A-SUMMABILITY AND APPROXIMATION OF CONTINUOUS PERIODIC FUNCTIONS

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Dedicated to Professor D. D. Stancu on his 80th birthday

Abstract. The aim of this paper is to present a generalization of the classical Korovkin approximation theorem by using a matrix summability method, for sequences of positive linear operators defined on the space of all real-valued continuous and \(2\pi\)-periodic functions. This approach is motivated by the works of O. Duman [4] and C. Orhan, Ö.G. Atlıhan [1].

1. Introduction

One of the most recently studied subject in approximation theory is the approximation of continuous function by linear positive operators using \(A\)-statistical convergence or a matrix summability method ([1], [3], [5], [7]).

In this paper, following [1], we will give a Korovkin type approximation theorem for a sequence of positive linear operators defined on the space of all real-valued continuous and \(2\pi\)-periodic functions via \(A\)-summability. Particular cases are also punctuated.

First of all, we recall some notation and definitions used in this paper.

Let \(A := (A_n)_{n \geq 1}\), \(A^n = (a_{kj})_{k,j \in \mathbb{N}}\) be a sequence of infinite non-negative real matrices.

For a sequence of real numbers, \(x = (x_j)_{j \in \mathbb{N}}\), the double sequence

\[\mathcal{A}x := \{(Ax)^n_k : k, n \in \mathbb{N}\}\]
defined by $(Ax)_k^n := \sum_{j=1}^\infty a_{kj}^n x_j$ is called the $A$-transform of $x$ whenever the series converges for all $k$ and $n$. A sequence $x$ is said to be $A$-summable to a real number $L$ if $Ax$ converges to $L$ as $k$ tends to infinity uniformly in $n$ (see [2]).

We denote by $C_{2\pi}(\mathbb{R})$ the space of all $2\pi$-periodic and continuous functions on $\mathbb{R}$. Endowed with the norm $\| \cdot \|_{2\pi}$ this space is a Banach space, where

$$\|f\|_{2\pi} := \sup_{t \in \mathbb{R}} |f(t)|, \quad f \in C_{2\pi}(\mathbb{R}).$$

We also have to recall the classical Bohman-Korovkin theorem.

**Theorem A.** If $\{L_j\}$ is a sequence of positive linear operators acting from $C_{2\pi}(\mathbb{R})$ into $C_{2\pi}(\mathbb{R})$ such that

$$\lim_{j \to \infty} \|L_jf_i - f_i\|_{2\pi} = 0 \quad (i = 1, 2, 3),$$

where $f_1(t) = 1$, $f_2(t) = \cos t$, $f_3(t) = \sin t$ for all $t \in \mathbb{R}$, then, for all $f \in C_{2\pi}(\mathbb{R})$ we have

$$\lim_{j \to \infty} \|L_jf - f\|_{2\pi} = 0.$$

Recently, the statistical analog of Theorem A has been studied by O. Duman [4]. It will be read as follows.

**Theorem B.** Let $A = (a_{kj})$ be a non-negative regular summability matrix, and let $\{L_j\}$ be a sequence of positive linear operators mapping $C_{2\pi}(\mathbb{R})$ into $C_{2\pi}(\mathbb{R})$. Then, for all $f \in C_{2\pi}(\mathbb{R})$,

$$st_A - \lim_{j \to \infty} \|L_jf - f\|_{2\pi} = 0$$

if and only if

$$st_A - \lim_{j \to \infty} \|L_jf_i - f_i\|_{2\pi} = 0 \quad (i = 1, 2, 3),$$

where $f_1(t) = 1$, $f_2(t) = \cos t$, $f_3(t) = \sin t$ for all $t \in \mathbb{R}$.
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2. A Korovkin type theorem

Theorem 2.1. Let $A = (A^n)_{n \geq 1}$ be a sequence of infinite non-negative real matrices such that
\[
\sup_{n,k} \sum_{j=1}^{\infty} a^n_{kj} < \infty \tag{2.1}
\]
and let $\{L_j\}$ be a sequence of positive linear operators mapping $C_{2\pi}(\mathbb{R})$ into $C_{2\pi}(\mathbb{R})$.

Then, for all $f \in C_{2\pi}(\mathbb{R})$ we have
\[
\lim_{k \to \infty} \sum_{j=1}^{\infty} a^n_{kj} \|L_j f - f\|_{2\pi} = 0, \tag{2.2}
\]
uniformly in $n$ if and only if
\[
\lim_{k \to \infty} \sum_{j=1}^{\infty} a^n_{kj} \|L_j f_i - f_i\|_{2\pi} = 0 \quad (i = 1, 2, 3), \tag{2.3}
\]
uniformly in $n$, where $f_1(t) = 1$, $f_2(t) = \cos t$, $f_3(t) = \sin t$ for all $t \in \mathbb{R}$.

Proof. Since $f_i (i = 1, 2, 3)$ belong to $C_{2\pi}(\mathbb{R})$, the implication (2.2) $\Rightarrow$ (2.3) is obvious.

Now, assume that (2.3) holds. Let $f \in C_{2\pi}(\mathbb{R})$ and let $I$ be a closed subinterval of length $2\pi$ of $\mathbb{R}$. Fix $x \in I$. By the continuity of $f$ at $x$, it follows that for any $\varepsilon > 0$ there exists a number $\delta > 0$ such that
\[
|f(t) - f(x)| < \varepsilon \text{ for all } t \text{ satisfying } |t - x| < \delta. \tag{2.4}
\]

By the boundedness of $f$ follows
\[
|f(t) - f(x)| \leq 2\|f\|_{2\pi} \text{ for all } t \in \mathbb{R}. \tag{2.5}
\]

Further on, we consider the subinterval $(x - \delta, 2\pi + x - \delta]$ of length $2\pi$. We show that
\[
|f(t) - f(x)| < \varepsilon + 2\|f\|_{2\pi} \psi(t) \text{ holds for all } t \in (x - \delta, 2\pi + x - \delta], \tag{2.6}
\]
where $\psi(t) := \sin^2 \left(\frac{t - x}{2}\right)$.

To prove (2.6) we examine two cases.
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Case 1. Let \( t \in (x-\delta, x+\delta) \). In this case we get \( |t-x| < \delta \) and the relation (2.6) follows by (2.4).

Case 2. Let \( t \in [x+\delta, 2\pi + x-\delta] \). In this case we have \( \delta \leq t-x \leq 2\pi - \delta \) and \( \delta \in (0, \pi] \). We get

\[
\sin^2 \frac{\delta}{2} \leq \sin^2 \left( \frac{t-x}{2} \right) \leq \sin^2 \left( \frac{\pi - \delta}{2} \right),
\]

for all \( \delta \in (0, \pi] \) and \( t \in [x+\delta, 2\pi + x-\delta] \).

Then, from (2.5) and (2.7) we obtain

\[
|f(t) - f(x)| \leq \frac{2\|f\|_{2\pi} \psi(t)}{\sin^2 \frac{\delta}{2}}
\]

for all \( t \in [x+\delta, 2\pi + x-\delta] \).

Since the function \( f \in C_{2\pi}(\mathbb{R}) \) is \( 2\pi \)-periodic, the inequality (2.6) holds for all \( t \in \mathbb{R} \).

Now, applying the operator \( L_j \), we get

\[
|L_j(f; x) - f(x)| \leq L_j([f-f(x)]; x) + |f(x)||L_j(f_1; x) - f_1(x)|
\]

\[
< L_j \left( \varepsilon + \frac{2\|f\|_{2\pi} \psi; x}{\sin^2 \frac{\delta}{2}} \right) + \|f\|_{2\pi}|L_j(f_1; x) - f_1(x)|
\]

\[
= \varepsilon L_j(f_1; x) + \frac{2\|f\|_{2\pi} L_j(\psi; x) + \|f\|_{2\pi}|L_j(f_1; x) - f_1(x)|}{\sin^2 \frac{\delta}{2}}
\]

\[
\leq \varepsilon + (\varepsilon + \|f\|_{2\pi})|L_j(f_1; x) - f_1(x)| + \frac{2\|f\|_{2\pi} L_j(\psi; x)}{\sin^2 \frac{\delta}{2}}.
\]

Since

\[
L_j(\psi; x) \leq \frac{1}{2} \{ |L_j(f_1; x) - f_1(x)| + |\cos x||L_j(f_2; x) - f_2(x)| + |\sin x||L_j(f_3; x) - f_3(x)| \},
\]

(2.8)
(see [8], Theorem 4) we obtain

\[
|L_j(f; x) - f(x)| < \varepsilon + \left( \frac{\varepsilon + \|f\|_{2\pi}}{\sin^2 \frac{\delta}{2}} \right) \left\{ |L_j(f_1; x) - f_1(x)| + |L_j(f_2; x) - f_2(x)| + |L_j(f_3; x) - f_3(x)| \right\}
\]

\[
\leq \varepsilon + K \{ \|L_j f_1 - f_1\|_{2\pi} + \|L_j f_2 - f_2\|_{2\pi} + \|L_j f_3 - f_3\|_{2\pi} \},
\]

where

\[
K := \varepsilon + \frac{\|f\|_{2\pi}}{\sin^2 \frac{\delta}{2}}.
\]

Taking supremum over \(x\), for all \(j \in \mathbb{N}\) we obtain

\[
\|L_j f - f\|_{2\pi} \leq \varepsilon + K \{ \|L_j f_1 - f_1\|_{2\pi} + \|L_j f_2 - f_2\|_{2\pi} + \|L_j f_3 - f_3\|_{2\pi} \}.
\]

Consequently, we get

\[
\sum_{j=1}^{\infty} a_{kj}^n \|L_j f - f\|_{2\pi} \leq \varepsilon \sum_{j=1}^{\infty} a_{kj}^n + K \sum_{j=1}^{\infty} a_{kj}^n \|L_j f_1 - f_1\|_{2\pi}
\]

\[
+ K \sum_{j=1}^{\infty} a_{kj}^n \|L_j f_2 - f_2\|_{2\pi} + K \sum_{j=1}^{\infty} a_{kj}^n \|L_j f_3 - f_3\|_{2\pi}.
\]

By taking limit as \(k \to \infty\) and by using (2.1), (2.3) we obtain the desired result. \(\square\)

Using the concept of \(A\)-statistical convergence, O. Duman and E. Erkuş [6] obtained a Korovkin type approximation theorem by positive linear operators defined on \(C_{2\pi}(\mathbb{R}^m)\), the space of all real-valued continuous and \(2\pi\)-periodic functions on \(\mathbb{R}^m\) \((m \in \mathbb{N})\) endowed with the norm \(\| \cdot \|_{2\pi}\) of the uniform convergence. The same result stands for \(A\)-summability.

**Theorem 2.2.** Let \(A = (A_n)_{n \geq 1}\) be a sequence of infinite non-negative real matrices such that

\[
\sup_{n,k} \sum_{j=1}^{\infty} a_{kj}^n < \infty
\]

and let \(\{L_j\}\) be a sequence of positive linear operators mapping \(C_{2\pi}(\mathbb{R}^m)\) into \(C_{2\pi}(\mathbb{R}^m)\).

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Then, for all $f \in C_{2\pi}(\mathbb{R}^m)$ we have

$$\lim_{k \to \infty} \sum_{j=1}^{\infty} a^{n}_{kj} \|L_j f - f\|_{2\pi} = 0,$$

uniformly in $n$, if and only if

$$\lim_{k \to \infty} \sum_{j=1}^{\infty} a^{n}_{kj} \|L_j f_p - f_p\|_{2\pi} = 0 \quad (p = 1, 2, \ldots, (2m + 1)),$$

uniformly in $n$, where $f_1(t_1, t_2, \ldots, t_m) = 1$, $f_p(t_1, t_2, \ldots, t_m) = \cos t_{p-1}$ ($p = 2, 3, \ldots, m + 1$), $f_q(t_1, t_2, \ldots, t_m) = \sin t_{q-m-1}$ ($q = m + 2, \ldots, 2m + 1$).

3. Particular cases

Taking $A^n = I$, $I$ being the identity matrix, Theorem 2.1 reduces to Theorem A.

If $A^n = A$, for some matrix $A$, then $A$-summability is the ordinary matrix summability by $A$.

Note that statistical convergence is a regular summability method. Considering Theorem B and our Theorem 2.1 we obtain the next result.

**Corollary 3.1.** Let $A = (A^n)_{n \in \mathbb{N}}$ be a sequence of non-negative regular summability matrices and let $\{L_j\}$ be a sequence of positive linear operators mapping $C_{2\pi}(\mathbb{R})$ into $C_{2\pi}(\mathbb{R})$.

Then, for all $f \in C_{2\pi}(\mathbb{R})$ we have

$$\text{st}_{A_n} \lim_{j \to \infty} \|L_j f - f\|_{2\pi} = 0, \text{ uniformly in } n$$

if and only if

$$\text{st}_{A_n} \lim_{j \to \infty} \|L_j f_i - f_i\|_{2\pi} = 0 \quad (i = 1, 2, 3), \text{ uniformly in } n,$$

where $f_1(t) = 1$, $f_2(t) = \cos t$, $f_3(t) = \sin t$ for all $t \in \mathbb{R}$. 

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References


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