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ON THE USE OF ABEL-JENSEN TYPE COMBINATORIAL FORMULAS FOR CONSTRUCTION AND INVESTIGATION OF SOME ALGEBRAIC POLYNOMIAL OPERATORS OF APPROXIMATION

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Abstract. The aim of this paper is to present some Abel-Jensen type combinatorial formulas useful for construction and investigation of some algebraic polynomial linear positive operators of approximation of univariate functions from the space $C[0,1]$.

In the first part of the paper we present the Abel generalization (1.1) of the classical binomial formula. Then we mention the two Abel type formulas (1.2) and (1.3), as well as the Vandermonde-Jensen formula (1.5).

In the second section one extends to factorial powers the preceding formulas. Then we establish the combinatorial identities (2.2)-(2.5), which are used to give the basic polynomials, depending on two non-negative parameters α and β.

In the third section we use these polynomials for construction several linear positive operators, depending on four parameters, associated to function $f \in C[0,1]$. Some particular cases of these operators were investigated by several authors mentioned at the end of the paper. Finally, we want to mention that in the paper [10] of Cheney and Sharma was proved that the operator Q_n reproduces only the constant functions.

In the fourth section are investigated the approximation properties of the operator $Q_m^{\alpha,\beta}$, defined at (3.3). In the last section are given evaluations of the remainder term of the approximation formula (5.1).

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1. **Introduction**

We start with the celebraten generalization of the Newton binomial formula, given in 1826, by the outstanding mathematical genius represented by the Norwegian Niels Henrik Abel [1], namely

$$(u+v)^n = \sum_{k=0}^{n} \binom{n}{k} u(u-k\beta)^{k-1}(v+k\beta)^{n-k}, \tag{1.1}$$

where β is a non-negative parameter.

We mention also the Abel type formulas

$$(u+v+n\beta)^n = \sum_{k=0}^{n} \binom{n}{k} u(u+k\beta)^{k-1}(v+(n-k)\beta)^{n-k}, \tag{1.2}$$

$$(u+v+n\beta)^n = \sum_{k=0}^{n} \binom{n}{k} (u+k\beta)^{k} v(v+(n-k)\beta)^{n-k-1}. \tag{1.3}$$

Jensen [29] has obtained a new symmetrical identity of Abel

$$(u+v(u+v+n\beta))^{n-1} = \sum_{k=0}^{n} \binom{n}{k} u(u+k\beta)^{k-1} v(v+(n-k)\beta)^{n-k-1}. \tag{1.4}$$

In the paper [18] the American mathematician H.W. Gould gave the following generalization of the Vandermonde formula

$$\binom{u+v+n\beta}{n} = \sum_{k=0}^{n} \binom{u+k\beta}{k} \binom{v+(n-k)\beta}{n-k} \frac{v}{v+(n-k)\beta},$$

which can be written, by using the factorial powers, under the form

$$(u+v+n\beta)^{[n]} = \sum_{k=0}^{n} \binom{n}{k} (u+k\beta)^{[k]} v(v+(n-k)\beta)^{[n-k-1]}.$$

The factorial power of a non-negative order n and increment h of u is defined by the formula

$$u^{[n,h]} = u(u-h)\dots(u-(n-1)h), \quad u^{[0,h]} = 1.$$

When $h=1$ we write $u^{[n,1]} = u^{[n]}$.

We shall also consider the generalized Vandermonde-Jensen formula

$$\frac{u+v}{u+v+n\beta}\binom{u+v+n\beta}{n} = \sum_{k=0}^{n} \frac{u}{u+k\beta}\binom{u+k\beta}{k}\frac{v}{v+(n-k)\beta}\binom{v+(n-k)\beta}{n-k}. \tag{1.5}$$

Jensen [29] has made the remark that formula (1.5) and the following formula

$$\binom{u+v}{n} = \sum_{k=0}^{n} \frac{u}{u+k\beta}\binom{u+k\beta}{k}\binom{v-k\beta}{n-k} \tag{1.6}$$

have been given earlier by I.G. Hagen [6] in 1891, but without demonstration.

These two formulas are particular cases of the more general formula

$$\frac{a(u+v-n\beta)+bnu}{u(u+v)(v-n\beta)}\binom{u+v}{n} = \sum_{k=0}^{n} \frac{a+bk}{(u+k\beta)(v-k\beta)}\binom{u+k\beta}{k}\binom{v-k\beta}{n-k},$$

given by Hagen [6] without any proof.

Jensen [29] has given also the new and elegant identity

$$\sum_{k=0}^{n}\binom{u+k\beta}{k}\binom{v-k\beta}{n-k} = \sum_{k=0}^{n}\binom{u+v-k}{n-k}\beta^{k}, \tag{1.7}$$

which can be seen in the book: "Combinatorial Identities" [23] of H.W. Gould.

In order to prove the identity (1.7) we introduce first the following notation

$$G(u,v,n) = \sum_{k=0}^{n}\binom{u+k\beta}{k}\binom{v-k\beta}{n-k}.$$

It is easy to see that we can write successively

$$\begin{aligned} G(u,v,n) &= \sum_{k=0}^{n}\binom{u+k\beta}{k}\binom{v-k\beta}{n-k}\left\{\frac{u}{u+k\beta}+\beta\frac{k}{u+k\beta}\right\} \\ &= \sum_{k=0}^{n}\frac{u}{u+k\beta}\binom{u+k\beta}{k}\binom{v-k\beta}{n-k} \\ &+ \beta\sum_{k=1}^{n}\binom{u-1+k\beta}{k-1}\binom{v-k\beta}{n-k}. \end{aligned}$$

By using an identity (1.6) given in the paper of H.W. Gould ([18], pag. 71), as well as the Vandermonde-type convolution (1.10) from the same paper, we are able to write the equality:

$$\sum_{k=0}^{n} \frac{u}{u+k\beta} \binom{u+k\beta}{k} \binom{v-k\beta}{n-k} = \binom{u+v}{n},$$

so that we can obtain the relation

$$G(u,v,n) = \binom{u+v}{n} + \beta \sum_{k=0}^{n-1} \binom{u-1+\beta+k\beta}{k} \binom{v-\beta-k\beta}{n-1-k}.$$

Consequently we can write the Jensen recurrence formula

$$G(u,v,n) - \beta G(u-1+\beta, v-\beta, n-1) = \binom{u+v}{n}. \tag{1.8}$$

By using this we are able to write

$$\beta G(u-1+\beta, v-\beta, n-1) - \beta^2 G(u-2+2\beta, v-2\beta, n-2) = \beta \binom{u+v-1}{n-1}$$

and by successive application of (1.8) and summing the resulting relations we ultimately obtain

$$G(u,v,n) - \beta^r G(u-r+r\beta, v-r\beta, n-r) = \sum_{k=0}^{r-1} \binom{u+v-k}{n-k} \beta^k.$$

Letting $r = n+1$ we find the relation (1.7), which we intended to prove.

Now we want to point out that the identity (1.7) is a counterpart for the Abel-type series:

$$\sum_{k=0}^{n} \frac{(u+k\beta)^k}{k!} \cdot \frac{(v-k\beta)^{n-k}}{(n-k)!} = \sum_{k=0}^{n} \frac{(u+v)^k}{k!} \cdot \beta^{n-k}.$$

One way to prove this is to develop a recurrence relation, or to carry through a limiting process, as was noted in the paper of Gould [18].

Ending this section we mention, with Gian-Carlo Rota and Ronald Mullin ([49], pag. 168 and 195), that the Abel polynomials

$$p_n(x) = x(x-an)^{n-1}$$

are the basic polynomials of the Abel operator $E^a D$ (here D is the differentiation operator and E^a is the shift operator or the translation operator).

We have $DE^a = E^a D : x(x-na)^{n-1} \to nx(x-(n-1)a)^{n-2}$.

Finally, I consider that it is important to mention a generating relation from the monograph of Boas and Buck ([6], pag. 34) for the general difference polynomials

$$e^{xt} = \sum_{n=0}^{\infty} \frac{p_n(x)}{n!}[(e^t - 1)e^{\beta t}]^n,$$

where β is a real parameter.

For $\beta = 0$ we get the Newton binomial polynomials

$$p_n(x) = \binom{x}{n} = x(x-1)\dots(x-n+1)/n!,$$

while for $\beta = -\frac{1}{2}$ we obtain the Stirling interpolation polynomials.

2. Extensions to factorial powers of the Abel-Jensen combinatorial formulas

As we have mentioned above, we denote by $u^{[n,h]}$ the factorial power of order n ($n \geq 0$) and increment h of u, that is

$$u^{[n,h]} = u(u-h)\dots(u-(n-1)h), \quad u^{[0,h]} = 1, \quad u^{[n,1]} = u^{[n]}.$$

By extension to factorial powers the Abel combinatorial formula (1.1) we obtain

$$(u+v)^{[n,h]} = \sum_{k=0}^{n} \binom{n}{k} u(u-k\beta)^{[k-1,h]}(v+k\beta)^{[n-k,h]}, \tag{2.1}$$

where β is a non-negative parameter.

If we replace here $h = -\alpha$, where α is a non-negative parameter, we get the identity

$$(u+v)^{[n,-\alpha]} = \sum_{k=0}^{n} \binom{n}{k} u(u-k\beta)^{[k-1,-\alpha]}(v+k\beta)^{[n-k,-\alpha]}.$$

Now we select $u = x$ and $v = 1-x$ and we obtain the important identity

$$\sum_{k=0}^{n} \binom{n}{k} x(x-k\beta)^{[k-1,-\alpha]}(1-x+k\beta)^{[n-k,-\alpha]} = 1^{[n,-\alpha]}$$

$$= 1(1+\alpha)(1+2\alpha)\dots(1+(n-1)\alpha). \tag{2.2}$$

By using the Abel-Jensen combinatorial formula (1.4) we are able to write

$$\sum_{k=0}^{n} \binom{n}{k} x(x+k\beta)^{[k-1,-\alpha]}(1-x)(1-x+(n-k)\beta) = (1+n\beta)^{[n-1,-\alpha]}. \tag{2.3}$$

According to the combinatorial formula (1.2), we get the identity

$$\sum_{k=0}^{n} \binom{n}{k} x(x+k\beta)^{[k-1,-\alpha]}(1-x+(n-k)\beta)^{[n-k,-\alpha]} = (1+n\beta)^{[n,-\alpha]} \tag{2.4}$$

while from (1.4) we obtain

$$\sum_{k=0}^{n} \binom{n}{k} (x+k\beta)^{[k,-\alpha]}(1-x)(1-x+(n-k)\beta)^{[n-k-1,-\alpha]} = (1+n\beta)^{[n,-\alpha]}. \tag{2.5}$$

By using the combinatorial identities (2.2), (2.3), (2.4) and (2.5) we can introduce the basic polynomials

$$s_{m,k}^{\alpha,\beta}(x) = \frac{1}{1^{[m,-\alpha]}} \sum_{k=0}^{m} \binom{m}{k} x(x-k\beta)^{[k-1,-\alpha]}(1-x+k\beta)^{[m-k,-\alpha]}$$

$$q_{m,k}^{\alpha,\beta}(x) = \frac{1}{(1+m\beta)^{[m-1,-\alpha]}} \sum_{k=0}^{m} \binom{m}{k} x(x+k\beta)^{[k-1,-\alpha]}(1-x)(1-x+(n-k)\beta)^{[m-k,-\alpha]}$$

$$p_{m,k}^{\alpha,\beta}(x) = \frac{1}{(1+m\beta)^{[m,-\alpha]}} \sum_{k=0}^{m} \binom{m}{k} x(x+k\beta)^{[k-1,-\alpha]}(1-x+(m-k)\beta)^{[m-k,-\alpha]}$$

$$r_{m,k}^{\alpha,\beta}(x) = \frac{1}{(1+m\beta)^{[m,-\alpha]}} \sum_{k=0}^{m} \binom{m}{k} (x+k\beta)^{[k,-\alpha]}(1-x)(1-x+(m-k)\beta)^{[m-k-1,-\alpha]}.$$

3. Linear positive operators constructed by means of the basic polynomials considered in the preceding section

For any function $f \in C[0,1]$ we construct linear positive operators, depending on four parameters:

$$\begin{aligned}
(S_m^{\alpha,\beta,\gamma,\delta} f)(x) &= \sum_{k=0}^{m} s_{m,k}^{\alpha,\beta}(x) f\left(\frac{k+\gamma}{m+\delta}\right), \\
(Q_m^{\alpha,\beta,\gamma,\delta} f)(x) &= \sum_{k=0}^{m} q_{m,k}^{\alpha,\beta}(x) f\left(\frac{k+\gamma}{m+\delta}\right), \\
(P_m^{\alpha,\beta,\gamma,\delta} f)(x) &= \sum_{k=0}^{m} p_{m,k}^{\alpha,\beta}(x) f\left(\frac{k+\gamma}{m+\delta}\right), \\
(R^{\alpha,\beta,\gamma,\delta} f)(x) &= \sum_{k=0}^{m} r_{m,k}^{\alpha,\beta}(x) f\left(\frac{k+\gamma}{m+\delta}\right),
\end{aligned} \tag{3.1}$$

where $0 \leq \gamma \leq \delta$.

In the case $\beta = \gamma = \delta = 0$ these operators reduce to the Stancu operator S_m^{α}, introduced and investigated in the paper [55]:

$$(S_m^{\alpha} f)(x) = \sum_{k=0}^{m} s_{m,k}^{\alpha}(x) f\left(\frac{k}{m}\right),$$

where

$$s_{m,k}^{\alpha}(x) = \binom{m}{k} \frac{x^{[k,-\alpha]}(1-x)^{[m-k,-\alpha]}}{1^{[m,-\alpha]}}.$$

This operator was further investigated and applied by several authors. B. Della Vecchia [12], A. Di Lorenzo - M.R. Occorsio [13], F. Frenţiu [17], I. Horova and Budikova [27], G. Mastroianni and G. Occorsio [40], [41], I.A. Rus [50], S. Toader [61] and others.

If we select $\alpha = \gamma = \delta = 0$ then we arrive at the operators of Cheney and Sharma [10] P_m and Q_m.

For $\gamma = \delta = 0$ the operator (3.1) becomes

$$(Q_m^{\alpha,\beta} f)(x) = \sum_{k=0}^{m} q_{m,k}^{\alpha,\beta}(x) f\left(\frac{k}{m}\right), \tag{3.2}$$

where

$$q_{m,k}^{\alpha,\beta}(x) = \frac{\binom{m}{k} x(x+k\beta)^{[k-1,-\alpha]}(1-x)(1-x+(m-k)\beta)^{[m-k-1,-\alpha]}}{(1+m\beta)^{[m-1,-\alpha]}}.$$

When $\alpha = 0$ this operator will be the second operator of Cheney-Sharma [10], defined by the formula

$$(Q_m f)(x;\beta) = \sum_{k=0}^{m} q_{m,k}(x;\beta) f\left(\frac{k}{m}\right),$$

where

$$q_{m,k}(x;\beta) = \binom{m}{k} \frac{x(x+k\beta)^{k-1}(1-x)(1-x+(m-k)\beta)^{m-1-k}}{(1+m\beta)^{m-1}}.$$

It will be easy to prove that this operator is similar with the Bernstein operator B_m, and preserves the linear functions.

4. **Convergence properties of the sequence** $(Q_m^{\alpha,\beta})$

For the convergence of the sequence of operators $Q_m^{\alpha,\beta}$, defined at (3.3), we shall use the classical theorem of Bohman-Korovkin [7], [33], which can be stated as follows:

If we have a sequence of linear positive operators $L_m : C[a,b] \to C[a,b]$ and we have $(L_m s_k)$ converges uniformly to s_k on $[a,b]$ for $k = 0, 1$ and 2, where $s_k(x) = x^k$, then the sequence $(L_m f)$ converges uniformly to f on $[a,b]$ for each $f \in C[a,b]$. In our case $[a,b] = [0,1]$ and we have the operators Q_m, defined at (3.3).

According to Abel-Jensen combinatorial formula (2.3) we can see that $Q_m^{\alpha,\beta} e_0 = e_0$.

In the case of the next test function e_1 we have

$$(Q_m^{\alpha,\beta} e_1)(x) = \frac{1}{(1+\alpha+m\beta)^{[m-1,-\alpha]}} (Z_m^{\alpha,\beta} e_1)(x) \tag{4.1}$$

where

$$(Z_m^{\alpha,\beta} e_1)(x) = \sum_{k=1}^{m} \frac{k}{m} \binom{m}{k} x(x+\alpha+k\beta)^{[k-1,-\alpha]}(1-x)(1-x+\alpha+(m-k)\beta)^{[m-k-1,-\alpha]}$$

$$= \sum_{k=1}^{m} \binom{m-1}{k-1} x(x+\alpha+k\beta)^{[k-1,-\alpha]}(1-x)(1-x+\alpha+(m-k)\beta)^{[m-1-k,-\alpha]}.$$

By changing the index of summation $k-1=j$, we get

$$(Z_m^{\alpha,\beta}e_1)(x) = x\sum_{j=0}^{m-1} \binom{m-1}{j} (x+\alpha+\beta+j\beta)^{[j,-\alpha]}(1-x)(1-x+(m-1-j)\beta)^{[m-2j,-\alpha]}. \tag{4.2}$$

Now we shall use an extension to factorial powers of the Abel combinatorial formula

$$(u+v+n\beta)^{[n,h]} = \sum_{k=0}^{n} \binom{n}{k} (u+k\beta)^{[k,h]} v(v+(n-k)\beta h)^{[n-k-1,h]}. \tag{4.3}$$

We have to replace here $n=m-1$, $h=-\alpha$, $u=x+\alpha+\beta$, $v=1-x$ and we arrive at the following identity

$$(1+\alpha+m\beta)^{[m-1,-\alpha]}$$

$$= \sum_{k=0}^{m-1} \binom{m-1}{b} (x+\alpha+\beta+b\beta)^{[k,-\alpha]}(1-x)(1-x+(m-k-1)\beta)^{[m-2-k,-\alpha]}.$$

According to (3.4), (3.5) and (3.6) we can write:

$$Q_n^{\alpha,\beta}e_1 = e_1.$$

Consequently, our operator reproduces the linear functions.

Going on to the next test function e_2 we find that

$$(Q_m^{\alpha,\beta}e_2)(x) = \frac{1}{m}\sum_{k=1}^{m}\left[\frac{k}{m} + \frac{k(k-1)}{m}\right] q_{m,k}^{\alpha,\beta}(x)$$

$$= \frac{1}{m}(Q_m^{\alpha,\beta}e_1)(x) + \frac{1}{m}\sum_{k=2}^{m}\binom{m-1}{m-2} x(x+\alpha+k\beta)^{[k-1,-\alpha]}(1-x)(1-x+\alpha+(m-k)\beta)^{[m-k-1,-\alpha]}$$

$$= \frac{x}{m} + \frac{m-1}{m}\sum_{j=0}^{m-2}\binom{m-2}{j}(x+\alpha+2\beta+j\beta)^{[j+1,-\alpha]}(1-x+\alpha+(m-2-j)\beta)^{[m-3-j,-\alpha]}.$$

Now if we use again the extension to factorial powers of the Abel combinatorial formula, we can see that $Q_m^{\alpha,\beta}e_2$ tends uniformly to e_2 on $[0,1]$, when m tends to infinity.

By applying the Bohman-Korovkin [7], [3]] convergence criterion we can state the following result: if $f \in C[0,1]$ and the parameters α and β are non-negative and depend on m such that $\alpha = \alpha(m) \to 0$ and $m\beta(m) \to 0$, when m tends to infinity, then the sequence $(Q_m^{\alpha,\beta} f)$ converges uniformly to f on $[0,1]$.

5. **Evaluations of the remainder term**

Because the operator $Q_m^{\alpha,\beta}$ reproduces the linear functions, we can state that the approximation formula

$$f(x) = (Q_m^{\alpha,\beta} f)(x) + (R_m^{\alpha,\beta} f)(x) \tag{5.1}$$

has the degree of exactness $N = 1$.

Assuming that the function f has a continuous second derivative on the interval $[0,1]$, we can represent the remainder of this formula under the following integral form

$$(R_m^{\alpha,\beta} f)(x) = \int_0^1 G_m^{\alpha,\beta}(t;x) f''(t)dt, \tag{5.2}$$

where

$$G_m^{\alpha,\beta}(t;x) = (R_m^{\alpha,\beta}\varphi_x)(t), \quad \varphi_x(t) = (x-t)_+ = \frac{x-t+|x-t|}{2},$$

understanding that $R_m^{\alpha,\beta}$ operates on φ_x as a function of x.

The above integral representation of the remainder can be obtained if we make use of the well-known theorem of Peano.

For the Peano kernel, associated to our operator, we have

$$G_m^{\alpha,\beta}(t;x) = (x-t)_+ - \sum_{k=0}^{m} q_{m,k}^{\alpha,\beta}(x) \left(\frac{k}{m} - t\right)_+ . \tag{5.3}$$

In order to find an explicit expression of this kernel, we assume that $x \in \left[\frac{s-1}{m}, \frac{s}{m}\right]$ and we can write

$$G_m^{\alpha,\beta}(t;x) = x - t - \sum q_{m,k}^{\alpha,\beta}(x) \left(\frac{k}{m} - t\right) \tag{5.4}$$

for $t \in \left[\frac{j-1}{m}, \frac{j}{m}\right]$, where $1 \le j \le s-1$.

If we consider that $t \in \left[\frac{s-1}{m}, x\right]$, then we obtain

$$G_m^{\alpha,\beta}(t;x) = x - t - \sum_{k \geq s} q_{m,k}^{\alpha,\beta}(x)\left(\frac{k}{m} - t\right), \tag{5.5}$$

while for $t \in \left[x, \frac{s}{m}\right]$ we get

$$G_m^{\alpha,\beta}(t;x) = -\sum_{k \geq s} q_{m,k}^{\alpha,\beta}(x)\left(\frac{k}{m} - t\right).$$

In the case $t \in \left[\frac{j-1}{m}, \frac{j}{m}\right]$, where $j > s$, we have

$$G_m^{\alpha,\beta}(t;x) = -\sum_{k \geq j} q_{m,k}^{\alpha,\beta}(x)\left(\frac{k}{m} - t\right).$$

Because the degree of exactness of the formula (5.1) is one, by replacing $f(x) = x - t$, the corresponding remainder vanishes and we obtain

$$x - t - \sum_{k=0}^{m} q_{m,k}^{\alpha,\beta}(x)\left(\frac{k}{m} - t\right)$$

$$= \sum_{k=0}^{j-1} q_{m,k}^{\alpha,\beta}(x)\left(\frac{k}{m} - t\right) + \sum_{k=j}^{m} q_{m,k}^{\alpha,\beta}(x)\left(\frac{k}{m} - t\right).$$

Therefore we can write

$$x - t = \sum_{k=j}^{m} q_{m,k}^{\alpha,\beta}(x)\left(\frac{k}{m} - t\right) = -\sum_{k=0}^{j-1} q_{m,k}^{\alpha,\beta}(x)\left(x - \frac{k}{m}\right).$$

Consequently, the representation (5.4) can be replaced by

$$G_m^{\alpha,\beta}(t;x) = -\sum_{k=0}^{j-1} q_{m,k}^{\alpha,\beta}(x)\left(t - \frac{k}{m}\right),$$

if $t \in \left[\frac{j-1}{m}, \frac{j}{m}\right]$ and $1 \leq j \leq s-1$, while (5.5) can be replaced by

$$G_m^{\alpha,\beta}(t;x) = \sum_{k=0}^{s-1} q_{m,k}^{\alpha,\beta}(x)\left(t - \frac{k}{m}\right),$$

when $t \in \left[\frac{s-1}{m}, x\right]$.

Because on the interval $[0,1]$ we have $G_m^{\alpha,\beta}(t;x) \leq 0$, we can apply the mean value theorem to the integral and we obtain

$$(R_m^{\alpha,\beta}f)(x) = f''(\xi)\int_0^1 G_m^{\alpha,\beta}(t;x)dt, \quad \xi \in (0,1), \tag{5.6}$$

under the hypothesis that $f \in C^2[0,1]$.

If in the approximation formula

$$f(x) = (Q_m^{\alpha,\beta}f)(x) + f''(\xi)\int_0^1 G_m^{\alpha,\beta}(t;x)dt$$

we replace $f(x) = e_2(x) = x^2$, we get

$$\int_0^1 G_m^{\alpha,\beta}(t;x)dt = \frac{1}{2}[x^2 - (Q_m^{\alpha,\beta}e_2)(x)] = \frac{1}{2}(R_m^{\alpha,\beta}e_2)(x).$$

Consequently, we can see that the remainder of the approximation formula (5.1) can be expressed under the following form

$$(R_m^{\alpha,\beta}f)(x) = \frac{1}{2}(R_m^{\alpha,\beta}e_2)(x)f''(\xi), \tag{5.7}$$

where $0 < \xi < 1$.

Therefore we can state the following result:

If we have the function $f \in C^2[0,1]$, then the remainder of the approximation formula (5.1) can be represented under the integral form (5.6).

We mention that in the particular case $\alpha = \beta = 0$, when $Q_m = B_m$, the corresponding approximation formula was established by D.D. Stancu in 1963 in the paper [54].

Now we want to make the remark that because $Q_m^{\alpha}f$ is interpolatory at both sides of the basic interval $[0,1]$, it is clear that $(R_m^{\alpha}e_2)(x)$ had to contain the factor $x(x-1)$.

Since $R_m f \neq 0$, if $\beta = 0$, for any convex function f of the first order, we can apply a criterion of T. Popoviciu [46] and we can find that the remainder $R_m^{\alpha}f$ is of a simple form. Therefore we can state the following result:

If the second-order divided differences of the function f are bounded on the interval $[0,1]$, then there exist three distinct points $t_{m,1}, t_{m,2}, t_{m,3}$ in the interval $[0,1]$, *which might depend on f, such that the remainder of the approximation formula (5.1) can be represented under the following form*

$$(R_m^\alpha f)(x) = (R_m^\alpha e_2)(x)[t_{m,1}, t_{m,2}, t_{m,3}; f],$$

where the nodes are certain distinct points of the interval $[0,1]$.

It is clear that if $f \in C^2[0,1]$ and we apply the mean-value theorem of divided differences, then we can obtain formula (5.7).

In the case $\alpha = 0$ we can see that we have

$$(R_m f)(x) = \frac{x(x-1)}{2m} f''(\xi),$$

which represents the remainder in the case of the Bernstein approximation operator B_m.

This result was obtained by D.D. Stancu in 1963 in the paper [54].

References

[1] Abel, N.H., *Démonstration d'une expression de laquelle la formule binôme est un cas particulier*, Journ. für Reine und Angewendte Mathematik, **1**(1826), 159-160, Oeuvres Complètes, Christiania, Groendahls, 1839.

[2] Agratini, O., *Operatori de aproximare*, Univ. Babeş-Bolyai, Cluj-Napoca, 1998.

[3] Agratini, O., *Aproximare prin operatori liniari*, Presa Univ. Clujeană, Cluj-Napoca, 2000.

[4] Agratini, O., *Binomial polynomials and their applications in approximation theory*, Conf. Semin. Mat. Univ. Bari, **281**(2001), 1-22.

[5] Altomare, F., Campiti, M., *Korovkin-Type Approximation Theory and Its Applications*, Walter de Gruyter, Berlin, 1994.

[6] Boas, R.P., Buck, R.C., *Polynomial expansions of analytic functions*, Springer-Verlag, Berlin, 1964.

[7] Bohman, H., *On approximation of continuous and analytic functions*, Ark. Mat., **2**(1952), 43-56.

[8] Carlitz, L., *Some formulas of Jensen and Gould*, Duke Math. J., **27**(1960), 319-321.

[9] Carlitz, L., *A binomial identity arising from a sorting problem*, SIAM Rev., **6**(1964), 20-30.

[10] Cheney, E.W., Sharma, A., *On a generalization of Bernstein polynomials*, Riv. Mat. Univ., Parma, **5**(1964), 77-84.

[11] Coman, Gh., *Analiză numerică*, Ed. Libris, Cluj-Napoca, 1995.

[12] Della Vecchia, B., *On the approximation of functions by means of the operators of D.D. Stancu*, Studia Univ. Babeş-Bolyai, Mathematica, **37**(1992), 3-36.

[13] Di Lorenzo, A., Occorsio, M.R., *Polinomi di Stancu*, Inst. Applic. Matem., Napoli, Rapp. Tecnico (1995), 42 pag.

[14] DeVore, R.A., *The approximation of Continuous Functions by Positive Linear Operators*, Springer-Verlag, Berlin, 1972.

[15] Erdérlyi, A. (editor), *Higher transcendental functions*, McGraw-Hill, New York, vol. 1, 2(1953), vol. 3(1955).

[16] Françon, J., *Preuves combinatoires des identités d'Abel*, Discrete Mathematics, **4**(1974), p. 231.

[17] Frenţiu, M., *On the asymptotic aspect of the approximation of functions by means of the D.D. Stancu operators*, Babeş-Bolyai Univ. Preprint, **8**(1987), 57-64.

[18] Gould, H.W., *Some generalizations of Vandermonde's convolution*, Amer. Math. Monthly, **63**(1956), 84-91.

[19] Gould, H.W., *Final analysis of Vandermonde's convolution*, Amer. Math. Monthly, **64**(1957), 409-415.

[20] Gould, H.W., *Generalization of a theorem of Jensen concerning convolutions*, Duke Math. Journ., **27**(1960), 71-76.

[21] Gould, H.W., *A series transformation for finding convolution identities*, Duke Math. Journ., **28**(1961), 193-202.

[22] Gould, H.W., *Generalization of an integral formula of Bateman*, Duke Math. J., **29**(1962), 475-480.

[23] Gould, H.W., *Combinatorial Identities*, Morgantown West Virginia, 1972 (published by the author).

[24] Gould, H.W., Kaucky, J., *Evaluation of a class of binomial coefficient summations*, J. Combinatorial Theory, **1**(1966), 233-247.

[25] Hatvany, A.C., *Stancu curves in CAGD*, Rev. Analyse Numér. Théorie de l'Approximation, **31**(2002), 71-87.

[26] Hagen, J.G., *Synopsis der höheren Mathematik*, t. 1(1891), p. 67.

[27] Horova, I., Budikova, M., *A note on D.D. Stancu operators*, Ricerche di Matem., **44**(1995), 397-407.

[28] Hurwitz, A., *Über Abel's Verallgemeinerung der binomischen Formel*, Acta Mathematica, **26**(2002), 199-203.

[29] Jensen, J.L.W., *Sur une identité d'Abel et sur d'autre formules analogues*, Acta Mathematica, **26**(1902), 307-318.

[30] Ismail, M.E.H., *Polynomials of binomial type and approximation theory*, J. Approx. Theory, **23**(1978), 177-186.

[31] King, J.P., *A class of positive linear operators*, Canad. Math. Bull., **11**(1968), 51-59.

[32] Knuth, D.E., *The Art of Computer-Programming*, vol. 1: *Fundamental Algorithms*, Addison-Wesley Publ. Comp., Reading, Massachusetts, USA, 1968, p. 75.

[33] Korovkin, P.P., *On convergence of linear positive operators in the space of continuous functions*, Doklady Akad. Nauk SSSR, **90**(1953), 961-964 (Russian).

[34] Korovkin, P.P., *Linear Operators and Approximation Theory*, Delhi, Hindustan Publ. Corp., 1960, 222 pp.

[35] Lagrange, R., *Mémoire sur les suites de polynômes*, Acta Mathem., **51**(1928), 201-309.

[36] Lomeli, H.E., Garcia, C.L., *Variations on a Theorem of Korovkin*, Amer. Math. Monthly, **113**(2006), 744-749.

[37] Lupaş, A., *Approximation operators of binomial type*, New Developments in Approximation Theory, **132**(1999), 175-198.

[38] Mac Mahon, R.A., *Combinatorial Analysis*, Cambridge Univ. Press, 1915, 1916.

[39] Manole, C., *Dezvoltări în serii de polinoame Appell generalizate cu aplicaţii la aproximarea funcţiilor*, Teză de doctorat, Univ. Babeş-Bolyai, Fac. Matematică, Cluj-Napoca, 1984.

[40] Mastroianni, G., Occorsio, M.R., *Sulle derivate dei polinomi di Stancu*, Rend. Accad. Sci. M.F.N., Ser. IV, **45**(1978), 273-281.

[41] Mastroianni, G., Occorsio, M.R., *Una generalizatione dell'operatore di Stancu*, Accad. Sci. M.F.N., Ser. IV, **45**(1978), 495-511.

[42] Moldovan, G., *Generalizări ale polinoamelor lui S.N. Bernstein*, Rezumatul tezei de doctorat, Cluj, 1971.

[43] Mullin, R., Rota, G.C., *On the theory of binomial enumeration*, Graph Theory and its Applications, Academic Press, N.Y., 1970.

[44] Netto, M.E., *Lehrbuch der Combinatorik*, 1901, p. 52. Sec. Ed. 1927, Reprinted by Chelsea Publ., New York, 1954.

[45] Popoviciu, T., *Remarques sur les polynômes binomiaux*, Bull. Soc. Sci. Cluj, **6**(1931), 146-148.

[46] Popoviciu, T., *Sur le reste dans certaines formules linéaires d'approximation de l'analyse*, Mathematica (Cluj), **1(24)**(1959), 95-142.

[47] Riordan, J., *An Introduction to Combinatorial Analysis*, Wiley, New York, 1958.

[48] Riordan, J., *Combinatorial Identities*, Wiley, New York, 1968.

[49] Rota, G.-C., Mullin, R., *On the Foundations of Combinatorial Theory*, Graph Theory and its Applications (ed. B. Harris), Academic Press, New York, 1970.

[50] Rus, A.I., *Iterates of Stancu operators via contraction principle*, Proc. of Intern. Sympos. on Numerical Analysis and Approximation Theory, Cluj-Napoca.

[51] Ryser, H.J., *Combinatorial Mathematics*, New York, 1963.

[52] Salié, H., *Über Abel's Verallgemeinerung der binomischen Formel*, Ber. Verh. Sächs. Akad. Wiss, Leipzig Math. Nat. Kl., **98**(1951), 19-22.

[53] Skalsky, M., *A note on a convolution-type combinatorial identity*, Amer. Math. Monthly, **74**(1967), 836-838.

[54] Stancu, D.D., *Evaluation of the remainder term in approximation formulas by Bernstein polynomials*, Math. Comput., **17**(1963), 270-278.

[55] Stancu, D.D., *Approximation of functions by a new class of linear polynomial operators*, Rev. Roumaine Math. Pures Appl., **13**(1968), 1173-1194.

[56] Stancu, D.D., *Approximation of functions by means of a new generalized Bernstein operator*, Calcolo, **20**(1983), 211-229.

[57] Stancu, D.D., *Use of an identity of A. Hurwitz for construction of a linear positive operator of approximation*, Rev. Analyse Numér. Théorie de l'Approximation, **31**(2002), 115-118.

[58] Stancu, D.D., Cismaşiu, C., *On an approximating linear positive operator of Cheney-Sharma*, Rev. Analyse Numér. Théorie de l'Approximation, **26**(1987), 221-227.

[59] Stubhaug, A., *Niels Henrik Abel and his Times*, Springer, 2000.

[60] Tascu, J., *Approximation of bivariate functions by operators of Stancu-Hurwitz type*, Facta Universitatis Nis, Ser. Math.-Inform., **20**(2005), 33-39.

[61] Toader, S., *An approximation operator of Stancu type*, Rev. Anal. Numér. Théorie de l'Approx., **34**(2005), 115-121.

[62] Trîmbiţaş, R.T., *Analiză Numerică. O introducere bazată pe MATLAB*, Presa Universitară Clujeană, Cluj-Napoca, 2005.

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