DIFFERENTIAL SUBORDINATIONS AND SUPERORDINATIONS
FOR ANALYTIC FUNCTIONS DEFINED BY CONVOLUTION
STRUCTURE

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Abstract. In the present investigation, we obtain some subordination and
superordination results involving Hadamard product for certain normalized
analytic functions in the open unit disk. Relevant connections of the re-
results, which are presented in this paper, with various other known results
also pointed out.

1. Introduction

Let $H$ be the class of analytic functions in $U := \{z : |z| < 1\}$ and $H(a,n)$ be
the subclass of $H$ consisting of functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots.$$  

Let $A$ be the subclass of $H$ consisting of functions of the form

$$f(z) = z + a_2 z^2 + \ldots.$$  

Let $p, h \in H$ and let $\phi(r, s, t; z) : \mathbb{C}^3 \times U \rightarrow \mathbb{C}.$

If $p$ and $\phi(p(z), zp'(z), z^2 p''(z); z)$ are univalent and if $p$ satisfies the second
order superordination

$$b(z) \prec \phi(p(z), zp'(z), z^2 p''(z); z), \quad (1.1)$$

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then $p$ is a solution of the differential superordination (1.1). (If $f$ is subordinate to $F$, then $F$ is superordinate to $f$.) An analytic function $q$ is called a subordinant if $q \prec p$ for all $p$ satisfying (1.1). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants $q$ of (1.1) is said to be the best subordinant. Recently Miller and Mocanu [12] obtained conditions on $h$, $q$ and $\phi$ for which the following implication holds:

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) \prec p(z).$$

For two functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, the Hadamard product (or convolution) of $f$ and $g$ is defined by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n =: (g * f)(z).$$

For $\alpha_j \in \mathbb{C}$ ($j = 1, 2, \ldots, l$) and $\beta_j \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$ ($j = 1, 2, \ldots, m$), the generalized hypergeometric function $\mathcal{H}_m^l(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)$ is defined by the infinite series

$$\mathcal{H}_m^l(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_l)_n}{(\beta_1)_n \cdots (\beta_m)_n} \frac{z^n}{n!}$$

($l \leq m + 1; l, m \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$), where $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n := \frac{\Gamma(a + n)}{\Gamma(a)} = \begin{cases} 1, & (n = 0); \\ a(a + 1)(a + 2) \cdots (a + n - 1), & (n \in \mathbb{N} := \{1, 2, 3 \ldots\}). \end{cases}$$

Corresponding to the function

$$h(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := z \mathcal{H}_m^l(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z),$$

the Dziok-Srivastava operator [6] (see also [7, 22]) $H_m^l(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)$ is defined by the Hadamard product

$$H_m^l(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)f(z) := h(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) * f(z)$$

$$= z + \sum_{n=2}^{\infty} \frac{(\alpha_1)_{n-1} \cdots (\alpha_l)_{n-1}}{(\beta_1)_{n-1} \cdots (\beta_m)_{n-1}} \frac{a_n z^n}{(n - 1)!}.$$
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For brevity, we write

\[ H_l^m[\alpha_1]f(z) := H_l^m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) f(z). \]

It is easy to verify from (1.2) that

\[ z(H_l^m[\alpha_1]f(z))' = \alpha_1 H_l^m[\alpha_1 + 1] f(z) - (\alpha_1 - 1) H_l^m[\alpha_1] f(z). \]  

(1.3)

Special cases of the Dziok-Srivastava linear operator includes the Hohlov linear operator [8], the Carlson-Shaffer linear operator \( L(a,c) \) [5], the Ruscheweyh derivative operator \( D^n \) [17], the generalized Bernardi-Libera-Livingston linear integral operator (cf. [2], [9], [10]) and the Srivastava-Owa fractional derivative operators (cf. [15], [16]).

Using the results of Miller and Mocanu [12], Bulboacă [4] considered certain classes of first order differential superordinations as well as superordination-preserving integral operators (see [3]). Further, using the results of Mocanu [12] and Bulboacă [4] many researchers [1, 18, 19, 20, 21] have obtained sufficient conditions on normalized analytic functions \( f \) by means of differential subordinations and superordinations.

Recently, Murugusundaramoorthy and Magesh [13, 14] obtained sufficient conditions for a normalized analytic functions \( f(z) \) in \( \mathcal{U} \) such that \( (f * \Psi)(z) \neq 0 \) and \( f \) to satisfy

\[ q_1(z) < \left( \frac{H_l^m[\alpha_1]f(z)}{f(z)} \right)^\delta < q_2(z), \quad q_1(z) < \left( \frac{f * \Phi)(z)}{f * \Psi)(z)} \right. < q_2(z) \]

and

\[ q_1(z) < \frac{H_l^m[\alpha_1 + 1](f * \Phi)(z)}{H_l^m[\alpha_1](f * \Psi)(z)} < q_2(z) \]

where \( q_1, q_2 \) are given univalent functions in \( \mathcal{U} \) with \( q_1(0) = 1 \) and \( q_2(0) = 1 \).

The main object of the present paper is to find sufficient condition for certain normalized analytic functions \( f(z) \) in \( \mathcal{U} \) such that \( (f * \Psi)(z) \neq 0 \) and \( f \) to satisfy

\[ q_1(z) < \frac{H_l^m[\alpha_1](f * \Phi)(z)}{H_l^m[\alpha_1 + 1](f * \Psi)(z)} < q_2(z), \]

where \( q_1, q_2 \) are given univalent functions in \( \mathcal{U} \) and \( \Phi(z) = z + \sum_{n=2}^{\infty} \lambda_n z^n, \quad \Psi(z) = z + \sum_{n=2}^{\infty} \mu_n z^n \) are analytic functions in \( \mathcal{U} \) with \( \lambda_n \geq 0, \mu_n \geq 0 \) and \( \lambda_n \geq \mu_n \). Also, we obtain the number of known results as their special cases.
2. Subordination and Superordination results

For our present investigation, we shall need the following:

**Definition 2.1.** [12] Denote by \( Q \), the set of all functions \( f \) that are analytic and injective on \( U - E(f) \), where

\[
E(f) = \{ \zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty \}
\]

and are such that \( f'(\zeta) \neq 0 \) for \( \zeta \in \partial U - E(f) \).

**Lemma 2.2.** [11] Let \( q \) be univalent in the unit disk \( U \) and \( \theta \) and \( \phi \) be analytic in a domain \( D \) containing \( q(U) \) with \( \phi(w) \neq 0 \) when \( w \in q(U) \). Set

\[
\psi(z) := zq'(z)\phi(q(z)) \quad \text{and} \quad h(z) := \theta(q(z)) + \psi(z).
\]

Suppose that

1. \( \psi(z) \) is starlike univalent in \( U \) and
2. \( \Re \left\{ \frac{zh'(z)}{\psi(z)} \right\} > 0 \) for \( z \in U \).

If \( p \) is analytic with \( p(0) = q(0) \), \( p(U) \subseteq D \) and

\[
\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)),
\]

then

\[
p(z) \prec q(z)
\]

and \( q \) is the best dominant.

**Lemma 2.3.** [4] Let \( q \) be convex univalent in the unit disk \( U \) and \( \vartheta \) and \( \varphi \) be analytic in a domain \( D \) containing \( q(U) \). Suppose that

1. \( \Re \{ \vartheta'(q(z))/\varphi(q(z)) \} > 0 \) for \( z \in U \) and
2. \( \psi(z) = zq'(z)\varphi(q(z)) \) is starlike univalent in \( U \).

If \( p(z) \in \mathcal{H}[q(0),1] \cap Q \), with \( p(U) \subseteq D \), and \( \vartheta(p(z)) + zp'(z)\varphi(p(z)) \) is univalent in \( U \) and

\[
\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(p(z)) + zp'(z)\varphi(p(z)),
\]

then \( q(z) \prec p(z) \) and \( q \) is the best subordinant.

Using Lemma 2.2, we first prove the following theorem.
Theorem 2.4. Let \( \Phi, \Psi \in A, \gamma_4 \neq 0, \gamma_1, \gamma_2, \gamma_3 \) be the complex numbers and \( q \) be convex univalent in \( U \) with \( q(0) = 1 \). Further assume that

\[
Re \left\{ \frac{\gamma_3}{\gamma_4} + \frac{2\gamma_2}{\gamma_4} q(z) + \left( 1 + \frac{z q''(z)}{q'(z)} \right) \right\} > 0 \quad (z \in U). \tag{2.3}
\]

If \( f \in A \) satisfies

\[
\Upsilon(f, \Phi, \Psi, \gamma_1, \gamma_2, \gamma_3, \gamma_4) \prec \gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 z q'(z), \tag{2.4}
\]

where

\[
\Upsilon(f, \Phi, \Psi, \gamma_1, \gamma_2, \gamma_3, \gamma_4) := \begin{cases} \gamma_1 + \gamma_2 \left( \frac{H_m^{\alpha}[\alpha_1][f \ast \Phi](z)}{H_m^{\alpha}[\alpha_1 + 1][f \ast \Psi](z)} \right)^2 + \gamma_3 \frac{H_m^{\alpha}[\alpha_1][f \ast \Phi](z)}{H_m^{\alpha}[\alpha_1 + 1][f \ast \Psi](z)} \\ + \gamma_4 \left( \alpha_1 \frac{H_m^{\alpha}[\alpha_1 + 1][f \ast \Phi](z)}{H_m^{\alpha}[\alpha_1 + 1][f \ast \Psi](z)} - (\alpha_1 + 1) \frac{H_m^{\alpha}[\alpha_1 + 2][f \ast \Phi](z)}{H_m^{\alpha}[\alpha_1 + 1][f \ast \Psi](z)} + 1 \right) \end{cases} \tag{2.5}
\]

then

\[
\frac{H_m^{\alpha}[\alpha_1][f \ast \Phi](z)}{H_m^{\alpha}[\alpha_1 + 1][f \ast \Psi](z)} \prec q(z)
\]

and \( q \) is the best dominant.

Proof. Define the function \( p \) by

\[
p(z) := \frac{H_m^{\alpha}[\alpha_1][f \ast \Phi](z)}{H_m^{\alpha}[\alpha_1 + 1][f \ast \Psi](z)} \quad (z \in U). \tag{2.6}
\]

Then the function \( p \) is analytic in \( U \) and \( p(0) = 1 \). Therefore, by making use of (2.6), we obtain

\[
\gamma_1 + \gamma_2 \left( \frac{H_m^{\alpha}[\alpha_1][f \ast \Phi](z)}{H_m^{\alpha}[\alpha_1 + 1][f \ast \Psi](z)} \right)^2 + \gamma_3 \frac{H_m^{\alpha}[\alpha_1][f \ast \Phi](z)}{H_m^{\alpha}[\alpha_1 + 1][f \ast \Psi](z)}
+ \gamma_4 \left( \alpha_1 \frac{H_m^{\alpha}[\alpha_1 + 1][f \ast \Phi](z)}{H_m^{\alpha}[\alpha_1 + 1][f \ast \Psi](z)} - (\alpha_1 + 1) \frac{H_m^{\alpha}[\alpha_1 + 2][f \ast \Phi](z)}{H_m^{\alpha}[\alpha_1 + 1][f \ast \Psi](z)} + 1 \right) 
\times \frac{H_m^{\alpha}[\alpha_1 + 1][f \ast \Phi](z)}{H_m^{\alpha}[\alpha_1 + 1][f \ast \Psi](z)}
= \gamma_1 + \gamma_2 p^2(z) + \gamma_3 p(z) + \gamma_4 z p'(z). \tag{2.7}
\]

By using (2.7) in (2.4), we have

\[
\gamma_1 + \gamma_2 p^2(z) + \gamma_3 p(z) + \gamma_4 z p'(z) \prec \gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 z q'(z). \tag{2.8}
\]
By setting
\[ \theta(w) := \gamma_1 + \gamma_2 \omega^2(z) + \gamma_3 \omega \quad \text{and} \quad \phi(w) := \gamma_4, \]

it can be easily observed that \( \theta(w) \) and \( \phi(w) \) are analytic in \( \mathbb{C} - \{0\} \) and that \( \phi(w) \neq 0 \). Hence the result now follows by an application of Lemma 2.2.

When \( l = 2, m = 1, \alpha_1 = a, \alpha_2 = 1 \) and \( \beta_1 = c \) in Theorem 2.4, we state the following corollary.

**Corollary 2.5.** Let \( \Phi, \Psi \in \mathcal{A} \). Let \( \gamma_4 \neq 0, \gamma_1, \gamma_2, \gamma_3 \) be the complex numbers and \( q \) be convex univalent in \( \mathcal{U} \) with \( q(0) = 1 \) and (2.3) holds true. If \( f \in \mathcal{A} \) satisfies

\[
\Upsilon_1(f, \Phi, \Psi, \gamma_1, \gamma_2, \gamma_3, \gamma_4) < \gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 z q'(z)
\]

where

\[
\Upsilon_1(f, \Phi, \Psi, \gamma_1, \gamma_2, \gamma_3, \gamma_4) := \begin{cases} 
\gamma_1 + \gamma_2 \left( \frac{L(a,c)(f \ast \Phi)(z)}{L(a+1,c)(f \ast \Psi)(z)} \right)^2 + \gamma_3 \frac{L(a,c)(f \ast \Phi)(z)}{L(a+1,c)(f \ast \Psi)(z)} \\
+ \gamma_4 \left( \frac{L(a,c)(f \ast \Phi)(z)}{L(a+1,c)(f \ast \Psi)(z)} - (a + 1) \frac{L(a+2,c)(f \ast \Psi)(z)}{L(a+1,c)(f \ast \Psi)(z)} + 1 \right) \\
\quad \times \left( \frac{L(a,c)(f \ast \Phi)(z)}{L(a+1,c)(f \ast \Psi)(z)} \right)
\end{cases}
\]

(2.9)

then

\[
\frac{L(a,c)(f \ast \Phi)(z)}{L(a+1,c)(f \ast \Psi)(z)} < q(z)
\]

and \( q \) is the best dominant.

By fixing \( \Phi(z) = \frac{z}{1-z} \) and \( \Psi(z) = \frac{z}{1-z} \) in Theorem 2.4, we obtain the following corollary.

**Corollary 2.6.** Let \( \gamma_4 \neq 0, \gamma_1, \gamma_2, \gamma_3 \) be the complex numbers and \( q \) be convex univalent in \( \mathcal{U} \) with \( q(0) = 1 \) and (2.3) holds true. If \( f \in \mathcal{A} \) satisfies

\[
\begin{align*}
\gamma_1 + \gamma_2 & \left( \frac{H^l_m[\alpha_1] f(z)}{H^l_m[\alpha_1+1] f(z)} \right)^2 + \gamma_3 \frac{H^l_m[\alpha_1] f(z)}{H^l_m[\alpha_1+1] f(z)} \\
+ \gamma_4 \left( \frac{\alpha_1 H^l_m[\alpha_1+1] f(z)}{H^l_m[\alpha_1] f(z)} - (\alpha_1 + 1) \frac{H^l_m[\alpha_1+2] f(z)}{H^l_m[\alpha_1+1] f(z)} + 1 \right) \left( \frac{H^l_m[\alpha_1] f(z)}{H^l_m[\alpha_1+1] f(z)} \right)
\end{align*}
\]

\[< \gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 z q'(z),\]

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then
\[ \frac{H_m^l[0_1]f(z)}{H_m^l[0_1 + 1]f(z)} < q(z) \]
and \( q \) is the best dominant.

By taking \( l = 2, m = 1, 0_1 = 1, 0_2 = 1 \) and \( 0_1 = 1 \) in Theorem 2.4, we state the following corollary.

**Corollary 2.7.** Let \( \Phi, \Psi \in \mathcal{A} \). Let \( \gamma_4 \neq 0, \gamma_1, \gamma_2, \gamma_3 \) be the complex numbers and \( q \) be convex univalent in \( U \) with \( q(0) = 1 \) and (2.3) holds true. If \( f \in \mathcal{A} \) satisfies
\[
\gamma_1 + \gamma_2 \left( \frac{(f + \Phi)(z)}{z(f + \Phi)'(z)} \right)^2 + \frac{(f + \Phi)(z)}{z(f + \Phi)'(z)} \left[ (\gamma_3 - \gamma_4) + \gamma_4 z(f + \Phi)'(z) - \gamma_4 (f + \Phi)'(z) \right] < \gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 q'(z),
\]
then
\[ \frac{(f + \Phi)(z)}{z(f + \Psi)'(z)} < q(z) \]
and \( q \) is the best dominant.

By fixing \( \Phi(z) = \Psi(z) \) in Corollary 2.7, we obtain the following corollary.

**Corollary 2.8.** Let \( \Phi \in \mathcal{A} \). Let \( \gamma_4 \neq 0, \gamma_1, \gamma_2, \gamma_3 \) be the complex numbers and \( q \) be convex univalent in \( U \) with \( q(0) = 1 \) and (2.3) holds true. If \( f \in \mathcal{A} \) satisfies
\[
\gamma_1 + \gamma_2 \left( \frac{(f + \Phi)(z)}{z(f + \Phi)'(z)} \right)^2 + \frac{(f + \Phi)(z)}{z(f + \Phi)'(z)} \left[ (\gamma_3 - \gamma_4) + \gamma_4 z(f + \Phi)'(z) - \gamma_4 (f + \Phi)'(z) \right] < \gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 q'(z),
\]
then
\[ \frac{(f + \Phi)(z)}{z(f + \Phi)'(z)} < q(z) \]
and \( q \) is the best dominant.

By fixing \( \Phi(z) = \frac{z}{1 - z} \) in Corollary 2.8, we obtain the following corollary.

**Corollary 2.9.** Let \( \gamma_4 \neq 0, \gamma_1, \gamma_2, \gamma_3 \) be the complex numbers and \( q \) be convex univalent in \( U \) with \( q(0) = 1 \) and (2.3) holds true. If \( f \in \mathcal{A} \) satisfies
\[
\gamma_1 + \gamma_2 \left( \frac{f(z)}{zf'(z)} \right)^2 + \frac{f(z)}{zf'(z)} \left[ (\gamma_3 - \gamma_4) + \gamma_4 zf'(z) - \gamma_4 fz'(z) \right] < \gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 q'(z),
\]
then
\[ \frac{f(z)}{zf'(z)} < q(z) \]
and \( q \) is the best dominant.
then
\[ \frac{f(z)}{zf'(z)} \prec q(z) \]
and \( q \) is the best dominant.

**Remark 2.10.** For the choices of \( \gamma_1 = \gamma_2 = 0 \) and \( \gamma_3 = 1 \) in Corollary 2.9, we get the result obtained by Shanmugam et.al [19].

By taking \( q(z) = 1 + Az \), \( 1 + Bz \) \((-1 \leq B < A \leq 1) \) in Theorem 2.4, we have the following corollary.

**Corollary 2.11.** Assume that (2.3) holds. If \( f \in A \) and
\[ \Upsilon(f, \Phi, \Psi, \gamma_1, \gamma_2, \gamma_3, \gamma_4) \prec \gamma_1 + \gamma_2 \left( \frac{1 + Az}{1 + Bz} \right)^2 + \gamma_3 \frac{1 + Az}{1 + Bz} + \gamma_4 \frac{(A - B)z}{(1 + Bz)^2}, \]
then
\[ \frac{H_m^{(\alpha_1)}(f * \Phi)(z)}{H_m^{(\alpha_1 + 1)}(f * \Psi)(z)} \prec \frac{1 + Az}{1 + Bz} \]
and \( 1 + \frac{Az}{1 + Bz} \) is the best dominant.

Now, by applying Lemma 2.3, we prove the following theorem.

**Theorem 2.12.** Let \( \Phi, \Psi \in A \). Let \( \gamma_1, \gamma_2, \gamma_3 \) and \( \gamma_4 \neq 0 \) be the complex numbers. Let \( q \) be convex univalent in \( U \) with \( q(0) = 1 \). Assume that
\[ \text{Re} \left\{ \frac{\gamma_3}{\gamma_4} + \frac{2\gamma_2}{\gamma_4} q(z) \right\} \geq 0. \] (2.10)
Let \( f \in A \), \( \frac{H_m^{(\alpha_1)}(f * \Phi)(z)}{H_m^{(\alpha_1 + 1)}(f * \Psi)(z)} \in H[q(0), 1] \cap Q \). Let \( \Upsilon(f, \Phi, \Psi, \gamma_1, \gamma_2, \gamma_3, \gamma_4) \) be univalent in \( U \) and
\[ \gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 z q'(z) \prec \Upsilon(f, \Phi, \Psi, \gamma_1, \gamma_2, \gamma_3, \gamma_4), \] (2.11)
where \( \Upsilon(f, \Phi, \Psi, \gamma_1, \gamma_2, \gamma_3, \gamma_4) \) is given by (2.5), then
\[ q(z) \prec \frac{H_m^{(\alpha_1)}(f * \Phi)(z)}{H_m^{(\alpha_1 + 1)}(f * \Psi)(z)} \]
and \( q \) is the best subordinant.

**Proof.** Define the function \( p \) by
\[ p(z) := \frac{H_m^{(\alpha_1)}(f * \Phi)(z)}{H_m^{(\alpha_1 + 1)}(f * \Psi)(z)}, \] (2.12)
Simple computation from (2.12), we get,

\[ \Upsilon(f, \Phi, \Psi, \gamma_1, \gamma_2, \gamma_3, \gamma_4) = \gamma_1 + \gamma_2 p^2(z) + \gamma_3 p(z) + \gamma_4 z p'(z), \]

then

\[ \gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 z q'(z) < \gamma_1 + \gamma_2 p^2(z) + \gamma_3 p(z) + \gamma_4 z p'(z). \]

By setting \( \vartheta(w) = \gamma_1 + \gamma_2 w^2 + \gamma_3 w \) and \( \phi(w) = \gamma_4 \), it is easily observed that \( \vartheta(w) \) is analytic in \( \mathbb{C} \). Also, \( \phi(w) \) is analytic in \( \mathbb{C} - \{0\} \) and that \( \phi(w) \neq 0 \).

Since \( q(z) \) is convex univalent function, it follows that

\[ \Re \left\{ \frac{\vartheta'(q(z))}{\phi'(q(z))} \right\} = \Re \left\{ \frac{\gamma_3}{\gamma_4} + \frac{2\gamma_2}{\gamma_4} q(z) \right\} > 0, \quad z \in \mathcal{U}. \]

Now Theorem 2.12 follows by applying Lemma 2.3. \( \square \)

When \( l = 2, m = 1, \alpha_1 = a, \alpha_2 = 1 \) and \( \beta_1 = c \) in Theorem 2.12, we state the following corollary.

**Corollary 2.13.** Let \( \Phi, \Psi \in \mathcal{A} \). Let \( \gamma_1, \gamma_2, \gamma_3 \) and \( \gamma_4 \neq 0 \) be the complex numbers. Let \( q \) be convex univalent in \( \mathcal{U} \) with \( q(0) = 1 \) and (2.10) holds true. If \( f \in \mathcal{A} \)

\[ \frac{L(a, c)(f * \Phi)(z)}{L(a + 1, c)(f * \Psi)(z)} \in H[q(0), 1] \cap Q. \]

Let \( \Upsilon_1(f, \Phi, \Psi, \gamma_1, \gamma_2, \gamma_3, \gamma_4) \) be univalent in \( \mathcal{U} \) and

\[ \gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 z q'(z) \prec \Upsilon_1(f, \Phi, \Psi, \gamma_1, \gamma_2, \gamma_3, \gamma_4), \]

where \( \Upsilon_1(f, \Phi, \Psi, \gamma_1, \gamma_2, \gamma_3, \gamma_4) \) is given by (2.9), then

\[ q(z) \prec \frac{L(a, c)(f * \Phi)(z)}{L(a + 1, c)(f * \Psi)(z)} \]

and \( q \) is the best subordinant.

When \( l = 2, m = 1, \alpha_1 = 1, \alpha_2 = 1 \) and \( \beta_1 = 1 \) in Theorem 2.12, we derive the following corollary.

**Corollary 2.14.** Let \( \Phi, \Psi \in \mathcal{A} \). Let \( \gamma_1, \gamma_2, \gamma_3 \) and \( \gamma_4 \neq 0 \) be the complex numbers. Let \( q \) be convex univalent in \( \mathcal{U} \) with \( q(0) = 1 \) and (2.10) holds true. If \( f \in \mathcal{A} \),

\[ \frac{L(a, c)(f * \Phi)(z)}{L(a + 1, c)(f * \Psi)(z)} \in H[q(0), 1] \cap Q. \]

Let

\[ \gamma_1 + \gamma_2 \left( \frac{(f * \Phi)(z)}{z(f * \Psi)'(z)} \right)^2 + \frac{(f * \Phi)(z)}{z(f * \Psi)'(z)} \left[ (\gamma_3 - \gamma_4) + \gamma_4 \frac{z(f * \Phi)'(z)}{(f * \Phi)(z)} - \gamma_4 \frac{z(f * \Psi)''(z)}{(f * \Psi)'(z)} \right] \]

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be univalent in $\mathcal{U}$ and

$$
\gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 q'(z)
$$

$$
\prec \gamma_1 + \gamma_2 \left( \frac{(f * \Phi)(z)}{z(f * \Psi)'(z)} \right)^2 + \frac{(f * \Phi)(z)}{z(f * \Psi)'(z)} \left[ (\gamma_3 - \gamma_4) + \gamma_4 \frac{z(f * \Phi)'(z)}{(f * \Phi)(z)} - \gamma_4 \frac{z(f * \Psi)'(z)}{(f * \Psi)'(z)} \right],
$$

then

$$
q(z) \prec \frac{(f * \Phi)(z)}{z(f * \Psi)'(z)}
$$

and $q$ is the best subordinant.

By fixing $\Phi(z) = \Psi(z)$ in Corollary 2.14, we obtain the following corollary.

**Corollary 2.15.** Let $\Phi \in A$. Let $\gamma_1, \gamma_2, \gamma_3$ and $\gamma_4 \neq 0$ be the complex numbers. Let $q$ be convex univalent in $\mathcal{U}$ with $q(0) = 1$ and (2.10) holds true. If $f \in A$, then

$$
\frac{(f * \Phi)(z)}{z(f * \Psi)'(z)} \in H[q(0), 1] \cap Q.
$$

Let

$$
\gamma_1 + \gamma_2 \left( \frac{(f * \Phi)(z)}{z(f * \Psi)'(z)} \right)^2 + \frac{(f * \Phi)(z)}{z(f * \Psi)'(z)} \left[ (\gamma_3 - \gamma_4) + \gamma_4 \frac{z(f * \Phi)'(z)}{(f * \Phi)(z)} - \gamma_4 \frac{z(f * \Psi)'(z)}{(f * \Psi)'(z)} \right],
$$

be univalent in $\mathcal{U}$ and

$$
\gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 q'(z)
$$

$$
\prec \gamma_1 + \gamma_2 \left( \frac{(f * \Phi)(z)}{z(f * \Psi)'(z)} \right)^2 + \frac{(f * \Phi)(z)}{z(f * \Psi)'(z)} \left[ (\gamma_3 - \gamma_4) + \gamma_4 \frac{z(f * \Phi)'(z)}{(f * \Phi)(z)} - \gamma_4 \frac{z(f * \Psi)'(z)}{(f * \Psi)'(z)} \right],
$$

then

$$
q(z) \prec \frac{(f * \Phi)(z)}{z(f * \Psi)'(z)}
$$

and $q$ is the best subordinant.

By fixing $\Phi(z) = \frac{1}{z}$ in Corollary 2.15, we obtain the following corollary.

**Corollary 2.16.** Let $\gamma_1, \gamma_2, \gamma_3$ and $\gamma_4 \neq 0$ be the complex numbers. Let $q$ be convex univalent in $\mathcal{U}$ with $q(0) = 1$ and (2.10) holds true. If $f \in A$, then $\frac{f(z)}{zf'(z)} \in H[q(0), 1] \cap Q$. Let

$$
\gamma_1 + \gamma_2 \left( \frac{f(z)}{zf'(z)} \right)^2 + \frac{f(z)}{zf'(z)} \left[ (\gamma_3 - \gamma_4) + \gamma_4 \frac{zf'(z)}{f(z)} - \gamma_4 \frac{zf'''(z)}{f'(z)} \right]
$$

be univalent in $\mathcal{U}$ and

$$
\gamma_1 + \gamma_2 \left( \frac{f(z)}{zf'(z)} \right)^2 + \frac{f(z)}{zf'(z)} \left[ (\gamma_3 - \gamma_4) + \gamma_4 \frac{zf'(z)}{f(z)} - \gamma_4 \frac{zf'''(z)}{f'(z)} \right],
$$

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then
\[ q(z) < \frac{f(z)}{zf'(z)} \]
and \( q \) is the best subordinant.

By taking \( q(z) = (1 + Az)/(1 + Bz) \) \((-1 \leq B < A \leq 1)\) in Theorem 2.12, we obtain the following corollary.

**Corollary 2.17.** Assume that (2.10) holds true. If \( f \in A \), \( H_l[\alpha_1](f \ast \Phi)(z) \in H[l(0), 1] \cap Q \). Let \( \Upsilon(f, \Phi, \Psi, \gamma_1, \gamma_2, \gamma_3, \gamma_4) \) be univalent in \( U \) and
\[
\gamma_1 + \gamma_2(1 + Az)^2 + \gamma_3(1 + Bz) + \gamma_4 \frac{(A - B)z}{(1 + Bz)^2} < \Upsilon(f, \Phi, \Psi, \gamma_1, \gamma_2, \gamma_3, \gamma_4),
\]
then
\[
\frac{1 + Az}{1 + Bz} \prec \frac{H_l[\alpha_1](f \ast \Phi)(z)}{H_l[\alpha_1 + 1](f \ast \Psi)(z)}
\]
and \( \frac{1 + Az}{1 + Bz} \) is the best subordinant.

### 3. Sandwich results

We conclude this paper by stating the following sandwich results.

**Theorem 3.1.** Let \( q_1 \) and \( q_2 \) be convex univalent in \( U \), \( \gamma_1, \gamma_2, \gamma_3 \) and \( \gamma_4 \neq 0 \) be the complex numbers. Suppose \( q_2 \) satisfies (2.3) and \( q_1 \) satisfies (2.10). Let \( \Phi, \Psi \in A \). Moreover suppose \( \frac{H_l[\alpha_1](f \ast \Phi)(z)}{H_l[\alpha_1 + 1](f \ast \Psi)(z)} \in \mathcal{H}[1, 1] \cap Q \) and \( \Upsilon(f, \Phi, \Psi, \gamma_1, \gamma_2, \gamma_3, \gamma_4) \) is univalent in \( U \). If \( f \in A \) satisfies
\[
\gamma_1 + \gamma_2q_1^2(z) + \gamma_3q_1(z) + \gamma_4zq_1'(z) < \Upsilon(f, \Phi, \Psi, \gamma_1, \gamma_2, \gamma_3, \gamma_4)
\]
\[
< \gamma_1 + \gamma_2q_2^2(z) + \gamma_3q_2(z) + \gamma_4zq_2'(z),
\]
where \( \Upsilon(f, \Phi, \Psi, \gamma_1, \gamma_2, \gamma_3, \gamma_4) \) is given by (2.5), then
\[
q_1(z) \prec \frac{H_l[\alpha_1](f \ast \Phi)(z)}{H_l[\alpha_1 + 1](f \ast \Psi)(z)} \prec q_2(z)
\]
and \( q_1, q_2 \) are respectively the best subordinant and best dominant.
By taking
\[ q_1(z) = \frac{1 + A_1 z}{1 + B_1 z} \quad (-1 \leq B_1 < A_1 \leq 1) \]
and
\[ q_2(z) = \frac{1 + A_2 z}{1 + B_2 z} \quad (-1 \leq B_2 < A_2 \leq 1) \]
in Theorem 3.1 we obtain the following result.

**Corollary 3.2.** Let \( \Phi, \Psi \in A \). If \( f \in A \),
\[ \frac{(f * \Phi)(z)}{z(f * \Psi)'(z)} \in \mathcal{H}[1, 1] \cap Q \]
and \( \Upsilon(f, \Phi, \Psi, \gamma_1, \gamma_2, \gamma_3, \gamma_4) \) is univalent in \( U \). Further
\[ \gamma_1 + \gamma_2 \left( \frac{1 + A_1 z}{1 + B_1 z} \right)^2 + \frac{1 + A_1 z}{1 + B_1 z} + \gamma_4 \frac{(A_1 - B_1)z}{(1 + B_1 z)^2} \]
\[ \prec \Upsilon(f, \Phi, \Psi, \gamma_1, \gamma_2, \gamma_3, \gamma_4) \]
\[ \prec \gamma_1 + \gamma_2 \left( \frac{1 + A_2 z}{1 + B_2 z} \right)^2 + \frac{1 + A_2 z}{1 + B_2 z} + \gamma_4 \frac{(A_2 - B_2)z}{(1 + B_2 z)^2} \]
where \( \Upsilon(f, \Phi, \Psi, \gamma_1, \gamma_2, \gamma_3, \gamma_4) \) is given by (2.5), then
\[ \frac{1 + A_1 z}{1 + B_1 z} \prec \frac{(f * \Phi)(z)}{z(f * \Psi)'(z)} \prec \frac{1 + A_2 z}{1 + B_2 z} \]
and \( \frac{1 + A_1 z}{1 + B_1 z}, \frac{1 + A_2 z}{1 + B_2 z} \) are respectively the best subordinant and best dominant.

We remark that Theorem 3.1 can easily restated, for the different choices of \( \Phi(z), \Psi(z), l, m, \alpha_1, \alpha_2, \ldots, \alpha_l, \beta_1, \beta_2, \ldots, \beta_m \) and for \( \gamma_1, \gamma_2, \gamma_3, \gamma_4 \).

**References**


DIFFERENTIAL SUBORDINATIONS AND DIFFERENTIAL SUPERORDINATIONS


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