ON RANDOM FIXED POINTS
IN RANDOM CONVEX STRUCTURES

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Abstract. In this paper, we present some random fixed point theorems in random convex structures.

1. Introduction and preliminaries

Random fixed point theory has received much attention for the last two decades, since the publication of the paper by Bharucha-Reid [2]. Also random best approximation attracted authors after the papers by Sehgal and Singh [15], Papageorgiou [13], Lin [11], and Beg et al. [1].

On the other hand, in the past years, because of practical necessities, the attempts of generalizing the notion of convexity introduced by J. Von Neumann and O. Morgenstern [12], M. Stone [16] were brought up-to-date by S.P. Gudder [5]. Consequently, Gudder (1979) introduced the notion of convex structure and of F-convex set with applications in quantum mechanics, colour vision and petroleum engineering. Subsequently, fixed point theorems for nonexpansive mappings using the convex structures introduced by Gudder was proved by Petrusel [14] and later by Ganguly and Jadhav [6] for approximation theorems.

Again, away from this, Takahashi [17] also introduced a notion of convexity in metric spaces and presented fixed point theorems for nonexpansive mappings. This motivated Guay et al. [7] to discuss the results on convex metric spaces. These
works along with those on random approximations motivated Beg et al. [3] to present random fixed point theorems and related results in random convex metric spaces.

It is a need for further research to study a relationship between convex structures introduced by Takahashi [17] and Gudder [5] respectively. In this vein, we are presenting random fixed point theorems in random convex structures, following Gudder [5], Petrusel [14], Beg and Shahzad [3].

Before we present our theorems, we will introduce some basic preliminaries.

Let \((\Omega, \Sigma)\) be a measurable space, \((X, d)\) a metric space, \(2^X\) the family of all subsets of \(X\), \(K(X)\) family of all nonempty compact subsets of \(X\) and \(CB(X)\) family of all nonempty closed bounded subsets of \(X\).

A mapping \(T : \Omega \rightarrow 2^X\) is called measurable if for any open subset \(C\) of \(X\),

\[ T^{-1}(C) = \{ \omega \in \Omega : T(\omega) \cap C \neq \emptyset \} \in \Sigma. \]

A mapping \(\xi : \Omega \rightarrow X\) is said to be a measurable selector of \(T\) if \(\xi\) is measurable and for any \(\omega \in \Omega\), \(\xi(\omega) \in T(\omega)\).

A mapping \(f : \Omega \times X \rightarrow X\) is called a random operator if for any \(x \in X\), \(f(., x)\) is measurable.

A measurable mapping \(\xi : \Omega \rightarrow X\) is called a random fixed point of a random multivalued (single valued) operator \(T : \Omega \times X \rightarrow CB(X)(f : \Omega \times X \rightarrow X)\) if for every \(\omega \in \Omega\), \(\xi(\omega) \in T(\omega, \xi(\omega))\) \((\xi(\omega) = f(\omega, \xi(\omega)))\).

A random operator \(T : \Omega \times X \rightarrow CB(X)\) is called Lipschitzian if

\[ H(T(\omega, x), T(\omega, y)) \leq L(\omega) \, d(x, y) \]

for any \(x, y \in X\) and \(\omega \in \Omega\), where \(L : \Omega \rightarrow [0, \infty)\) is a measurable map and \(H\) is the Pompeiu-Hausdorff metric on \(CB(X)\), induced by the metric \(d\). When \(L(\omega) < 1\), \((L(\omega) = 1)\) for each \(\omega \in \Omega\), \(T\) is called contraction (nonexpansive).

We present, for the convenience of readers, the following definitions which also appear in Petrusel [14].

**Definition 1.1.** Let \(X\) be a set and \(F : [0,1] \times X \times X \rightarrow X\) a mapping. Then the pair \((X, F)\) forms a convex prestructure.
Definition 1.2. Let \((X, F)\) be a convex prestructure. If \(F\) satisfies the following conditions:

1. \(F(\lambda, x, F(\mu, y, z)) = F(\lambda + (1 - \lambda)\mu, F(\lambda(\lambda + (1 - \lambda)\mu) - 1, x, y), z)\) for every \(\lambda, \mu \in [0, 1]\) with \(\lambda + (1 - \lambda)\mu \neq 0\) and \(x, y, z \in X\).
2. \(F(\lambda, x, x) = x\) for any \(x \in X\) and \(\lambda \in [0, 1]\), then \((X, F)\) forms a semi-convex structure.

If \((X, F)\) is a semi-convex structure, then \(F(1, x, y) = x\) for any \(x, y \in X\).

Definition 1.3. A semi-convex structure \((X, F)\) is said to form a convex structure if \(F\) also satisfies the conditions:

1. \(F(\lambda, x, y) = F(1 - \lambda, y, x)\) for every \(\lambda \in [0, 1]\), \(x, y \in X\).
2. If \(F(\lambda, x, y) = F(\lambda, x, z)\) for some \(\lambda \neq 0\), \(x \in X\), then \(y = z\).

Definition 1.4. Let \((X, F)\) be a semi-convex structure. A subset \(Y\) of \(X\) is called \(F\) - semi-starshaped if there exists a \(p \in Y\), so that for any \(x \in Y\) and \(\lambda \in [0, 1]\), \(F(\lambda, x, p) \in Y\).

Definition 1.5. Let \((X, F)\) be a convex structure. A subset \(Y\) of \(X\) is called:

1. \(F\) - starshaped if there exists a \(p \in Y\), so that for any \(x \in Y\) and \(\lambda \in [0, 1]\), \(F(\lambda, x, p) \in Y\).
2. \(F\) - convex if for any \(u, v \in Y\) and \(\lambda \in [0, 1]\), we have \(F(\lambda, u, v) \in Y\).

For \(F(\lambda, u, v) = \lambda u + (1 - \lambda)v\), we obtain the known notions of starshaped and convexity from linear spaces.

Petrușel [14] noted with an example that a set can be a \(F\) - semi convex structure without being a convex structure. So, it follows that the results on fixed point theory and on best approximation theory obtained for semi-convex and semi-starshaped structures will be more general than those on \(F\) - convex structure.

Definition 1.6 (Random Semi-Convex Structure). Let \(F : \Omega \times X \times X \times [0, 1] \rightarrow X\) be a mapping having the following properties:

1. For each \(\omega \in \Omega\), \(F(\omega, \ldots)\) is a semi-convex structure on \(X\),
2. For each $x, y \in X$, $\lambda \in [0, 1]$, $F(. , x , y , \lambda)$ is measurable.

The mapping $F$ is called a random semi-convex structure on $X$.

**Example 1** [5]. The mapping $F : [0, 1] \times R^*_+ \times R^*_+ \rightarrow R^*_+$ given by

$$F(\lambda , u , v) = \lambda u^1 v^{1-\lambda}$$

together with the set of strict positive real numbers form a convex structure.

**Example 2** [14]. The mapping $F : [0, 1] \times R \times R \rightarrow R$ given by

$$F(\lambda , u , v) = [\lambda u^{2k} + (1 - \lambda)v^{2k}]^{1/2k}, k \in N^*$$

together with the set of real numbers form a semi-convex structure without being a convex structure.

2. **Main results**

**Theorem 2.1.** Let $X$ be a separable random Banach space with semi-convex structure $F$, where the mapping $F : \Omega \times X \times X \times [0, 1] \rightarrow X$ satisfies the following conditions:

1. $F$ is $\phi$ - contractive relative to the second argument, i.e., there exists a mapping $\phi : [0, 1] \rightarrow [0, 1]$ so that:

$$||F(\omega, x, p, \lambda) - F(\omega, y, p, \lambda)|| \leq \phi(\lambda)||x - y||,$$

for any $x, y, p \in X$ and $\lambda \in [0, 1]$ and $\omega \in \Omega$.

2. $F$ is continuous relative to the first argument.

Let $Y$ be a compact and $F$ - semi-starshaped subset of $X$ and the mapping $T : \Omega \times Y \rightarrow Y$ be nonexpansive random operator. Then $T$ has a random fixed point.

**Proof.** Choose $p \in Y$ so that for any $u \in Y$ and $\lambda \in [0, 1]$, we have $F(\omega, u, p, \lambda) \in Y$ for each $\omega \in \Omega$. Let $\{K_n\}$ be a sequence of measurable mappings $K_n : \Omega \rightarrow (0, 1)$ and $K_n(\omega) \rightarrow 1$ as $n \rightarrow \infty$.

Define the random operator $T_n : \Omega \times Y \rightarrow Y$ by

$$T_n(\omega, x) = F(\omega, T(\omega, x), p, K_n(\omega))$$
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$T_n$ is, because of $F$ - semi-starshaped of $Y$, well defined. The operator $T_n$ is a contraction. Indeed

$$||T_n(\omega, x_1) - T_n(\omega, x_2)|| = ||F(\omega, T(\omega, x_1), p, K_n(\omega)) - F(\omega, T(\omega, x_2), p, K_n(\omega))|| \leq \phi(K_n(\omega)) ||T(\omega, x_1) - T(\omega, x_2)||$$

for all $x, y \in Y$ and $\omega \in \Omega$. By Hans [8], $T_n$ has a unique random fixed point $\xi_n$.

For each $n$, define $G_n : \Omega \to K(X)$ by $G_n(\omega) = CL\{\xi_i(\omega) : i \geq n\}$.

Define $G : \Omega \to K(X)$ by $G(\omega) = \bigcap_{n=1}^{\infty} G_n(\omega)$. Since $G$ is measurable (see Himmelberg [9], Theorem 4.1), by Kuratowski and Ryll-Nardzewski theorem in [10] we have that $G$ has a measurable selector $\xi$. Because $Y$ is compact, $\{\xi_n(\omega)\}$ has a subsequence $\{\xi_n, (\omega)\}$ converging to $\xi(\omega)$. By the continuity of $T$ and $F$, $T(\omega, \xi_n(\omega))$ converges to $T(\omega, \xi(\omega))$. Thus, $T(\omega, \xi_n(\omega)) = \xi(\omega)$ for each $\omega \in \Omega$.

Next we have the following:

**Theorem 2.2.** Let $X$ be a separable random Banach space with a semi-convex structure $F$, where the mapping $F : \Omega \times X \times X \times [0, 1] \to X$ satisfies the conditions:

1. $F$ is $\phi$ - contractive relative to the second argument.
2. $F$ is continuous relative to the first argument.

Let $Y$ be a weakly compact and $F$ - semi-starshaped subset of $X$ and the mapping $T : \Omega \times Y \to Y$ be nonexpansive and weakly continuous mapping. In these conditions the mapping $T$ has a random fixed point.

**Proof.** As in Theorem 2.1, define $\{K_n\}$ and the random operator $T_n$. As before, each $T_n$ is a contraction mapping on $Y$. Since the weak topology of $X$ is Hausdorff and $Y$ is weakly compact, we have that $Y$ is weakly closed and therefore, strongly closed (See Dotson, Theorem 2 [4]). Hence $Y$ is a complete metric space (with the norm topology of the Banach space $X$). By Hans [8], $T_n$ has a unique random fixed point $\xi_n \in Y$.

By the Eberlein-Smulian [4] theorem, $Y$ is weakly sequentially compact. Thus there is a subsequence $\{\xi_n(\omega)\}$ such that $\xi_n(\omega) \weak Y \xi(\omega) \in Y$ (denotes weak convergence).

Since $T$ is weakly continuous and $F$ - continuous, we have

$$T(\omega, \xi_n(\omega)) \weak Y T(\omega, \xi(\omega))$$
Thus, $T(\omega, \xi(\omega)) = \xi(\omega)$, for each $\omega \in \Omega$.

**Acknowledgements.** We are grateful to Professor Adrian Petruşel, Department of Applied Mathematics, Babes-Bolyai University Cluj-Napoca, Romania, for his kind help and valuable advice in the preparation of this paper. We also thank Dr. M.S. Rathore, Department of Mathematics, Govt. P.G. College, Sehore (M.P.), India, for his valuable encouragement during the preparation of paper.

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