac{(1 - \beta)zh'(z)}{\gamma + (1 - \gamma)H(z)}. \tag{2.11}
\]

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Taking \( u = h(z) = u_1 + iv_2 = zh'(z) = v_1 + iv_2 \) in (2.11), we form the function \( \Psi(u, v) \) as follows:

\[
\Psi(u, v) = (1 - \beta)u + \frac{(1 - \beta)v}{\gamma + (1 - \gamma)H(z)} .
\] (2.12)

It is clear that the function \( \Psi(u, v) \) defined in \( D = C \times C \) by (2.12) satisfies conditions (i) and (ii) of Lemma easily. To verify condition (iii), we proceed as follows:

\[
\text{Re } \Psi(iu_2, v_1) = (1 - \beta)v_1 \left[ \frac{\gamma}{\gamma + (1 - \gamma)h_1(x, y)} \right]^2 + \left[ \frac{(1 - \gamma)h_2(x, y)}{\gamma + (1 - \gamma)h_1(x, y)} \right]^2 ,
\]

where \( H(z) = h_1(x, y) + ih_2(x, y) \), \( h_1(x, y) \) and \( h_2(x, y) \) being the functions of \( x \) and \( y \) and \( \text{Re } H(z) = h_1(x, y) > 0 \). By putting \( v_1 \leq -\frac{1}{\delta}(1 + u_2^2) \), we obtain

\[
\text{Re } \Psi(iu_2, v_1) \leq -\frac{(1 - \beta)(1 + u_2^2)}{2 \left[ \frac{\gamma}{\gamma + (1 - \gamma)h_1(x, y)} \right]^2 + \left[ \frac{(1 - \gamma)h_2(x, y)}{\gamma + (1 - \gamma)h_1(x, y)} \right]^2} < 0 .
\]

Hence \( \text{Re } h(z) > 0(z \in U) \) and \( f(z) \in K_{n+1}(\beta, \gamma) \). The proof of Theorem 2.3 is complete. \( \square \)

Using the same method as in Theorem 2.3 with the fact that \( f(z) \in K_{n+1}(\beta, \gamma) \Leftrightarrow zf'(z) \in K_n(\beta, \gamma) \), we can deduce from Theorem 2.3 the following:

**Theorem 2.4.** \( K_n(\beta, \gamma) \subset K_{n+1}(\beta, \gamma)(0 \leq \beta, \gamma < 1, n \in N_0) \).

### 3. Integral operator

For \( c > -1 \) and \( f(z) \in A \), we recall here the generalized Bernardi-Libera-Livingston integral operator as:

\[
L_c(f) = \frac{c + 1}{z^c} \int_{0}^{z} t^{c-1} f(t) dt .
\] (3.1)

The operator \( L_c(f) \) when \( c \in N \) was studied by Bernardi [2]. For \( c = 1 \), \( L_1(f) \) was investigated earlier by Libera [5] and Livingston [7].

The following theorems deal with the generalized Bernardi-Libera-Livingston integral operator \( L_c(f) \) defined by (3.1).

**Theorem 3.1.** Let \( c > -\gamma \). If \( f(z) \in S_n^*(\gamma) \), then \( L_c(f) \in S_n^*(\gamma) \).
Differentiating (3.4) with respect to $z$

From (3.1), we have

Proof.

Set

$$z(I^n L_c(f))' = (c + 1)I^n f(z) - cI^n L_c(f). \quad (3.2)$$

Set

$$\frac{z(I^n L_c(f))'}{I^n L_c(f)} = \frac{1 + (1 - 2\gamma)w(z)}{1 - w(z)}, \quad (3.3)$$

where $w(z)$ is analytic or meromorphic in $U$, $w(0) = 0$. Using (3.2) and (3.3) we get

$$\frac{I^n f(z)}{I^n L_c(f)} = \frac{c + 1 + (1 - c - 2\gamma)w(z)}{(c + 1)(1 - w(z))}. \quad (3.4)$$

Differentiating (3.4) with respect to $z$ logarithmically, we obtain

$$\frac{z(I^n f(z))'}{I^n f(z)} = \frac{1 + (1 - 2\gamma)w(z)}{1 - w(z)} + \frac{zw'(z)}{1 - w(z)} + \frac{(1 - c - 2\gamma)zw'(z)}{1 + c + (1 - c - 2\gamma)w(z)}. \quad (3.5)$$

Now we claim that $|w(z)| < 1(z \in U)$. Otherwise, there exists a point $z_0 \in U$ such that $\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1$. Then by Jack’s lemma [3], we have $z_0w'(z_0) = kw(z_0)(k \geq 1)$.

Putting $z = z_0$ and $w(z_0) = e^{i\theta}$ in (3.5), we have

$$\text{Re} \left\{ \frac{1 + (1 - 2\gamma)w(z_0)}{1 - w(z_0)} \right\} = \text{Re} \left\{ (1 - \gamma)\frac{1 + w(z_0)}{1 - w(z_0)} + \gamma \right\} = \gamma,$$

and

$$\text{Re} \left\{ \frac{z_0(I^n f(z_0))'}{I^n f(z_0)} - \gamma \right\} = \text{Re} \left\{ \frac{2(1 - \gamma)ke^{i\theta}}{(1 - e^{i\theta})[1 + c + (1 - c - 2\gamma)e^{i\theta}]} \right\}$$

$$= 2k(1 - \gamma)\text{Re} \left\{ \frac{(e^{i\theta} - 1)[1 + c + (1 - c - 2\gamma)e^{-i\theta}]}{2(1 - \cos \theta)[(1 + c)^2 + 2(1 + c)(1 - c - 2\gamma)\cos \theta + (1 - c - 2\gamma)^2]} \right\}$$

$$= \frac{-2k(1 - \gamma)(c + \gamma)}{(1 + c)^2 + 2(1 + c)(1 - c - 2\gamma)\cos \theta + (1 - c - 2\gamma)^2} \leq 0,$$

which contradicts the hypothesis that $f(z) \in S_\alpha^\gamma(\gamma)$. Hence $|w(z)| < 1$ for $z \in U$, and it follows from (3.3) that $L_c(f) \in S_\alpha^\gamma(\gamma)$. The proof of Theorem 3.1 is complete. \(\square\)

**Theorem 3.2.** Let $c > -\gamma$. If $f(z) \in C_n(\gamma)$, then $L_c(f) \in C_n(\gamma)$.

**Proof.** $f \in C_n(\gamma) \Leftrightarrow zf' \in S_\alpha^\gamma(\gamma) \Rightarrow L_c(zf') \in S_\alpha^\gamma(\gamma) \Leftrightarrow z(L_c f)' \in S_\alpha^\gamma(\gamma) \Leftrightarrow L_c(f) \in C_n(\gamma).$ \(\square\)

**Theorem 3.3.** Let $c > -\gamma$. If $f(z) \in K_n(\beta, \gamma)$, then $L_c(f) \in K_n(\beta, \gamma)$.
Proof. Let \( f(z) \in K_n(\beta, \gamma) \). Then, by definition, there exists a function \( g(z) \in S_n^\gamma(\gamma) \) such that

\[
\text{Re} \left\{ \frac{z(I^n f(z))'}{I^n g(z)} \right\} > \beta \quad (z \in U) .
\]

Put

\[
\frac{z(I^n L_c(f))'}{I^n L_c(g)} - \beta = (1 - \beta)h(z) ,
\]

where \( h(z) = 1 + c_1 z + c_2 z^2 + \ldots \). From (3.2), we have

\[
\frac{z(I^n f(z))'}{I^n g(z)} = \frac{I^n(z f'(z))}{I^n g(z)} = \frac{z(I^n L_c(z f'))'}{z(I^n L_c(g))'} + cI^n L_c(z f')
\]

\[
= \frac{z(I^n L_c(g))'}{I^n L_c(g)} + c .
\]

Since \( g(z) \in S_n^\gamma(\gamma) \), then from Theorem 3.1, we have \( L_c(g) \in S_n^\gamma(\gamma) \). Let

\[
\frac{z(I^n L_c(g))'}{I^n L_c(g)} = \gamma + (1 - \gamma)H(z) ,
\]

where \( \text{Re} H(z) > 0(z \in U) \). Using (3.7), we have

\[
\frac{z(I^n f(z))'}{I^n g(z)} = \frac{z(I^n L_c(z f'))'}{I^n L_c(g)} + c[1(1 - \beta)h(z) + \beta]
\]

\[
= \gamma + c + (1 - \gamma)H(z) .
\]

Also, (3.6) can be written as

\[
\frac{z(I^n L_c(f))'}{I^n L_c(g)} = I^n L_c(g)[\beta + (1 - \beta)h(z)] .
\]

Differentiating both sides of (3.9), we have

\[
z \left\{ \frac{z(I^n L_c(f))'}{I^n L_c(g)} \right\}' = z(I^n L_c(g))'\left[ \beta + (1 - \beta)h(z) + (1 - \beta)z h'(z)I^n L_c(g) \right],
\]

or

\[
z \left\{ \frac{z(I^n L_c(f))'}{I^n L_c(g)} \right\}' = z(I^n L_c(g))' \]
\[
= (1 - \beta)z h'(z) + [\beta + (1 - \beta)h(z)]\left[ \gamma + (1 - \gamma)H(z) \right].
\]
From (3.8), we have
\[
\frac{\left(I_n f(z)\right)'}{I_n g(z)} - \beta = (1 - \beta)h(z) + \frac{(1 - \beta)zh'(z)}{\gamma + c + (1 - \gamma)H(z)}.
\]  
(3.10)

We form the function \(\Psi(u, v)\) by taking \(u = h(z)\) and \(v = zh'(z)\) in (3.10) as:
\[
\Psi(u, v) = (1 - \beta)u + \frac{(1 - \beta)v}{\gamma + c + (1 - \gamma)H(z)}.
\]  
(3.11)

It is clear that the function \(\Psi(u, v)\) defined by (3.11) satisfies the conditions (i), (ii) and (iii) of Lemma 2.1. Thus we have \(I_n(f(z)) \in K_n(\beta, \gamma)\). The proof of Theorem 3.3 is complete. \(\square\)

Similarly, we can prove:

**Theorem 3.4.** Let \(c > -\gamma\). If \(f(z) \in K_n^*(\beta, \gamma)\), then \(I_n(f(z)) \in K_n^*(\beta, \gamma)\).

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