STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LII**, Number 3, September 2007

BOOK REVIEWS

Barry Simon, ***Orthogonal Polynomials on the Unit Circle***,
Part 1: *Classical Theory,* xxv+pages 1-466 (ISBN:0-8218-3446-0),
Part 2: *Spectral Theory*, xxi+pages 467-1044 (ISBN:0-8218-3675-7),
American Mathematical Society, Colloquium Publications, Volume 54, Providence, Rhode Island 2005, ISBN:0-8218-3757-5 (set).

This monumental two volume treatise contains a comprehensive study of orthogonal polynomials on the unit circle (OPUC) corresponding to nontrivial (i.e., with infinite support) probability measures on the unit circle $\partial\mathbb{D}$, in the complex plane, where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the unit disk. This is viewed as a counterpart of the well established theory of orthogonal polynomials on the real line (OPRL) as presented, for instance, in the classical treatises of G. Szegö (first edition, AMS 1939) and G. Freud (Pergamon Press 1971). At the same time, OPUC theory supplies the study of OPRLs with new tools and methods. If μ is a nontrivial probability measure on $\partial\mathbb{D}$, then $1, z, z^2, \ldots$ are linearly independent in the Hilbert space $L^2(\partial\mathbb{D}, \mu)$, so that the Gram-Schmidt orthogonal procedure produces an orthogonal system $\{\Phi_n\}$ given by $\Phi_n = P_n[z^n]$, where P_n denotes the projection onto $\{1, z, \ldots z^{n-1}\}^{\perp}$. By normalization one obtains the orthonormal system $\varphi_n = \Phi_n/\|\Phi_n\|$. The orthogonal polynomials Φ_n satisfy Szegő's recurrence $\Phi_{n+1}(z) = z\Phi_n(z) - \bar{\alpha}_n\Phi_n^*(z)$, where $\Phi_n^*(z) = z^n\overline{\Phi_n(1/\bar{z})}$ is the reversed polynomials of Φ_n. The parameters $\alpha_0, \alpha_1, \ldots$, called Verblunsky coefficients, all belong to $\mathbb{D}$ (i.e., $|\alpha_j| < 1$). These were considered by S. Verblunsky in two remarkable papers published in 1934 and 1935 in Proceedings of the London Mathematical Society, where he proved, among other things, that there is a bijective correspondence between the nontrivial probability measures on $\partial\mathbb{D}$ and the sequences $\{\alpha_j\}_{j=0}^{\infty}$ in $\mathbb{D}$. The author gives four proofs for this result. The irony is that the results of Verblunsky were largely overlooked by the mathematical community, some of them being rediscovered later.

One of the central topic of the book, and of the whole theory of OPUC, is how the properties of the measure μ correspond to properties of the Verblunsky coefficients, and viceversa. The major result in this respect is Szegő Theorem asserting that $\prod_{j=0}^{\infty}(1 - |\alpha_j|^2) = \exp\left(\int_0^{2\pi} \log(w(\theta))\frac{d\theta}{2\pi}\right)$, where $w(\theta)d\theta$ is the absolutely continuous part of μ. In particular, $\sum_j |\alpha_j|^2 < \infty$ if and only if $\int \log w(\theta)d\theta > -\infty$. Four proofs of this remarkable result are given. An extension of this result is the so called Strong Szegő Theorem, which can be rephrased as an assertion about the asymptotics of Toeplitz determinants. Verblunsky coefficients are related also with other important quantities in the theory of OPUC as, for instance, with Fourier coefficients $\hat{\mu}_n$ of the measure μ: $\sum_j |\alpha_j| < \infty$ if and only if $\sum_j |\hat{\mu}_j| < \infty$ (Baxter's Theorem (1961)).

Another important tools, relating OPUC with the theory of analytic functions are the Carathéodori functions defined by $F(z) = \int [e^{i\theta} + z][e^{i\theta} + z]^{-1} d\mu(\theta)$ and the associated Schur functions f given by the relation $F(z) = [1 + zf(z)][1 + zf(z)]^{-1}$. Section 1.3 contains an overview of their properties. Schur proved that Schur functions admit continued fraction expansion with coefficients $|\gamma_j| < 1$, called Schur parameters. A remarkable result proved by Ya. L. Geronimus in the fifties of the last century asserts that Schur parameters agree with Verblunsky coefficients: $\gamma_j = \alpha_j$, $j \in \mathbb{N}$. Again five proofs of this result are included in the book. The method of Schur functions and some real variable methods were used by S. Kruschev in a series of papers published between 1993 and 2005 to obtain some important results on OPUC.

The book is very well organized and covers a lot of important results. In fact, the aim of the inclusion of several proofs for some important results is to emphasize that different approaches shed new light on the subject and, at the same time, allow to systematize and organize the study of OPUCs. For the convenience of the reader, beside the section on Carathéodori and Schur functions we did yet mention, a section containing a survey of principal results and methods from OPRL theory, as well as one on spectral theory of operators on Hilbert space are included. At the end of the second volume there are four very useful appendices: A. *Reader's Guide: Topics and Formulae*; B. *Perspective* (OPRL vs OPUC); C. *Twelve Great Theorems*; D. *Conjectures and Open Questions*. Concerning the "twelve great theorems", the author quote a nice remark of his father who said once that "to pick ten people from twenty for some positive reasons, you don't make ten friends but ten enemies". Apparently, the only way to make ten friends from twenty people is to pick ten for some negative reasons.

The bibliography counts 1119 items with references to the pages where each of them is quoted. The remarks and the historical notes following each section present the evolution of the subject, putting in evidence some corner points and seminal papers.

Undoubtedly that, as Szegő's book, published first in 1939 as the volume 23 in the same prestigious series, this book will become a standard reference in the field tracing the way for future investigations on orthogonal polynomials and their applications. Combining methods from various areas of analysis (calculus, real analysis, functional analysis, complex analysis) as well as by the importance of the orthogonal polynomials in applications, the book will have a large audience including researchers in mathematics, physics, engineering.

S. Cobzaş

Athanase Papadopoulos, ***Metric spaces, convexity and nonpositive curvature,*** IRMA Lectures in Mathematics and Theoretical Physics, Vol. 6, European Mathematical Society, Zürich 2005, xi+287 pp, ISBN: 3-03719-010-8.

Geodesic metric spaces form a class of metric spaces in which convexity of subsets can be defined as well as other related analytic concepts. Buseman spaces are geodesic metric spaces whose length function satisfies a convexity condition. Beside their intrinsic geometric interest, geodesic metric spaces are important for their applications to complex analysis and nonlinear analysis - fixed point theory for nonexpansive mappings, generalized differentiability and optimization. Classical examples of geodesic metrics are the Riemannian metric, the Poincaré metric on the hyperbolic ball $\mathbb{H}^n$, the Carathéodori and the Kobayashi distances for complex manifolds, Thusrston's metric on complex projective surfaces, the Teichmüller metric and Teichmüller spaces. The book starts with a short historical overview emphasizing some corner points in its development - the pioneering work of J. Hadamard, the contributions of K. Menger, A. Wald, H. Busemann and A. D. Alexandrov.

In order to make the book self-contained the author systematically develop in the first two chapters 1. *Lengths of paths in metric spaces,* 2. *Length spaces and geodesic spaces,* the basic construction and the properties of length spaces and geodesic spaces, including convexity - geodesic convexity and Menger convexity, this last being defined through the betweeness relation. Chapter 3. *Maps between metric spaces,* is concerned with Lipschitz maps and fixed points for contractive and for nonexpansive mappings on geodesic spaces. The analog of Hausdorff distance for subsets of a geodesic metric space, called the Busemann-Hausdorff distance, with applications to limits of subsets is considered in Chapter 4. *Distances.*

Chapters 5. *Convexity in vector spaces,* 6. *Convex functions,* and 7. *Strictly convex normed spaces,* are concerned with convexity in vector and normed spaces, emphasizing connections with geodesic metric spaces and the geometry of Minkowski space (finite dimensional normed spaces).

The rest of the book, chapters 8. *Busemann spaces,* 9. *Locally convex spaces* (meaning geodesic metric spaces such that every point has a neighborhood which is a Busemann space), 10. *Asymptotic rays and the visual boundary,* 11. *Isometries,* 12. *Busemann functions, co-rays and horospheres,* is devoted to the theory of Busemann spaces. Again convexity is the main topic and the unifying idea of the development of Busemann spaces.

Written in a clear and pleasant style, with numerous examples from geometry and analysis, the book is accessible to graduate students interested in this topic of intense current research due to its intrinsic importance and to its numerous applications as well.

S. Cobzaş

The essential of John Nash, Edited by Harold W. Kuhn and Sylvia Nasar, Princeton University Press, 2007, ISBN-13: 978-0-691-09610-0, ISBN-10: 0-691-09610-4.

The brilliant mathematician John Forbes Nash was born in 1928 in Bluefield, West Wirginia. After finishing high school in Bluefield he went to Carnegie Tech in Pittsburgh with major chemical engineering, but he eventually switched to mathematics. After graduation he was offered fellowship at either Harvard or Princeton, and he choose the more generous Princeton fellowship, which was also nearer to his hometown. At Princeton he wrote in 1950 a 27 pages Ph.D. thesis on *Non-cooperative games* which initiated a new era in game theory with applications to economics, social behavior, war questions, etc. The basic idea was that which is called now the Nash equilibrium, which finally led to a Nobel Prize in Economics attributed to him in 1994, shared with John Harsanyi and Reinhard Selten. As it is mentioned in the motivation of the Nobel committee the Nash equilibrium has become "the analytical structure for studying all situations of conflict and cooperation".

Other important contributions of John Nash concern the imbedding of Riemannian manifolds in the Euclidean space, the Nash implicit function theorem, real algebraic varieties, and parabolic partial differential equations.

But after these astonishing and deep contributions J. Nash suffered at the age of thirty one of mental illness, being diagnosed as paranoid schizophrenia, so he had to retire from MIT where he was affiliated. After a long period of absence (30 years) he recovered himself, due in good part to the recognition of his achievements by the Nobel prize committee, started to travel and to work again. The illness prevented him to obtain a Fields medal which he fully deserved for his outstanding results on elliptic and parabolic partial differential equations. Because in 1958 these results were still unpublished, the Fields Committee postponed Nash as a virtual winner of the 1962 Medal, but the mental illness destroyed his career for a long period of time.

His situation is known to the general public due to the book *A beautiful mind* by Sylvia Nasar (one of the editors of the present volume) and by the movie with the same name with Russell Crowe starring as John Nash.

While the biography by Sylvia Nasar was concerned mainly with the life of J. Nash, the present volume deals with his scientific work. It contains contributions by Harold W. Kuhn (the other editor of the book), a Princeton fellow of J. Nash and a life friend, an introduction by Sylvia Nasar, the press release of the Swedish Academy on Nobel prize and an autobiography written by J. Nash with this occasion, a nice collection of photos, a short note by John Milnor on the Hex game (known also as the Nash game), the facsimile of the Ph.D. thesis of J. Nash and several of his seminal papers.

The book is written in a pleasant and informal style, being addressed to a large audience. It's nice that Princeton University Press released this cheeper paperback version making the book accessible to a large public.

P. T. Mocanu

J. Baik, T. Kriecherbauer, K.T.-R. McLaughlin, P.D. Miller, ***Discrete Orthogonal Polynomials. Asymptotics and Applications,*** Annals of Mathematics Studies, Number 164, Princeton University Press, 2007, 170 pp. ISBN-13: 978-0-691-12734-7, ISBN-10: 0-691-12734-4.

As the title of the book suggests, the main aim of this monograph is to present asymptotic properties of polynomials that are orthogonal with respect to pure point measures supported on finite sets of nodes. Further on, the authors use these results to establish various statistical properties of discrete orthogonal polynomial ensembles. In particular, the authors apply their results to the problem of computing asymptotic of statistics for random rhombus tilings of a large hexagon. They also obtain new results for the continuum limit of the Toda lattice. Working with a general class of weights that contains Krawtchouk and Hahn weights as special cases, the authors compute the asymptotic of the associated discrete orthogonal polynomials for all values of the variable in the complex plane.

The starting point is the following basic interpolation problem: given a natural number N, a set X_N of nodes and a set of corresponding weights $\{w_{N,n}\}$, consider the possibility of finding a 2×2 matrix $P(z; N, k)$, $k \in \mathbb{Z}$, with certain properties. After a comprehensive introduction providing the reader with a thorough mathematical background, the main results for the discrete orthogonal polynomials and for corresponding applications are presented in Chapter 2 and Chapter 3, respectively. Chapters 4 and 5 contain the complete asymptotic analysis of the matrix $P(z; N, k)$ in the limit N tends to infinity. In the last two chapters the authors prove the theorems stated in Chapters 2 and 3.

The contents of the book is enriched with 3 Appendices. I mention the first of them summarizes construction of the solution of a limiting Riemann-Hilbert problem by means of hyperelliptic function theory and the second of them gives a proof of the determination of the equilibrium measure of the Hahn weight. At the same time good references are inserted.

The authors' style is pleasant offering detailed and clear explanations of every concept and method.

The book includes the authors' own research results developed over the last years. Their approach and proofs are straightforward constructive making this monograph accessible and valuable to undergraduate and graduate students, PhD students and researchers involved in the asymptotic analysis of systems of discrete orthogonal polynomials.

Octavian Agratini

David L. Applegate, Robert E. Bixby, Vasek Chvtal and William J. Cook ***The Traveling Salesman Problem:A Computational Study,*** Princeton University Press, 2007, 606 pp., ISBN13: 978-0-691-12993-8.

The present book contains an exhaustive and interesting presentation of various questions related to the famous Travelling Salesman Problem (TSP). Its goal is to set down the techniques that have led to the solution of a number of large instances of the TSP, including the full set of examples from TSPLIB challenge collection.

The first and the second chapters of the book cover history and applications of the TSP. This part is very interesting and accessible to a wide audience. A short definition of the TSP is the following: given a set of cities along with the cost of travel between each part of them, the travelling salesman problem is to find the way of visiting all the cities and returning to the start point with minimal cost. The origins of the study of the TSP as a mathematical problem are somewhat obscure. M. Flood said that the TSP was posed, in 1934, by Hassler Whitney in a seminar talk at Princeton University. Therefore as father of the TSP can be considered Hassler Whitney. The third chapter is dedicated to present the work of Danzig, Fulkerson and Johnson for solving the 49-city problem and, indirectly, to make a short presentation of the cutting plane method. The fourth chapter contents a history of TSP computation. It includes, among other things, the bases of the branch-and-bound method, the Gomory's cuts, the TSP's cuts, the Lin-Kernighan's heuristic, the Crowder-Padberg's code, the dynamic programming etc. It is mentioned that the dynamic programming algorithm can solve any n-city TSP instance in time that is at most proportional to $n^2 \cdot 2^n$.

Chapters 5 - 10 describe (some) methods for finding cutting planes for the TSP. In the chapter 11 it is presented a separation method that disdains all understanding of the TSP polytope and bashes on regardless of all prescribed templates.

The twelfth chapter presents machinery to handle the flow of cuts and edges into and out of the linear programming relaxations, as well as methods used for interacting with a linear programming solver. The actual solution of the linear programming problems is described in the chapter 13.

The branch-and-cut algorithm embeds the cutting-plane method within a branch-and-bound search. Its specialization to TSP is described in the chapter 14. There is a growing literature devoted to the study of heuristic algorithms for the TSP and to their various aspects. The fifteenth chapter includes some of them.

The algorithmic components described in this book are brought together in the *Concorde* computer code for the TSP. *Concorde* is described in the chapter 16. Also, in this chapter a report on computational studies learning to the solution of the full set of TSPLIB instances is given.

Chapter 17 contents a short survey of recent work dedicated to the TSP by various research group.

The book includes a bibliography of 561 titles.

Liana Lupşa