STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LII**, Number 3, September 2007

ON THE DEGREE OF APPROXIMATION IN VORONOVSKAJA'S THEOREM

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Dedicated to Professor Petru Blaga at his 60th *anniversary*

Abstract. The present article continues earlier research by P. Piţul, I. Raşa and the author on quantitative versions of E.V. Voronovskaja's 1932 result concerning the asymptotic behavior of Bernstein polynomials.

1. Introduction and historical remarks

In two recent notes P. Piţul, I. Raşa and the author discussed E.V. Voronovskaja's [20] classical theorem on the asymptotic behavior of Bernstein polynomials $B_n(f;\cdot)$ for twice continuously differentiable functions given on $[0,1]$. We recall that for $f \in \mathbb{R}^{[0,1]}, n \geq 1$ and $x \in [0,1]$ one puts

$$\begin{aligned} B_n(f;x) &:= \textstyle\sum_{k=0}^n f\left(\frac{k}{n}\right) \cdot p_{n,k}(x) \\ &:= \textstyle\sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}. \end{aligned}$$

Normally Voronovskaja's result is given today in its local version as follows.

Theorem 1.1. *(see R.A. DeVore & G.G. Lorentz* [3]*)*

If f is bounded on $[0,1]$, differentiable in some neighborhood of x and has second derivative $f''(x)$ for some $x \in [0,1]$, then

$$n \cdot [B_n(f;x) - f(x)] - \frac{x(1-x)}{2} \cdot f''(x) \to 0, n \to \infty.$$

If $f \in C^2[0,1]$, the convergence is uniform.

Received by the editors: 16.04.2007.

2000 *Mathematics Subject Classification.* 41A10, 41A17, 41A25, 41A36.

Key words and phrases. Bernstein polynomials, Voronovskaja type theorem.

In an article following directly that of Voronovskaja S.N. Bernstein [1] generalized the uniform version as given below.

Theorem 1.2. *If $q \in \mathbb{N}$ is even, $f \in C^q[0,1]$, then, uniformly in $x \in [0,1]$,*

$$n^{\frac{q}{2}} \cdot \left\{ B_n(f;x) - f(x) - \sum_{r=1}^{q} B_n((e_1 - x)^r; x) \cdot \frac{f^{(r)}(x)}{r!} \right\} \to 0, n \to \infty.$$

Before we continue here is a word of warning:

"Voronovskaja" ist just one possible way to spell the Russian ВОРОНОВСКАЯ in Latin characters. Other possibilities to be observed in the literature are Voronovskaya, Woronowskaja, Woronowskaya and even Voronovsky (as given on the original 1932 article).

It is the aim of this note to present new quantitative results which also cover Bernstein's above case, among others. Before going into details we give three examples of quantitative Voronovskaja theorems. Some authors call them "strong Voronovskaja-type theorems" because, in addition to the convergence of $n \cdot [B_n g - g]$ towards Ag, they also express a degree of approximation depending on smoothness properties of the function g.

Example 1.3. *Write $A(f;x) := \frac{x(1-x)}{2} \cdot f''(x), \varphi(x) := \sqrt{x(1-x)}$. Then for*

- $g \in C^4[0,1] : \|n \cdot [B_n g - g] - Ag\|_\infty \leq \frac{24}{n}(\|g'''\| + \|g^{(4)}\|)$ *(see [8]);*
- $g \in C^3[0,1] : \|n \cdot [B_n g - g] - Ag\|_\infty \leq \frac{C}{\sqrt{n}} \cdot \|\varphi^3 \cdot g'''\|$ *(see [4]).*

A full quantitative pointwise version of Voronovskaja's uniform result reads

- $f \in C^2[0,1] : |n \cdot [B_n(f;x) - f(x)] - A(f;x)| \leq \frac{x(1-x)}{2} \cdot \tilde{\omega}(f''; \frac{1}{3 \cdot \sqrt{n}})$ (see [6]).

All the proofs are based on Taylor's formula; the last one uses the "Peano remainder without Landau" as recalled in the next section. During the writing of this note it was brought to the author's attention that already in 1985 V.S. Videnskij published in [19] (see Theorem 15.2 on p. 49) the following:

- $f \in C^2[0.1] : |n \cdot [B_n(f;x) - f(x)] - A(f;x)| \leq x(1-x) \cdot \omega(f''; \sqrt{\frac{2}{n}})$.

Videnskij's inequality follows from ours given in [6]; later in this note an even more precise pointwise estimate will be given.

2. An auxiliary result

Here we recall an estimate of the remainder in Taylor's formula which (strange enough!) we were unable to locate in the literature.

Theorem 2.1. *(see* [6]*) Let $\omega(f;\varepsilon)$ denote the classical first order modulus of continuity of $f \in C[a,b], \varepsilon > 0$. The least concave majorant $\tilde{\omega}(f;\varepsilon)$ is given by*

$$\tilde{\omega}(f;\varepsilon) = \begin{cases} \sup\limits_{\substack{0\le x\le\varepsilon\le y\le b-a\\ x\neq y}} \frac{(\varepsilon-x)\omega(f;y)+(y-\varepsilon)\omega(f;x)}{y-x}, & 0 \le \varepsilon \le b-a; \\ \omega(f;b-a), \quad \varepsilon > b-a. \end{cases}$$

Suppose that $f \in C^q[a,b], q \ge 0$. Then for the remainder in Taylor's formula we have

$$|R_q(f;x,t)| \le \frac{|t-x|^q}{q!}\tilde{\omega}\left(f^{(q)}; \frac{|t-x|}{q+1}\right).$$

Here $x \in [a,b]$ is fixed, and $t \in [a,b]$.

Remark 2.2. *Since $\tilde{\omega}(f^{(q)}; \frac{|t-x|}{q+1}) = o(1), t \to x$, this is a more explicit form of Peano's remainder in Taylor's formula.*

3. A general quantitative Voronovskaja-type theorem

As mentioned before, Bernstein's generalization can be turned into a quantitative statement. However, we intend to be more general. For historical reasons we recall the following

Theorem 3.1. *(R.G. Mamedov* [13]*)*
Let $q \in \mathbb{N}$ be even, $f \in C^q[0,1]$, and $L_n : C[0,1] \to C[0,1]$ be a sequence of positive linear operators such that

$$L_n(e_0;x) = 1, x \in [0,1];$$

$$\lim_{n\to\infty} \frac{L_n((e_1-x)^{q+2j};x)}{L_n((e_1-x)^q;x)} = 0 \text{ for at least one } j \in \{1,2,\ldots\}.$$

Then

$$\frac{1}{L_n((e_1-x)^q;x)}\{L_n(f;x) - f(x) - \sum_{r=1}^{q} L_n((e_1-x)^r;x)\cdot\frac{f^{(r)}(x)}{r!}\} \to 0, n \to \infty.$$

The quantitative version of the above result will be based upon

Theorem 3.2. *Let $q \in \mathbb{N}_0, f \in C^q[0,1]$ and $L : C[0,1] \to C[0,1]$ be a positive linear operator. Then*

$$|L(f;x) - \sum_{r=0}^{q} L((e_1-x)^r;x) \cdot \frac{f^{(r)}(x)}{r!}| \leq \frac{L(|e_1-x|^q;x)}{q!} \cdot \tilde{\omega}(f^{(q)}; \frac{1}{q+1} \cdot \frac{L(|e_1-x|^{q+1};x)}{L(|e_1-x|^q;x)}.$$

Sketch of proof. For x fixed and $t \in [0,1]$ write Taylor's formula as

$$f(t) = \sum_{r=0}^{q} \frac{f^{(r)}(x)}{r!}(t-x)^r + R_q(f;x,t), \text{ i.e.,}$$

$$f(t) - \sum_{r=0}^{q} \frac{f^{(r)}(x)}{r!}(t-x)^r = R_q(f;x,t).$$

Applying L to both sides (as functions of t) yields

$$|L(f;x) - \textstyle\sum_{r=0}^{q} \frac{f^{(r)}(x)}{r!} \cdot L((e_1-x)^r;x)|$$

$$\leq L(|R_q(f;x,\cdot)|;x)$$

$$\leq L(\tfrac{|e_1-x|^q}{q!} \cdot \tilde{\omega}(f^{(q)}; \tfrac{|e_1-x|}{q+1};x)$$

$$\leq \tfrac{L(|e_1-x|^q;x}{q!} \cdot \tilde{\omega}(f^{(q)}; \tfrac{1}{q+1} \cdot \tfrac{L(|e_1-x|^{q+1};x)}{L(|e_1-x|^q;x)}).$$

For further details (such as the intermediate use of a K-functional) see [6].□

Corollary 3.3. *(Mamedov's situation) Suppose that we consider a sequence (L_n) of positive linear operators, q is even, $L_n(e_0;x) = 1$ and that for at least one $j \in \mathbb{N}$ one has*

$$\lim_{n\to\infty} \frac{L_n((e_1-x)^{q+2j};x)}{L_n((e_1-x)^q;x)} = 0.$$

In this case $L_n(|e_1-x|^q;x) = L_n((e_1-x)^q;x)$.

Using the Cauchy-Schwarz inequality for positive linear functionals (possibly repeatedly) we obtain

$$\frac{L_n(|e_1-x|^{q+1};x)}{L_n(|e_1-x|^q;x)} = \frac{L_n(|e_1-x|^{\frac{q}{2}} \cdot |e_1-x|^{\frac{q}{2}+1};x)}{L_n((e_1-x)^q;x)}$$

$$\leq \sqrt[2]{\frac{L_n((e_1-x)^{q+2};x)}{L_n((e_1-x)^q;x)}} \cdots \leq \sqrt[2j]{\frac{L_n((e_1-x)^{q+2j};x)}{L_n((e_1-x)^q;x)}}.$$

And from here Mamedov's statement follows, since according to his assumption the latter quantity tends to 0 as n goes to ∞. □

4. Some special cases

Here we briefly discuss the cases $q = 0, q = 1$ and $q = 2$.

Example 4.1. *In case $q = 0$ we may assume that $L(e_0; x) > 0$. Since otherwise, for $f \in C[0,1]$ arbitrary, we would have*

$$|L(f;x)| \leq L(|f|;x) \leq \|f\| \cdot L(e_0;x) = 0,$$

leading to a trivial inequality. Making the above assumption we get

$$|L(f;x) - L(e_0;x) \cdot f(x)| \leq L(e_0;x) \cdot \tilde{\omega}\left(f; \frac{L(|e_1 - x|;x)}{L(e_0;x)}\right);$$

thus

$$|L(f;x) - f(x) + f(x) - L(e_0;x) \cdot f(x)| \leq L(e_0;x) \cdot \tilde{\omega}\left(f; \frac{L(|e_1 - x|;x)}{L(e_0;x)}\right),$$

or

$$|L(f;x) - f(x)| \leq |L(e_0;x) - 1| \cdot |f(x)| + L(e_0;x) \cdot \tilde{\omega}\left(f; \frac{L(|e_1 - x|;x)}{L(e_0;x)}\right),$$

or, for $L(e_0;x) = 1$,

$$|L(f;x) - f(x)| \leq \tilde{\omega}(f; L(|e_1 - x|;x).$$

This is an inequality which can already be found in [5], Theorem 3.1.

Example 4.2. *For $q = 1$, i.e., $f \in C^1[0,1]$ we arrive at*

$$|L(f;x) - L(e_0;x) \cdot f(x) - L(e_1 - x;x) \cdot f'(x)| \leq L(|e_1 - x|;x) \cdot \tilde{\omega}\left(f'; \frac{1}{2} \cdot \frac{L((e_1 - x)^2;x)}{L(|e_1 - x|;x)}\right).$$

Proceeding as in the previous case we find

$$\begin{aligned} |L(f;x) - f(x)| &\leq |(L(e_0;x) - 1) \cdot f(x) + L(e_1 - x;x) \cdot f'(x)| \\ &\quad + L(|e_1 - x|;x) \cdot \tilde{\omega}(f'; \tfrac{1}{2} \tfrac{L((e_1-x)^2;x)}{L(|e_1-x|;x)}) \\ &\leq |L(e_0;x) - 1| \cdot |f(x)| + |L(e_1 - x;x)| \cdot |f'(x)| \\ &\quad + L(|e_1 - x|;x) \cdot \tilde{\omega}(f'; \tfrac{1}{2} \tfrac{L((e_1-x)^2;x)}{L(|e_1-x|;x)}). \end{aligned}$$

If L reproduces linear functions this simplifies to

$$|L(f;x)-f(x)| \le L(|e_1-x|;x)\cdot\tilde{\omega}(f';\frac{1}{2}\cdot\frac{L((e_1-x)^2;x)}{L(|e_1-x|;x)}).$$

A similar inequality was given in [5], *Section 4.*

Example 4.3. *For $q=2$ we get*

$$|L(f;x)-L(e_0;x)\cdot f(x)-L((e_1-x);x)\cdot f'(x)-\tfrac{1}{2}\cdot L((e_1-x)^2;x)\cdot f''(x)|$$

$$\le \tfrac{1}{2}\cdot L((e_1-x)^2;x)\cdot\tilde{\omega}(f'';\tfrac{1}{3}\cdot\tfrac{L(|e_1-x|^3;x)}{L((e_1-x)^2;x)}).$$

If $L(e_0;x)=1$ and $L((e_1-x);x)=0$, then this turns into

$$|L(f;x)-f(x)-\tfrac{1}{2}\cdot L((e_1-x)^2;x)\cdot f''(x)|$$

$$\le \tfrac{1}{2}\cdot L((e_1-x)^2;x)\cdot\tilde{\omega}(f'';\tfrac{1}{3}\cdot\tfrac{L(|e_1-x|^3;x)}{L((e_1-x)^2;x)})$$

(see [6], *proof of Theorem 6.2).*

For Bernstein operators B_n we arrive at

$$|B_n(f;x)-f(x)-\frac{x(1-x)}{2n}\cdot f''(x)| \le \frac{x(1-x)}{2n}\cdot\tilde{\omega}(f'';\frac{n}{3}\frac{B_n(|e_1-x|^3;x)}{x(1-x)})$$

$$\le \frac{x(1-x)}{2n}\cdot\tilde{\omega}(f'';\frac{1}{3\sqrt{n}})$$

(see [6], Proposition 7.2). This is the example mentioned earlier. But we can do better as is shown in the next section.

5. Application to Bernstein-type operators

Theorem 5.1. *For $f\in C^2[0,1], x\in[0,1]$ and $n\in\mathbb{N}$ one has*

$$|n\cdot[B_n(f;x)-f(x)]-\frac{x(1-x)}{2}\cdot f''(x)| \le \frac{x(1-x)}{2}\cdot\tilde{\omega}(f'';\sqrt{\frac{1}{n^2}+\frac{x(1-x)}{n}}).$$

Proof. We discriminate two cases.

(i) $x\in[\frac{1}{n},1-\frac{1}{n}]$. In this situation we showed in Remark 9.4 of [6] that, using the Cauchy-Schwarz inequality,

$$\frac{B_n(|e_1-x|^3;x)}{B_n((e_1-x)^2;x)} \le \sqrt{\frac{B_n((e_1-x)^4;x}{B_n((e_1-x)^2;x)}} \le 2\cdot\sqrt{\frac{x(1-x)}{n}}.$$

(ii) $x \in [0, \frac{1}{n}] \cup [1 - \frac{1}{n}, 1]$. In Remark 7.3 of [6] it was proved that in this case

$$\frac{B_n(|e_1 - x|^3; x)}{B_n((e_1 - x)^2; x)} \leq \frac{3}{n}.$$

Thus for all $x \in [0, 1]$ we get

$$\frac{B_n(|e_1 - x|^3; x)}{B_n((e_1 - x)^2; x)} \leq 3 \cdot \sqrt{\frac{1}{n^2} + \frac{x(1-x)}{n}}$$

which, using also

$$B_n((e_1 - x)^2; x) = \frac{x(1-x)}{n},$$

gives the inequality in the theorem. □

Remark 5.2. *The above inequality can also be written as*

$$B_n(|e_1 - x|^3; x) \leq 3 \cdot \sqrt{\frac{1}{n^2} + \frac{x(1-x)}{n}} \cdot B_n((e_1 - x)^2; x).$$

This shows that - in this particular case - the absolute moment of "high" order 3 may be estimated by the product of a function vanishing uniformly of order $o(\frac{1}{\sqrt{n}})$ *and the moment of "low" order 2. It would be interesting to know if this can be proved for more general positive linear operators.* □

A similar improvement close to the endpoints 0 and 1 is also possible for the so-called "genuine Bernstein-Durrmeyer operators" defined by

$$U_n(f; x) = f(0) \cdot p_{n,0}(x) + f(1) \cdot p_{n,n}(x) + \sum_{k=1}^{n-1} p_{n,k}(x) \cdot \int_0^1 p_{n-2,k-1}(t) \cdot f(t)dt.$$

A survey on these and related operators was recently prepared by D. Kacsó [9]. For our purposes the information given about them in [6] and [7] will suffice. We obtain

Theorem 5.3. *For* $f \in C^2[0, 1], x \in [0, 1]$ *and* $n \in \mathbb{N}, n \geq 2$, *the following inequality holds*

$$|(n+1)[U_n(f; x) - f(x)] - x(1-x)f''(x)| \leq \frac{x(1-x)}{n+1} \cdot \tilde{\omega}(f''; 4 \cdot \sqrt{\frac{1}{(n+1)^2} + \frac{x(1-x)}{n+1}}).$$

Proof. Again we consider two cases.

(i) $x \in [\frac{1}{n}, 1-\frac{1}{n}]$. In Section 7 of [6] we noted that

$$U_n((e_1-x)^2;x) = \frac{2x(1-x)}{n+1},$$

$$U_n((e_1-x)^4;x) = \frac{12x^2(1-x)^2(n-7)+24x(1-x)}{(n+1)(n+2)(n+3)}.$$

From this we get

$$\frac{U_n(|e_1-x|^3;x)}{U_n((e_1-x)^2;x)} \le \sqrt{\frac{U_n((e_1-x)^4;x)}{U_n((e_1-x)^2;x)}} = \sqrt{\frac{6x(1-x)(n-7)+12}{(n+2)(n+3)}}.$$

For $n \ge 2$ and $x \in [\frac{1}{n}, 1-\frac{1}{n}]$

$$\frac{6x(1-x)(n-7)+12}{(n+2)(n+3)} \le \frac{18x(1-x)}{n+1}, \text{ so that}$$

$$\frac{U_n(|e_1-x|^3;x)}{U_n((e_1-x)^2;x)} \le \sqrt{\frac{18x(1-x)}{n+1}}.$$

(ii) $x \in [0, \frac{1}{n}] \cup [1-\frac{1}{n}, 1]$. We only consider the left interval, the second one can be treated analogously. In this case we write

$$U_n(|e_1-x|^3;x) = U_n(|e_1-x|^3;x) - U_n((e_1-x)^3;x) + U_n((e_1-x)^3;x)$$

$$= U_n(|e_1-x|^3-(e_1-x)^3;x) + U_n((e_1-x)^3;x)$$

$$= U_n(2\cdot(e_1-x)^2(x-e_1)_+;x) + U_n((e_1-x)^3;x).$$

Here $(x-e_1)_+ := \max\{0, x-e_1\}$.

From the definition of U_n it follows that

$$\begin{aligned} &U_n(|e_1-x|^3;x) \\ &= 2x^3\cdot p_{n,0}(x) + 2\sum_{k=1}^{n-1} p_{n,k}(x)\cdot\int_0^x p_{n-2,k-1}(t)\cdot(x-t)^3dt + U_n((e_1-x)^3;x) \\ &\le 2x^3\cdot(1-x)^n + \frac{2}{n^2}x(1-x)\cdot\sum_{k=1}^{n-1} p_{n,k}(x) + \frac{6x(1-x)(1-2x)}{(n+1)(n+2)}. \end{aligned}$$

The representation of $U_n((e_1-x)^3;x)$ can be found in [7].

The latter expression is bounded from above by

$$\frac{2}{n^2}x(1-x)\cdot[(1-x)^{n-1}+1+3(1-2x)]$$
$$\leq \frac{10}{n^2}x(1-x)\leq \frac{45}{2}\cdot\frac{x(1-x)}{(n+1)^2} \text{ for } n\geq 2.$$

Thus in this case we get

$$\frac{U_n(|e_1-x|^3;x)}{U_n((e_1-x)^2;x)}\leq \frac{45}{2}\cdot\frac{x(1-x)}{(n+1)^2}\cdot\frac{n+1}{2x(1-x)}\leq\frac{12}{n+1}.$$

Combining the two cases entails, for $x\in[0,1]$ and $n\geq 2$,

$$\frac{U_n(|e_1-x|^3;x)}{U_n(e_1-x)^2;x)}\leq 12\cdot\sqrt{\frac{1}{(n+1)^2}+\frac{x(1-x)}{n+1}},$$

and from this the theorem follows. □

6. **Concluding remarks**

Remark 6.1. *Here we make some further remarks concerning the case when $q\geq 3$ is odd. In this case the right hand side in the inequality of Theorem 3.1 is*

$$\frac{L(|e_1-x|^q;x)}{q!}\cdot\tilde{\omega}(f^{(q)};\frac{1}{q+1}\cdot\frac{L((e_1-x)^{q+1};x)}{L(|e_1-x|^q;x)}.$$

Furthermore we assume that $L(e_0;x)=1$. A Hölder-type inequality for positive linear operators (see [6], Theorem 5.1) then implies for $1\leq s<r$:

$$L(|e_1-x|^s;x)^{\frac{1}{s}}\leq L(|e_1-x|^r;x)^{\frac{1}{r}}.$$

Taking $s=q-1\geq 2$ and $r=q$ gives

$$L((e_1-x)^{q-1};x)^{\frac{1}{q-1}}\leq L(|e_1-x|^q;x)^{\frac{1}{q}}$$

or

$$L((e_1-x)^{q-1};x)^{\frac{q}{q-1}}\leq L(|e_1-x|^q;x).$$

Thus the left side in Theorem 3.1 is bounded from above by

$$\frac{L(|e_1-x|^q;x)}{q!}\cdot\tilde{\omega}\left(f^{(q)};\frac{1}{q+1}\cdot\frac{L((e_1-x)^{q+1};x)}{L((e_1-x)^{q-1};x)^{\frac{q}{q-1}}}\right).$$

Now the moments inside $\tilde{\omega}(f^{(q)};\cdot)$ are both of even order and can more easily be evaluated. The absolute moment in front of $\tilde{\omega}(f^{(q)};\cdot)$ can also be estimated using Hölder's inequality.

Our quantitative Voronovskaja-type theorem is not only applicable to polynomial operators as can be seen from the following

Example 6.2. *(see* [7]*) For variation-diminishing spline operators S_{Δ_n} giving piecewise linear interpolators at equidistant knots in* $[0,1]$ *the quantitative Voronovskaja theorem in case $q=2$ reads*

$$|\frac{n^2}{z_n(x)(1-z_n(x))}[S_{\Delta_n}(f;x)-f(x)]-\frac{1}{2}\cdot f''(x)|\leq\frac{1}{2}\cdot\tilde{\omega}(f'';\frac{1}{3n}).$$

This is (again) obtained via representations of the second and fourth central moments as given, for example, in Lupaş' Romanian Ph. D. thesis [11] *on p. 46:*

$$S_{\Delta_n}((e_1-x)^2;x)=\frac{1}{n^2}z_n(x)\cdot(1-z_n(x)),$$

$$S_{\Delta_n}((e_1-x)^4;x)=\frac{1}{n^2}z_n(x)(1-z_n(x))\cdot[1-3z_n(x)(1-z_n(x))].$$

Here $z_n(x)=\{nx\}:=nx-[nx]$ is the fractional part of nx.

Voronovskaja-type results are also known for other cases of Schoenberg's variation-diminishing spline operators. See the cited articles by Marsden, Riemenschneider and Schoenberg for non-quantitative versions. It would be of interest to find quantitative statements also for cases other than S_{Δ_n}.

Remark 6.3. *We noted before (see Theorem 1.1) that Voronovskaja's theorem is pointwise in nature, i.e., it does not only hold for functions $f\in C^2[0,1]$. One example is the negative "entropy function"*

$$f(x)=x\log x+(1-x)\log(1-x), x\in(0,1), f(0):=0, f(1):=0.$$

Here $f''(x)=[x(1-x)]^{-1}, x\in(0,1)$, so that the local version of Voronovskaja's theorem gives

$$\lim_{n\to\infty}n\cdot[B_n(f;x)-f(x)]=\frac{1}{2}, x\in(0,1),$$

while $B_n(f;0)-f(0)=0=B_n(f;1)-f(1), n\in\mathbb{N}$.

In [2] *several interesting phenomena concerning the approximation of the entropy function* f *by* $B_n f$ *are discussed.*

Moreover, Lupaş showed in [12] *that for this function one has*

$$\frac{x(1-x)}{2} \leq n[B_n(f;x) - f(x)] \leq \sqrt{2nx(1-x)};$$

see also the related question in [18].

Remark 6.4. *It was not easy to find out details about the life of Elizaveta Vladimirovna Voronovskaja who was born in 1898 or 1899 in Sankt Peterburg (Russia) and died in 1972, most likely in Leningrad (Soviet Union). Voronovskaja held university degrees in mathematics and history and was influenced in her mathematical work by S.N. Bernstein and V.I. Smirnov . Her main scientific achievement is - besides the famous 1932 paper on the asymptotic behavior of Bernstein polynomials - the monography "The functional method and its applications" which was published in Russian in 1963 and in English in 1970. Since 1946 she was the chairperson of the department of higher mathematics in the Leningrad Institute of Aerospace Instrumentation (now St. Petersburg State University of Aerospace Instrumentation). The last years of her life she also worked as a chairperson, now in the department of higher mathematics in the Leningrad Institute of Telecommunications (now St. Petersburg State University of Telecommunications).*

This and more information on E.V. Voronovskaja including a photograph can be found at the following Russian internet pages (operative on April 20, 2007):

`http://www.spbstu.ru/public/m_v/N_002/Yarv/Voronovskaya.html`

`http://www.spbstu.ru/phmech/math/persons/HISTORY/Voronovskaia_E_V.html`

We are grateful to Vitalii Arestov (Ekaterinburg), Elena Berdysheva (Stuttgart-Hohenheim) and Andriy Bondarenko (Kiev) for helping us to locate these pages.

Acknowledgement. The author gratefully acknowledges the technical assistance of Ms. Bärbel Schmitz during the preparation of this note.

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