STATISTICAL APPROXIMATION BY AN INTEGRAL TYPE OF POSITIVE LINEAR OPERATORS

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Dedicated to Professor Petru Blaga at his 60th anniversary

Abstract. In this paper we will construct an integral type generalization of operators defined and investigated by M.A. Ozarslan, O. Duman, O. Dogru in [7]. We also present a statistical approximation result for these operators.

1. Introduction

In [7] the following positive linear operators defined on $C[0,b], 0 < b < 1,$

\[ T_n(f; x) = \frac{u_n}{F_n(x, t)} \sum_{v=0}^{\infty} f \left( \frac{v}{u_n(x)} \right) C_v^{(n)}(t) x^v, \quad f \in [0, b], \]

have been introduced, where $u_n \geq 0$, $x \in [0, b]$, $t \in (-\infty, 0]$. In the above $\{F_n(x, t)\}$ is the set of generating functions for the sequence of functions $\{C_v^{(n)}(t)\}_{v \in \mathbb{N}_0}$ in the form

\[ F_n(x, t) = \sum_{v=0}^{\infty} C_v^{(n)}(t) x^v \]

and $C_v^{(n)}(t) \geq 0$ for $t \in (-\infty, 0]$. This general sequence includes many well-known operators in approximation theory.

In the present paper we construct an integral type generalization of operators defined by (1.1) and we present a Korovkin type approximation theorem via A-statistical convergence.

At first we recall some notation on A-statistical convergence.

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Let $A := (a_{jn})$, $j, n = 1, 2, \ldots$, be an infinite summability matrix. For a given sequence $x = (x_n)$, the $A$-transform of $x$, denoted by $Ax := ((Ax)_j)$, is given by

$$(Ax)_j = \sum_{n=1}^{\infty} a_{jn} x_n,$$

provided that the series converges for each $j \in \mathbb{N}$.

We say that $A$ is regular if $\lim_{j}(Ax)_j = L$ whenever $\lim_{j} x_j = L$. Assume that $A$ is a non-negative regular summability matrix. A sequence $x = (x_n)$ is called $A$-statistically convergent to $L$ if for every $\varepsilon > 0$,

$$\lim_{j} \sum_{n:|x_n-L| \geq \varepsilon} a_{jn} = 0.$$ 

We denote this limit by $st_A \lim x = L$ (see [2]).

Observe that, if $A$ is the identity matrix, then $I$-statistical convergence reduces to ordinary convergence.

It is not hard to see that every convergent sequence is $A$-statistically convergent. E. Kolk [2] proved that $A$-statistical convergence is stronger than convergence when $A = (a_{jn})$ is a regular summability matrix such that

$$\lim_{j} \lim_{n} |a_{jn}| = 0.$$

2. Auxiliary results

In this section we define an integral type generalization of operators defined by (1.1) and present a statistical approximation result for these operators.

We introduce the sequence of operators $\{T^*_n\}$ as follows

$$(T^*_n f)(x) = \frac{u_n}{F_n(x, t)} \sum_{v=0}^{\infty} C^{(n)}(t)x^{v+c_{n,v}} \int_{v}^{v+c_{n,v}} f \left( \frac{\xi}{a_n(v)} \right) d\xi, \quad n \in \mathbb{N},$$

$x \in [0, b]$, where $f$ is an integrable function on the interval $(0, 1)$ and $(c_{n,v})$ is a sequence such that

$$(2.2) \quad 0 < c_{n,v} \leq 1, \quad (n, v) \in \mathbb{N} \times \mathbb{N},$$

158
where \( e \in \mathbb{N} \geq -3 \). Each \( Z^0 \) for any

**STATISTICAL APPROXIMATION BY AN INTEGRAL TYPE OF POSITIVE LINEAR OPERATORS**

The set \( \{ F_n(x, t) \} \), \( t \in (-\infty, 0] \), is described as in (1.2).

Assume that the next conditions hold

(i) \( F_{n+1}(x, t) = p(x)F_n(x, t), \ p(x) < M < \infty, \ x \in (0, 1) \),

(ii) \( Bt(-1)C_{v-1}^{(n+1)}(t) = \alpha_n(v)C_{v-1}^{(n)}(t) - \nu C_v^{(n)}(t), \ B \in [0, a], \ C_v^{(n)}(t) = 0 \) for \( v \in \mathbb{Z}^- := \{ \ldots, -3, -2, -1 \} \),

(iii) \( \max\{v, n\} \leq a_n(v) \leq a_n(v + 1) \).

In what follows we prove inequalities for the operators \( T_n^* \) given by (2.1).

We set \( e_j, e_{j}(x) = x^j, j \geq 0 \).

**Lemma 2.1.** Let \( T_n^* \) be the positive linear operator given by (2.1). Then, for each \( x \in [0, b], \ t \in (-\infty, 0] \) and \( n \in \mathbb{N} \) we have

\[
\| T_n e_1 - e_1 \|_{[0, b]} \leq \frac{u_n}{2n} + abM|t\frac{u_n}{n} + b|u_n - 1|,
\]

where \( e_1(x) = x \) and \( M, a \) are given as in (i) and (ii) respectively.

**Proof.** Using (2.1), (2.2), (2.3), (2.4), (i), (ii) and (iii) respectively we get

\[
(T_n^* e_1)(x) = \frac{u_n}{F_n(x, t)} \sum_{v=0}^{\infty} C_v^{(n)}(t)x^v \int_0^{\nu + c_{n,v}} \frac{\xi}{a_n(v)} d\xi
\]

\[
= \frac{u_n}{F_n(x, t)} \sum_{v=0}^{\infty} C_v^{(n)}(t)x^v \int_0^{\nu + c_{n,v}} \frac{\xi}{a_n(v)} \left( \frac{2}{v} \right)^v d\xi
\]

\[
= \frac{u_n}{F_n(x, t)} \sum_{v=0}^{\infty} \frac{C_v^{(n)}(t)}{a_n(v)} \cdot x^v \cdot \frac{1}{2} (c_{n,v}^2 + 2v c_{n,v})
\]

\[
\leq \frac{u_n}{2F_n(x, t)} \sum_{v=0}^{\infty} \frac{C_v^{(n)}(t)}{a_n(v)} x^v + \frac{u_n}{F_n(x, t)} \sum_{v=0}^{\infty} \frac{C_v^{(n)}(t)}{a_n(v)} x^v
\]

\[
\leq \frac{u_n}{2F_n(x, t)} \sum_{v=0}^{\infty} C_v^{(n)}(t)x^v + \frac{u_n}{F_n(x, t)} \sum_{v=1}^{\infty} \left[ C_v^{(n)}(t) - \frac{Bt}{a_n(v)} C_v^{(n+1)}(t) \right] x^v
\]

It follows that

\[
(T_n^* e_1)(x) = x \leq \frac{u_n}{2n} + u_n x - x + \frac{u_n}{F_n(x, t)} \sum_{v=1}^{\infty} \frac{Bt}{a_n(v)} C_v^{(n+1)}(t)x^v
\]
where

Consequently, we have

Hence, by taking the supremum over \( x \in [0, b] \) on both sides of the above inequality, the proof is completed.

**Lemma 2.2.** For each \( x \in [0, b] \), \( t \in (-\infty, 0] \) and \( n \in \mathbb{N} \) we have

\[
\|T^*_n e_2\|_{C[0,b]} \leq \frac{u_n}{2n} + abM|t|\frac{u_n}{n^2} + \frac{u_n}{n}b(\alpha M|t| + aM|t| + 2) + b^2|u_n - 1|,
\]

where \( e_2(x) = x^2 \) and \( M \) are as in Lemma 2.1.

**Proof.** We have from (2.1) that

\[
(T_n e_2)(x) = \frac{u_n}{F_n(x,t)} \sum_{v=0}^{\infty} C^{(n)}_v(t) x^v \int_{v}^{\infty+c_{n,v}} \frac{\xi^2}{\left(\frac{u_n}{\xi}\right)^2} d\xi
\]

\[
= \frac{u_n}{F_n(x,t)} \sum_{v=0}^{\infty} C^{(n)}_v(t) x^v \frac{1}{(a_n(v))^2} \cdot \frac{\xi^3}{3} v
\]

\[
= \frac{u_n}{F_n(x,t)} \sum_{v=0}^{\infty} C^{(n)}_v(t) x^v \frac{1}{[a_n(v)]^2} \cdot \frac{1}{3} (v^3 + 3v^2c_{n,v} + 3vc_{n,v}^2 + c_{n,v}^3 - v^3)
\]

\[
= \frac{u_n}{F_n(x,t)} \sum_{v=0}^{\infty} C^{(n)}_v(t) x^v \frac{v^2}{[a_n(v)]^2} c_{n,v} + \frac{u_n}{F_n(x,t)} \sum_{v=0}^{\infty} C^{(n)}_v(t) x^v \frac{v}{[a_n(v)]^2} c_{n,v}^2
\]

\[
+ \frac{u_n}{F_n(x,t)} \sum_{v=0}^{\infty} C^{(n)}_v(t) x^v \frac{1}{3[a_n(v)]^2} c_{n,v}^3
\]

\[
\leq \frac{u_n}{F_n(x,t)} \sum_{v=0}^{\infty} C^{(n)}_v(t) x^v \frac{v^2}{[a_n(v)]^2} + \frac{u_n}{F_n(x,t)} \sum_{v=0}^{\infty} C^{(n)}_v(t) x^v \frac{v}{[a_n(v)]^2}
\]
Statistical Approximation by an Integral Type of Positive Linear Operators

Using the recurrence formula (ii) twice, we may write

\[ C_v^{(n)}(t) = \frac{a_n(v-1)}{a_n(v)} C_v^{(n-1)}(t) - \frac{Bt}{a_n(v)} C_v^{(n+1)}(t) \]

Taking into account (2.5) we get respectively

\[ (T^*_n e_2)(x) - e_2(x) \leq \left( \sum_{v=2}^{\infty} \frac{a_n(v-1)}{a_n(v)} C_v^{(n-2)}(t) x^v - x^2 \right) \]

By using the requirement (iii) we have

\[ \frac{v+1}{a_n(v+1)} \leq \frac{1}{n}, \quad \frac{1}{a_n(v+2)} \leq \frac{1}{n}, \quad \frac{1}{a_n(v+1)} \leq \frac{1}{n}, \quad a_n(v-1) \leq a_n(v). \]

Considering (2.2), (i), (ii), (iii) and the above relations results

\[ \left| \frac{Bt u_n}{F_n(x, t)} \sum_{v=2}^{\infty} \frac{1}{a_n(v)} C_v^{(n+1)}(t) x^v \right| \leq \frac{B|x| u_n}{F_n(x, t)} \sum_{v=0}^{\infty} \frac{1}{a_n(v+1)} C_v^{(n+1)}(t) x^v \]

\[ \leq \frac{a |x| u_n}{n} \]

\[ \left| \frac{Bt u_n}{F_n(x, t)} \sum_{v=2}^{\infty} \frac{1}{a_n(v)} C_v^{(n+1)}(t) x^v \right| \leq \frac{B|x|^2 u_n}{F_n(x, t)} \sum_{v=0}^{\infty} \frac{1}{a_n(v+2)} C_v^{(n+1)}(t) x^v \]

\[ \leq \frac{a |x|^2 u_n}{n F_n(x, t)} \sum_{v=0}^{\infty} C_v^{(n+1)}(t) x^v = a |x|^2 \frac{u_n}{n} \]

\[ \left| \frac{u_n}{F_n(x, t)} \sum_{v=2}^{\infty} \frac{1}{a_n(v)} C_v^{(n)}(t) x^v \right| \leq \frac{x u_n}{n}, \]

\[ \frac{u_n}{F_n(x, t)} \sum_{v=2}^{\infty} \frac{a_n(v-1)}{a_n(v)} C_v^{(n)}(t) x^v - x^2 \leq x^2 (u_n - 1). \]
The above inequalities and (2.6) imply

\[(T^*_n e_2)(x) - x^2 \leq x^2(u_n - 1) + ax^2|t|p(x)\frac{u_n}{n} + \frac{xu_n}{n} + a|t|x p(x)\frac{u_n}{n} \]

\[+ x u_n + a|t|p(x)\frac{u_n}{n^2} + \frac{u_n}{3n^2}\]

Consequently, we have

\[\|T^*_n e_2 - e_2\|_{C[0,b]} \leq \frac{u_n}{3n^2} + BbM|t|\frac{u_n}{n^2} + \frac{bu_n}{n}(abM|t| + aM|t| + 2) + b^2|u_n - 1|\]

3. Statistical approximation

In this section, we provide a Korovkin type theorem via A-statistical convergence for the sequence of positive linear operators defined by (2.1).

**Lemma 3.1.** Let \(A = (a_{jn})\) be a non-negative regular summability matrix. Then we have

\[st_A - \lim_n \|T^*_n e_1 - e_1\|_{C[0,b]} = 0,\]

where \(T^*_n\) is defined by (2.1).

**Proof.** We conclude from Lemma 2.1 that

\[(2.7) \quad \|T^*_n e_1 - e_1\|_{C[0,b]} \leq \frac{u_n}{2n} + BbM|t|\frac{u_n}{n} + b|u_n - 1|\]

\[= \frac{u_n}{2n}(1 + 2bBM|t|) + b|u_n - 1| \leq B_1 \left(\frac{u_n}{2n} + |u_n - 1| \right),\]

where \(B_1 = \max\{(1 + 2bBM|t|), b\} \).

We can conclude according to (2.3) that

\[st_A - \lim_n \frac{u_n}{2n} = 0.\]

Now, for a given \(\varepsilon > 0\), define

\[U := \left\{ n : \frac{u_n}{2n} + |u_n - 1| \geq \frac{\varepsilon}{B_1} \right\},\]

\[U_1 := \left\{ n : \frac{u_n}{2n} \geq \frac{\varepsilon}{2B_1} \right\}, \quad U_2 := \left\{ n : |u_n - 1| \geq \frac{\varepsilon}{2B_1} \right\}.\]

We see that \(U \subseteq U_1 \cup U_2\).
The inequality (2.7) yields
\[ \sum_{n : \|T_n e_1 - e_1\| \geq \varepsilon} a_{jn} \leq \sum_{n \in U} a_{jn} \leq \sum_{n \in U_1} a_{jn} + \sum_{n \in U_2} a_{jn}, \]
and taking \( j \to \infty \) the result follows.

**Lemma 3.2.** Let \( A = (a_{jn}) \) be a non-negative regular summability matrix. Then we have
\[ st_A - \lim_n \|T_n^* e_2 - e_2\| = 0, \]
where \( T_n^* \) is defined by (2.1).

**Proof.** It follows from Lemma 2.2 that
\[ \|T_n^* e_2 - e_2\|_{C[0, b]} \leq \frac{u_n}{3n^2} (1 + 3bBM|t|) + \frac{u_n}{n} (b^2BM|t| + bBM|t| + 2) + b^2|u_n - 1|. \]
Hence, we get
\[ \|T_n^* e_2 - e_2\|_{C[0, b]} \leq B_2 \left( \frac{u_n}{3n^2} + \frac{u_n}{n} + |u_n - 1| \right), \]
where
\[ B_2 = \max\{(1 + 3bBM|t|), (b^2BM|t| + bBM|t| + 2), b^2\}. \]
By (2.3) we have
\[ st_A - \lim_n u_n = 1, \quad st_A - \lim_n \frac{u_n}{n} = 0 \quad \text{and} \quad st_A - \lim_n \frac{u_n}{3n^2} = 0. \]
For a given \( \varepsilon > 0 \) we define
\[ U := \left\{ n : \frac{u_n}{3n^2} + \frac{u_n}{n} + |u_n - 1| \geq \frac{\varepsilon}{B_2} \right\}, \]
\[ U_1 := \left\{ n : \frac{u_n}{3n^2} \geq \frac{\varepsilon}{3B_2} \right\}, \quad U_2 := \left\{ n : \frac{u_n}{n} \geq \frac{\varepsilon}{3B_2} \right\}, \]
\[ U_3 := \left\{ n : |u_n - 1| \geq \frac{\varepsilon}{3B_2} \right\}. \]
Then we have \( U \subseteq U_1 \cup U_2 \cup U_3 \).
By using (2.8) we can write successively
\[ \sum_{n : \|T_n e_1 - e_1\|_{C[0, b]} \geq \varepsilon} a_{jn} \leq \sum_{n \in U} a_{jn} \leq \sum_{n \in U_1} a_{jn} + \sum_{n \in U_2} a_{jn} + \sum_{n \in U_3} a_{jn}. \]
Taking limit as $j \to \infty$ the proof is complete.

We recall the following important result established by A.P. Gadjiev and C. Orhan.

**Theorem 3.1.** ([4], Theorem 1) If the sequence of positive linear operators $L_n : C_M[a, b] \to B[a, b]$ satisfies the conditions

1. $st - \lim ||L_ne_0 - e_0||_B = 0$
2. $st - \lim ||L_ne_1 - e_1||_B = 0$
3. $st - \lim ||L_ne_2 - e_2||_B = 0$

then for any function $f \in C_M[a, b]$ we have

4. $st - \lim ||L_nf - f||_B = 0$,

where $C_M[a, b] = \{ f : \mathbb{R} \to \mathbb{R}, f \text{ continuous on } [a, b] \text{ and bounded on the whole real axis} \}$ and $\|f\|_B := \sup_{a \leq x \leq b} |f(x)|$.

We mention that the above Theorem is given for statistical convergence, but the proof also works for A-statistical convergence.

Now we provide a Korovkin type approximation theorem for the operators $T_n^*$ via A-statistical convergence.

**Theorem 3.2.** Let $A = (a_{jn})$ be a non-negative regular summability matrix. Then, for all $f \in C[0, b]$, we have

$$st_A - \lim_n \|T_n^*f - f\|_{C[0,b]} = 0.$$

**Proof.** By $(T_n^*e_i)(x) \leq u_n$, Lemmas 3.1 and 3.2 we get

$$st_A - \lim_n \|T_n^*e_i - e_i\|_{C[0,b]} = 0, \quad i = 0, 1, 2.$$

The result follows from Theorem 1 in [4].
References


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