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COMPLETENESS WITH RESPECT TO THE PROBABILISTIC POMPEIU-HAUSDORFF METRIC

ŞTEFAN COBZAŞ

Dedicated to Professor Petru Blaga at his 60^{th} *anniversary*

Abstract. The aim of the present paper is to prove that the family of all closed nonempty subsets of a complete probabilistic metric space L is complete with respect to the probabilistic Pompeiu-Hausdorff metric H. The same is true for the families of all closed bounded, respectively compact, nonempty subsets of L. If L is a complete random normed space in the sense of Šerstnev, then the family of all nonempty closed convex subsets of L is also complete with respect to H. The probabilistic Pompeiu-Hausdorff metric was defined and studied by R.J. Egbert, Pacific J. Math. **24** (1968), 437-455, in the case of Menger probabilistic metric spaces, and by R.M. Tardiff, Pacific J. Math. **65** (1976), 233-251, in general probabilistic metric spaces. The completeness with respect to probabilistic Pompeiu-Hausdorff metric of the space of all closed bounded nonempty subsets of some Menger probabilistic metric spaces was proved by J. Kolumbán and A. Soós, Studia Univ. Babes-Bolyai, Mathematica, **43** (1998), no. 2, 39-48, and **46** (2001), no. 3, 49-66.

1. Introduction

The study of probabilistic metric spaces (PM spaces for short) was initiated by K. Menger [17] and A. Wald [28], in connection with some measurements problems in physics. The positive number expressing the distance between two points p, q of a metric space is replaced by a distribution function (in the sense of probability theory)

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$F_{p,q} : \overline{\mathbb{R}} \to [0,1]$, whose value $F_{p,q}(x)$ at the point $x \in \mathbb{R}$ can be interpreted as the probability that the distance between p and q be less than x. Since then the subject developed in various directions, an important one being that of fixed points in PM spaces. Important contributions to the subject have been done by A.N. Šerstnev and the Kazan school of probability theory, see [21, 22, 23, 24] and the bibliography in [19].

A clear and thorough presentation of the results up to 1983 is given in the book by B. Schweizer and A. Sklar [19]. Beside this book, at the present there are several others dealing with various aspects of analysis in probabilistic metric spaces and in probabilistic normed spaces - V. Istrăţescu [11], I. Istrăţescu and Gh. Constantin [4, 5], V. Radu [18], S.-S. Chang and Y. J. Cho [3], O. Hadžić [8], O. Hadžić and E. Pap [9]. In the present paper we shall follow the treatise [19].

The probabilistic Pompeiu-Hausdorff metric on the family of nonempty closed subsets of a PM space was defined by Egbert [6] in the case of Menger PM spaces, and by Tardiff [27] in general PM spaces (see also [19, §12.9]), by analogy with the classical case. Sempi [20] used the probabilistic Pompeiu-Hausdorff metric to prove the existence of a completion of a PM space. Some results have been obtained also by Beg and Ali [2].

As it is well known, the family of nonempty closed bounded subsets of a complete metric space is complete with respect to the Pompeiu-Hausdorff distance (see, e.g., [10, Chapter 1]). The aim of the present paper is to prove the probabilistic analog of this result for the family of all nonempty closed subsets of a probabilistic metric space. We shall prove that the families of all nonempty closed bounded, respectively compact, subsets of a complete probabilistic metric space L are also complete with respect to the probabilistic Pompeiu-Hausdorff metric. If L is a complete random normed space in the sense of Šerstnev, then the family of all nonempty closed convex subsets of L is complete with respect to the Pompeiu-Hausdorff metric too. In the case of Menger PM spaces (L, ρ, Min), and (L, ρ, W), with t-norms $\mathrm{Min}(s,t) = \min\{s,t\}$, $s,t \in [0,1]$, respectively $W(s,t) = \max\{s+t-1, 0\}$, the completeness of the space of all closed bounded nonempty subsets of L with respect to

the probabilistic Pompeiu-Hausdorff metric was proved by Kolumbán and Soós in [13] and [14]. In the case of a Menger PM space (L, ρ, Min), they proved also in [13] the completeness of the family of all compact nonempty subsets of L. These completeness results were applied in [13, 14, 15] to prove the existence of invariant sets for finite families of contractions in PM spaces of random variables (E-spaces in the sense of Sherwood [25], or [19, Ch. 9, Sect. 1]).

As in Aubin's book [1], I have adopted the term Pompeiu-Hausdorff metric. For a short comment on this fact, as well as on the similar case of the Painlevé-Kuratowski convergence for sequences of sets, see [1, page xiv].

2. **Preliminary notions**

Denote by Δ the set of *distribution functions*, meaning nondecreasing, left continuous functions $F : \overline{\mathbb{R}} \to [0, 1]$ with $F(-\infty) = 0$ and $F(\infty) = 1$. Let D be the subclass of Δ formed by all functions $F \in \Delta$ such that

$$\lim_{x \to -\infty} F(x) = 0 \quad \text{and} \quad \lim_{x \to \infty} F(x) = 1.$$

The *weak convergence* of a sequence (F_n) in Δ to $F \in \Delta$, denoted by $F_n \xrightarrow{w} F$, means that the equality

$$\lim_{n \to \infty} F_n(x) = F(x) \tag{2.1}$$

holds for every continuity point x of F. Since F is non-decreasing the set of its discontinuity points is at most countable, so that the set of continuity points of F is dense in $\mathbb{R}$. In order that $F_n \xrightarrow{w} F$ it is sufficient that the relation (2.1) holds for every x in an arbitrary dense subset of $\mathbb{R}$. An important result concerning weak convergence of distribution functions is Helly's First Theorem: every sequence in Δ contains a weakly convergent subsequence (see Loève [16, Sect. 11.2]).

The topology of weak convergence in Δ is metrizable. The first who realized this was P. Lévy (see the Appendix to Fréchet's book [7]), and for this reason the metrics generating the weak convergence in Δ are called Lévy metrics. Since the original Lévy metric characterizes the weak convergence only in D, Sibley [26] proposed a modification of Lévy metric that generates the weak convergence in Δ. We shall work

with a further modification proposed by Schweizer and Sklar [19] and denoted by d_L. The distance $d_L(F,G)$ between two functions $F,G \in \Delta$ is defined as the infimum of all numbers $h>0$ such that the inequalities

$$F(x-h)-h \le G(x) \le F(x+h)+h$$

and

$$G(x-h)-h \le F(x) \le G(x+h)+h$$

hold for every $x \in (-h^{-1};h^{-1})$. One shows that d_L is a metric on Δ and, for any sequence (F_n) in Δ and $F \in \Delta$, we have

$$F_n \xrightarrow{w} F \iff d_L(F_n,F) \to 0.$$

By Helly's First Theorem the space (Δ, d_L) is compact, hence complete (see [19, §4.2]).

The sets of *distance functions* are:

$$\Delta^+ = \{F \in \Delta : F(0)=0\} \quad \text{and} \quad D^+ = D \cap \Delta^+.$$

It follows that for $F \in \Delta^+$ we have $F(x)=0$, $\forall x \le 0$. The set Δ^+ is closed in the metric space Δ, hence compact and complete too.

Two important distance functions are

$$\epsilon_0(x)=0 \quad \text{for} \quad x \le 0 \qquad \text{and} \qquad \epsilon_\infty(x)=0 \quad \text{for} \quad x<\infty$$
$$=1 \quad \text{for} \quad x>0 \qquad\qquad\qquad\qquad =1 \quad \text{for} \quad x=\infty$$

The order in Δ^+ is defined as the punctual order: for $F,G \in \Delta^+$ we put

$$F \le G \iff \forall x>0 \;\; F(x) \le G(x).$$

It follows that ϵ_0 is the maximal element of Δ^+ and of D^+ as well, and ϵ_∞ is the minimal element of Δ^+.

In the following we shall define some functions, say F, on $\mathbb{R}$ and consider them automatically extended to $\overline{\mathbb{R}}$ by $F(-\infty)=0$ and $F(\infty)=1$.

If $\{F_i : i \in I\}$ is a family of functions in Δ^+ then the function $F : \overline{\mathbb{R}} \to [0,1]$ defined by

$$F(x) = \sup\{F_i(x) : i \in I\},\ x \in \mathbb{R},$$

is the supremum of the family $\{F_i\}$ in the ordered set $(\Delta^+, \leq)$: $F = \sup_{i\in I} F_i$.

To define the infimum of the family $\{F_i\}$ put

$$\Gamma(x) = \inf\{F_i(x) : i \in I\},\ x \in \mathbb{R}. \tag{2.2}$$

Since the function Γ is nondecreasing, but not necessarily left continuous on $\mathbb{R}$, we have to regularize it by taking the left limit

$$G(x) = \ell^-\Gamma(x) := \lim_{x' \nearrow x} \Gamma(x') = \sup_{x'<x} \Gamma(x'),\ x \in \mathbb{R}. \tag{2.3}$$

Then $G(x) \leq \Gamma(x),\ \forall x \in R$, the function G belongs to Δ^+ and $G = \inf_{i\in I} F_i$ - the infimum of the family $\{F_i\}$ in the ordered set $(\Delta^+, \leq)$.

A *triangle function* is a binary operation τ on $\Delta^+,\ \tau : \Delta^+ \times \Delta^+ \to \Delta^+,$ that is commutative, associative, non-decreasing in each place ($\tau(F_1, G_1) \leq \tau(F_2, G_2)$, if $F_1 \leq F_2$ and $G_1 \leq G_2$), and has ϵ_0 as identity: $\tau(F, \epsilon_0) = F,\ F \in \Delta^+$. The triangle function τ is called continuous if it is continuous with respect to the d_L-topology of Δ^+. It follows that τ is, in fact, uniformly continuous, since the metric space (Δ^+, d_L) is compact.

3. **Probabilistic metric spaces**

A *probabilistic metric space* (PM space) is a triple (L, ρ, τ), where L is a set, ρ is a mapping from $L \times L$ to Δ^+, and τ is a continuous triangle function. The value of ρ at $(p, q) \in L \times L$ is denoted by F_{pq}, i.e., $\rho(p, q) = F_{pq}$.

One supposes that the following conditions are satisfied for all $p, q, r \in L$:

(PM1) $F_{pp} = \epsilon_0$,

(PM2) $F_{pq} = \epsilon_0 \ \Rightarrow\ p = q$,

(PM3) $F_{pq} = F_{qp}$,

(PM4) $F_{pr} \geq \tau(F_{pq}, F_{qr})$.

The mapping ρ is called the *probabilistic metric* on L and the condition (PM4) is the probabilistic analogue of the triangle inequality.

The *strong topology* on a PM space is defined by the neighborhood system:

$$U_t(p) = \{q \in L : F_{pq}(t) > 1 - t\}, \quad t > 0. \tag{3.1}$$

Putting

$$\bar{U}_t(p) = \{q \in L : F_p(t) \geq 1 - t\} \tag{3.2}$$

we have $U_t(p) \subset \bar{U}_t(p)$ and $\bar{U}_{t'}(p) \subset U_t(p)$ for $t' < t$, showing that the family (3.2) of subsets of L forms also a neighborhood base for the strong topology of L.

Observe that $U_t(p) = L$, for $t > 1$, and $\bar{U}_t(p) = L$, for $t \geq 1$, so that we can restrict to $t \in (0, 1)$ when working with strong neighborhoods. In fact, we can suppose that t is as small as we need.

The strong topology on a PM space (L, ρ, τ) is derived from the uniformity $\mathcal{U}$ generated by the vicinities:

$$U_t = \{(p, q) \in L \times L : F_{pq}(t) > 1 - t\}, \quad t > 0. \tag{3.3}$$

The strong topology is metrizable since $\{U_{1/n} : n \in \mathbb{N}\}$ is a countable base for the uniformity $\mathcal{U}$. The probabilistic metric ρ is uniformly continuous mapping from $L \times L$ with the product topology to (Δ^+, d_L), meaning that

$$p_n \to p \text{ and } q_n \to q \text{ in } L \;\Rightarrow\; F_{p_n q_n} \xrightarrow{w} F_{pq}. \tag{3.4}$$

The convergence of a sequence (p_n) in L to $p \in L$ is characterized by

$$\begin{aligned} p_n \to p &\iff \forall t > 0\ \exists n_0\ \forall n \geq n_0 \quad p_n \in U_t(p) \\ &\iff F_{p_n p} \xrightarrow{w} \epsilon_0 \\ &\iff d_L(F_{p_n p}, \epsilon_0) \to 0. \end{aligned}$$

A sequence (p_n) in L is called a *Cauchy sequence*, or *fundamental*, if

$$F_{p_n p_m} \xrightarrow{w} \epsilon_0 \quad \text{for} \quad n, m \to \infty,$$

or, equivalently,

$$\forall t > 0\ \exists n_0 \text{ such that } \forall n, m \geq n_0\ \ (p_n, p_m) \in U_t\ (\iff F_{p_n p_m}(t) > 1 - t).$$

A convergent sequence in L is a Cauchy sequence, and the PM space L is called *complete* (with respect to the strong topology) if every Cauchy sequence is convergent.

For these and other questions concerning the strong topology of a PM space, see [19, Chapter 12].

Throughout this paper all the topological notions concerning a PM space will be considered with respect to the strong topology.

4. **The probabilistic Pompeiu-Hausdorff metric**

For a metric space (X, d), two nonempty bounded subsets A, B of X and a point $p \in X$, one introduces the following notations and notions :

$$d(p, B) = \inf\{d(p, q) : q \in B\}\ -\ \text{the distance from } p \text{ to } B,$$

$$h^*(A, B) = \sup\{d(p, B) : p \in A\}\ -\ \text{the excess of } A \text{ over } B,$$

and let

$$h(A, B) = \max\{h^*(A, B), h^*(B, A)\}$$

be the Pompeiu-Hausdorff distance between the sets A, B.

Denoting by $P_{fb}(X)$ the family of all nonempty closed bounded subset of X it follows that h is a metric on $P_{fb}(X)$, and the metric space $(P_{fb}(X), h)$ is complete if (X, d) is complete (see, e.g., [10, Chapter 1]).

In the case of a PM space (L, ρ, τ) the definitions are similar but, taking into account the fact that the probabilistic triangle inequality (PM4) is written in reversed form with respect to the usual triangle inequality, sup and inf will change their places.

For two nonempty subsets A, B of L and $p \in L$ denote by

$$F_{pB} = \sup\{F_{pq} : q \in B\} \iff F_{pB}(x) = \sup\{F_{pq}(x) : q \in B\},\ x \in \mathbb{R}, \tag{4.1}$$

the *probabilistic distance* from p to B, and let

$$F^*_{AB} = \inf\{F_{pB} : p \in A\}. \tag{4.2}$$

Taking into account the formulae (2.2) and (2.3), it follows

$$F^*_{AB} = \ell^- \Gamma^*_{AB},$$

where

$$\Gamma^*_{AB}(x) = \inf\{F_{pB}(x) : p \in A\},\ x \in \mathbb{R}.$$

The *probabilistic Pompeiu-Hausdorff distance* between the sets A, B is defined by

$H(A,B) = F_{AB}$, where

$$F_{AB}(x) = \min\{F^*_{AB}(x), F^*_{BA}(x)\},\ \ x \in \mathbb{R}. \tag{4.3}$$

The probabilistic Pompeiu-Hausdorff metric was defined and studied by Egbert [6] in the case of Menger PM spaces and by Tardiff [27] in general PM spaces (see also [19, §12.9]). The mapping $H(A,B) = F_{AB}$ satisfies the following properties, where cl denotes the closure with respect to the strong topology:

Proposition 4.1. ([19, Th. 12.9.2])

1. $F_{\{p\}\{q\}} = F_{pq}$ *for* $p, q \in L$;

2. For nonempty $A, B \subset L$, $F_{AB} = F_{BA}$, $F_{AB} = F_{\mathrm{cl}(A)\,\mathrm{cl}(B)}$, *and* $F_{AB} = \epsilon_0$ *if and only if* $\mathrm{cl}(A) = \mathrm{cl}(B)$.

In order that H satisfy the probabilistic triangle inequality (PM4), we have to impose a supplementary condition on the triangle function τ. The triangle function τ is called *sup-continuous* if

$$\tau(\sup_{i\in I} F_i, G) = \sup_{i\in I} \tau(F_i, G) \tag{4.4}$$

for any family $\{F_i : i \in I\} \subset \Delta^+$ of distance functions and any $G \in \Delta^+$.

Denote by $P_f(L)$ the family of all nonempty closed subsets of a PM space (L, ρ, τ).

Theorem 4.2. ([19, Th. 12.9.5]) *If the triangle function τ is sup-continuous then the mapping $H(A,B) = F_{AB}$, where F_{AB} is defined by* (4.3)*, is a probabilistic metric on $P_f(L)$.*

In the following proposition we collect some properties which will be used in the proof of the completeness of $P_f(L)$ with respect to the probabilistic Pompeiu-Hausdorff metric.

Proposition 4.3. *Let (L, ρ, τ) be a PM space with sup-continuous triangle function τ, and let $A, B \in P_f(L)$ and $p \in L$. Then*

1. $F_{pB} \geq \tau\left(F_{pA}, F^*_{AB}\right)$;

and

2. $F_{pB} \geq \Gamma^*_{AB} \geq F^*_{AB} \geq F_{AB}$.
3. *If $F_{AB}(s) > 1 - s$ for some s, $0 < s < 1$, then*

$$\forall p \in A \; \exists q \in B \text{ such that } F_{pq}(s) > 1 - s, \tag{4.5}$$

and

$$\forall q \in B \; \exists p \in A \text{ such that } F_{pq}(s) > 1 - s. \tag{4.6}$$

Proof. For $x \in \mathbb{R}$ we have

$$\forall a \in A \; \forall b \in B \quad F_{pB}(x) \geq F_{pb}(x) \geq \tau(F_{pa}F_{ab})(x).$$

Taking the supremum with respect to $b \in B$ and taking in account that τ is sup continuous and monotonic in each place, we get

$$\forall a \in A \;\; F_{pB}(x) \geq \tau(F_{pa}, F_{aB})(x) \geq \tau(F_{pA}, F^*_{AB})(x).$$

Taking now the supremum with respect to $a \in A$ one obtains the inequality 1.

The inequalities 2 are immediate from definitions.

To prove 3, observe that

$$F_{AB}(s) > 1 - s \iff F^*_{AB}(s) > 1 - s \text{ and } F^*_{BA}(s) > 1 - s.$$

It follows

$$\inf\{F_{p'B}(s) : p' \in A\} = \Gamma^*(s) \geq F^*_{AB}(s) > 1 - s,$$

so that

$$\sup\{F_{pq}(s) : q \in B\} > 1 - s,$$

implying (4.5).

The inequality (4.6) can be proved similarly. □

The completeness result will be obtained under a further restriction imposed to τ. We say that the triangle function τ satisfies the condition (W) if

$$\text{(W)} \qquad F(x) > \alpha \text{ and } G(x) > \beta \;\Rightarrow\; \tau(F,G)(x) > \max\{\alpha + \beta - 1, 0\},$$

for all $x > 0$, where $F, G \in \Delta^+$, and $\alpha, \beta \in \mathbb{R}$.

Remark. Considering the t-norm

$$W(x,y) = \max\{x + y - 1, 0\}, \quad (x,y) \in [0,1]^2,$$

(see [19, p. 5]) and the associated triangle function $\mathbf{W}$, defined for $F, G \in \Delta^+$ by

$$\mathbf{W}(F,G)(x) = W(F(x), G(x)),\ x \in \mathbb{R},$$

(see [19, p. 97]), the condition (W) essentially means that $\tau \geq \mathbf{W}$.

Now we are ready to state and prove the completeness result.

Theorem 4.4. *Let (L, ρ, τ) be a PM space with sup-continuous triangle function τ satisfying the condition (W).*

If the PM space L is complete then the space $P_f(L)$ is complete with respect to the probabilistic Pompeiu-Hausdorff metric.

Proof. Let (A_n) be a sequence in $P_f(L)$ that is fundamental with respect to the probabilistic Pompeiu-Hausdorff metric H.

Put

$$A = \bigcap_{n \geq 1} \operatorname{cl}\left(\bigcup_{m \geq n} A_m\right),$$

and show that $A \in P_f(L)$ (meaning that $A \subset L$ is nonempty closed) and that the sequence (A_n) converges to A with respect to the probabilistic Pompeiu-Hausdorff metric H.

Observe that

$$p \in A \quad \Longleftrightarrow \quad \exists\ n_1 < n_2 < ...\ \exists p_k \in A_{n_k} : \quad p_k \to p. \tag{4.7}$$

For $0 < t < 1/2$ fixed, choose $n_0 \in \mathbb{N}$ such that

$$\forall n, m \geq n_0 \quad F_{A_n A_m}(t) > 1 - t.$$

For $m \geq n_0$ fixed, put $n_1 := m$ and pick an element $p_1 \in A_{n_1}$.

Let now $n_2 > n_1$ be such that

$$\forall n, n' \geq n_2 \quad F_{A_n A_{n'}}(\frac{t}{2}) > 1 - \frac{t}{2}.$$

The inequalities

$$F^*_{A_{n_1} A_{n_2}}(t) \geq F_{A_{n_1} A_{n_2}}(t) > 1 - t$$

and the fact that p_1 belongs to A_{n_1} imply $F_{p_1 A_{n_2}}(t) > 1 - t$, so that there exists $p_2 \in A_{n_2}$ such that

$$F_{p_1 p_2}(t) > 1 - t.$$

Take now $n_3 > n_2$ such that

$$\forall n, n' \geq n_3 \quad F_{A_n A_{n'}}(\frac{t}{2^2}) > 1 - \frac{t}{2^2}.$$

Reasoning like above, we can find an element $p_3 \in A_{n_3}$ such that

$$F_{p_2 p_3}(\frac{t}{2}) > 1 - \frac{t}{2}.$$

Continuing in this way, we obtain a strictly increasing sequence of indices $n_1 < n_2 < ...$ and the elements $p_k \in A_{n_k},\ k \in \mathbb{N},$ such that

$$F_{p_k p_{k+1}}(\frac{t}{2^{k-1}}) > 1 - \frac{t}{2^{k-1}}, \tag{4.8}$$

for all $k \in \mathbb{N}$.

Claim I. $\quad \forall i \in \mathbb{N}\ \forall k \in \mathbb{N} \quad F_{p_k p_{k+i}}(\frac{t}{2^{k-1}}) > 1 - (\frac{1}{2^{k-1}} + \frac{1}{2^k} + ... + \frac{1}{2^{k+i-2}})t.$

We proceed by induction on i. For $i = 1$ the assertion is true by the choice of the elements p_k (see (4.8)).

Suppose that the assertion is true for i and prove it for $i+1$. Appealing to condition (W) we have

$$F_{p_kp_{k+i+1}}(\frac{t}{2^{k-1}}) \geq \tau\left(F_{p_kp_{k+1}}, F_{p_{k+1}p_{k+i+1}}\right)(\frac{t}{2^{k-1}}) > 1-(\frac{1}{2^{k-1}}+\frac{1}{2^k}+...+\frac{1}{2^{k+i-1}})t,$$

since $F_{p_kp_{k+1}}(\frac{t}{2^{k-1}}) > 1-\frac{1}{2^{k-1}}$ and, by the induction hypothesis,

$$F_{p_{k+1}p_{k+i+1}}(\frac{t}{2^{k-1}}) \geq F_{p_{k+1}p_{k+i+1}}(\frac{t}{2^k}) > 1-(\frac{1}{2^k}+...+\frac{1}{2^{k+i-1}})t.$$

Claim II. The sequence (p_k) is fundamental in the PM space L.

For $0<s<1$ choose $k_0\in\mathbb{N}$ such that $2^{-k_0+1}<s$. Then for any $k\geq k_0$ and arbitrary $i\in\mathbb{N}$ we have

$$F_{p_kp_{k+i}}(s) \geq F_{p_kp_{k+i}}(\frac{t}{2^{k-1}}) > 1-(\frac{t}{2^{k-1}}+...+\frac{t}{2^{k+i-1}}) > 1-\frac{t}{2^k} > 1-s.$$

Since the PM space L is complete, there exists $p\in L$ such that $p_k\to p$ in the strong topology of L. The choice of the elements p_k and (4.7) yield $p\in A$. Since the set A is obviously closed it follows $A\in P_f(L)$.

By Claim I we have

$$F_{p_1p_k}(t) > 1-(1+\frac{1}{2}+...+\frac{1}{2^{k-2}})t > 1-2t.$$

Let now t', $t<t'<2t$, be a continuity point of the distribution function F_{p_1p}. The continuity of the distance function (see (3.4)) and the inequalities

$$F_{p_1p_k}(t') \geq F_{p_1p_k}(t) > 1-2t$$

yield, for $k\to\infty$, $F_{p_1p}(t')\geq 1-2t$, so that

$$F_{p_1A}(t') = \sup_{q\in A} F_{p_1q}(t') \geq 1-2t.$$

As p_1 was arbitrarily chosen in A_m, it follows

$$\Gamma^*_{A_mA}(t') = \inf\{F_{p'A}(t') : p'\in A_m\} \geq 1-2t.$$

But then

$$F^*_{A_m A}(2t) = \sup_{t'} \Gamma^*_{A_m A}(t') \geq 1 - 2t,$$

where the supremum is taken over all continuity points t' of the function $F_{p_1 p}$ lying in the interval $(t, 2t)$. The fact that the set of these points is dense in the interval $(t, 2t)$ justifies the equality sign in the first of the above relations.

Taking into account that $m \geq n_0$ was arbitrarily chosen too, we finally obtain

$$\forall m \geq n_0 \quad F^*_{A_m A}(2t) \geq 1 - 2t, \tag{4.9}$$

Let now $p \in A$ and let $n_1 < n_2 < ...$ and $p_k \in A_{n_k}$ be such that $p_k \to p$ in the strong topology of the PM space L.

Choose $k_0 \in \mathbb{N}$ such that

$$\forall k \geq k_0 \quad F_{p_k p}(t) > 1 - t.$$

Proposition 4.3, the inequality $F_{pA_{n_{k_0}}} \geq F_{pp_{k_0}}$, and condition (W) give, for any $t',\ t < t' < 2t$,

$$F_{pA_m}(t') \geq F_{pA_m}(t) \geq \tau\left(F_{pA_{n_{k_0}}}, F^*_{A_{n_{k_0}} A_m}\right)(t) \geq \tau\left(F_{pp_{k_0}}, F^*_{A_{n_{k_0}} A_m}\right)(t) > 1 - 2t.$$

Since $p \in A$ was arbitrarily chosen, it follows

$$\forall t',\ t < t' < 2t, \quad \Gamma^*_{AA_m}(t') \geq 1 - 2t,$$

so that

$$\forall m \geq n_0 \quad F^*_{AA_m}(2t) \geq 1 - 2t. \tag{4.10}$$

The inequalities (4.9) and (4.10) yield

$$\forall m \geq n_0 \quad H(A_m, A)(2t) = F_{A_m A}(2t) \geq 1 - 2t,$$

i.e., the sequence (A_m) converges to A with respect to the probabilistic Pompeiu-Hausdorff metric H.

The proof of the completeness is complete. □

The *diameter* of a subset A of a PM space (L, ρ, τ) is defined by

$$D_A(t) = \ell^- \Phi_A(t)$$

where

$$\Phi_A(t) = \inf\{F_{pp'}(t) : p, p' \in A\}.$$

The set A is called *bounded* if $D_A \in D^+$, i.e. $\sup\{D_A(t) : t > 0\} = 1$ (see [19, pages 200-201]). This is equivalent to

$$\sup\{\Phi_A(t) : t > 0\} = 1. \tag{4.11}$$

Now we shall show that the families $P_{fb}(L)$ and $P_k(L)$ of all closed bounded nonempty subsets of a PM space L, respectively of all nonempty compact subsets of L, are complete in $P_f(L)$ with respect to the Pompeiu-Hausdorff metric, provided the PM space L is complete. To prove the assertion concerning the compact sets, we shall use the characterization of compactness in uniform spaces in terms of total boundedness (see [12, Ch. 6]). Let $(X,\mathcal{U})$ be a uniform space. For $U \in \mathcal{U}$ and a subset A of X put $U(A) = \{x \in X : \exists y \in A \text{ such that } (x, y) \in U\}$. It follows that $U(x) = U(\{x\})$ is a neighborhood of x and $\{U(x) : U \in \mathcal{U}\}$ forms a neighborhood base at x. A subset Y of X is called *totally bounded* if for every $U \in \mathcal{U}$ there exists a finite subset Z of X such that $Y \subset U(Z)$. Then a subset of a uniform space $(X,\mathcal{U})$ is compact if and only if it is complete and totally bounded ([12, Ch. 6, Th. 32]). If L is a PM space then, considering L as a uniform space with respect to the uniformity generated by the vicinities (3.3), denote by $P_{ftb}(L)$ the family of all nonempty, closed and totally bounded subsets of L.

Theorem 4.5. *If (L, ρ, τ) is a PM space with sup-continuous triangle function τ satisfying the condition (W), then the subspaces $P_{fb}(L)$ and $P_{ftb}(L)$ are closed in $P_f(L)$.*

Consequently, if the PM space L is complete then the subspaces $P_{fb}(L)$ and $P_k(L)$ are complete with respect to the probabilistic Pompeiu-Hausdorff metric.

Proof. Let (A_n) be a sequence of closed bounded nonempty sets converging to $A \in P_f(L)$ with respect to probabilistic Pompeiu-Hausdorff metric H. We have to show that A is bounded too, i.e. that

$$\sup\{\Phi_A(t) : t > 0\} = 1. \tag{4.12}$$

Let $0 < \epsilon < 1/3$ and let $m \in \mathbb{N}$ be such that

$$\forall n \geq m \quad F_{AA_n}(\epsilon) > 1 - \epsilon. \tag{4.13}$$

Since $\sup\{\Phi_{A_m}(t) : t > 0\} = 1$ there exists $t > 0$ such that $\Phi_{A_m}(t) > 1 - \epsilon$, so that

$$\forall q, q' \in A_m \quad F_{qq'}(t) > 1 - \epsilon. \tag{4.14}$$

We can suppose also that $t \geq \epsilon$. By (4.13) and (4.5), for any $p, p' \in A$ there exist $q, q' \in A_m$ such that

$$F_{pq}(\epsilon) > 1 - \epsilon \quad \text{and} \quad F_{p'q'}(\epsilon) > 1 - \epsilon. \tag{4.15}$$

Since $t \geq \epsilon$ we have $F_{pq}(t) \geq F_{pq}(\epsilon) > 1 - \epsilon$ and $F_{p'q'}(t) \geq F_{p'q'}(\epsilon) > 1 - \epsilon$, so that, by (4.14) and condition (W), we have

$$F_{qp'}(t) \geq \tau(F_{qq'}, F_{q'p'})(t) > 1 - 2e,$$

and

$$F_{pp'}(t) \geq \tau(F_{pq}, F_{qp'})(t) > 1 - 3\epsilon.$$

We have proved that for any $\epsilon,\ 0 < \epsilon < 1/3,$ there exists $t > 0$ such that $F_{pp'}(t) > 1 - \epsilon$ for all $p, p' \in A$. It follows $\Phi_A(t) \geq 1 - 3\epsilon,$ so that (4.12) holds.

Suppose now that (A_n) is a sequence of nonempty compact subsets of L converging with respect to the probabilistic Pompeiu-Hausdorff metric H to a set $A \in P_f(L)$. We shall show that A is totally bounded with respect to the uniformity having as vicinities the sets U_t given by (3.3).

Let $0 < \epsilon < 1/2$ and let $n \in \mathbb{N}$ be such that $F_{AA_n}(\epsilon) > 1 - \epsilon$. By (4.5) it follows

$$\forall p \in A\ \exists q \in A_n \text{ such that } F_{pq}(\epsilon) > 1 - \epsilon. \tag{4.16}$$

Now, since the set A_n is totally bounded, there exits a finite set $Z \subset L$ such that

$$\forall q \in A_n \ \exists z \in Z \text{ such that } F_{qz}(\epsilon) > 1 - \epsilon. \tag{4.17}$$

For an arbitrary $p \in A$ choose first an element $q \in A_n$ according to (4.16) and then, for this q select $z \in Z$ according to (4.17). Taking into account the condition (W) we get

$$F_{pq}(\epsilon) \geq \tau(F_{pq}, F_{qz})(\epsilon) > \max\{1 - 2\epsilon, 0\} = 1 - 2\epsilon.$$

It follows $A \subset U_{2\epsilon}(Z)$, i.e. the set A is totally bounded.

Now, if the PM space L is complete and A is closed in L, it follows that A is complete too, hence compact, as complete and totally bounded. □

Remark. As we have yet mentioned, in the case of Menger PM spaces (L, ρ, Min), and (L, ρ, W), the completeness of the space of all closed bounded subsets of L was proved by Kolumbán and Soós in [13] and [14], respectively. Since $\mathrm{Min} \geq W$, both of these results are contained in the above completeness result. The completeness of $P_k(L)$ in the case of a Menger PM space (L, ρ, Min) was proved in [13].

For a subset A of a PM space (L, ρ, τ) and $0 < \epsilon \leq 1$ let

$$A_\epsilon = \{q \in L : \exists p \in A \ F_{pq}(\epsilon) > 1 - \epsilon\} = \bigcup \{U_\epsilon(p) : p \in A\}.$$

As in the case of ordinary metric spaces we have:

Proposition 4.6. *(i)* $\mathrm{cl}\, A = \bigcap_{\epsilon > 0} A_\epsilon$

If τ satisfies (W) then

(ii) $A \subset B_\epsilon \ \Rightarrow \ \mathrm{cl}\, A \subset B_{2\epsilon}$.

Proof. Let $q \in \mathrm{cl}\, A$ and $\epsilon > 0$. Choosing $p \in U_\epsilon(q) \cap A$ it follows

$$q \in U_\epsilon(p) \subset A_\epsilon.$$

i.e. $\mathrm{cl}\, A \subset \cap_\epsilon A_\epsilon$. To prove the reverse inclusion we shall show that

$$\cap_{n \geq 1} A_{1/n} \subset \mathrm{cl}\, A.$$

If $q \in \cap_{n\geq 1} A_{1/n}$ then

$$\forall n \; \exists p_n \in A \text{ such that } F_{pp_n} > 1 - \frac{1}{n},$$

which implies that (p_n) converges to p in the strong topology of the PM space L, i.e. $p \in \operatorname{cl} A$.

To prove (ii), let $p \in \operatorname{cl} A$. It follows $U_\epsilon(p) \cap A \neq \emptyset$, so that $F_{pq}(\epsilon) > 1 - \epsilon$, for some $q \in A$.

Since $A \subset B_\epsilon$ it follows $F_{qr}(\epsilon) > 1 - \epsilon$, for some $r \in B$. But then, taking into account the condition (W), we have for $0 < \epsilon \leq 1/2$

$$F_{pr}(\epsilon) \geq \tau(F_{pq}, F_{qr})(\epsilon) > \max\{1 - 2\epsilon, 0\} = 1 - 2\epsilon,$$

showing that $p \in B_{2\epsilon}$. If $\epsilon > 1/2$ then $B_{2\epsilon} = L$. □

In the following proposition we give two expressions for the probabilistic Pompeiu-Hausdorff limit of a sequence of sets in $P_f(L)$, inspired by a well known result for the usual Pompeiu-Hausdorff metric (see [10, Proposition 1.3]).

Proposition 4.7. *Let (L, ρ, τ) be a PM space with sup-continuous triangle function τ satisfying the condition (W). If (A_n) is sequence in $P_f(L)$ converging to $A \in P_f(L)$ with respect to the probabilistic Pompeiu-Hausdorff metric H then*

$$A = \bigcap_{n\geq 1} \operatorname{cl}\left(\bigcup_{m\geq n} A_m\right) = \bigcap_{\epsilon>0} \bigcup_{n\geq 1} \bigcap_{m\geq n} (A_m)_\epsilon. \tag{4.18}$$

Proof. Show first that

$$A \subset \bigcap_{n\geq 1} \operatorname{cl}\left(\bigcup_{m\geq n} A_m\right). \tag{4.19}$$

Let $p \in A$ and let $n_1 \in \mathbb{N}$ be such that

$$\forall m \geq n_1 \; F_{AA_m}(\frac{1}{2}) > 1 - \frac{1}{2}.$$

By (4.5),

$$\exists p_1 \in A_{n_1} \; F_{pp_1}(\frac{1}{2}) > 1 - \frac{1}{2}.$$

Continuing in this way we obtain a sequence $n_1 < n_2 < ...$ of indices and the elements $p_k \in A_{n_k}$ such that

$$F_{pp_k}(\frac{1}{2^k}) > 1 - \frac{1}{2^k}.$$

It follows $p_k \to p$, so that

$$p \in \bigcap_{n\geq 1} \operatorname{cl}\left(\bigcup_{m\geq n} A_m\right).$$

Let now $0 < \epsilon < 1/2$ and let $n_0 \in \mathbb{N}$ be such that

$$\forall m \geq n_0 \; F_{AA_m}(\epsilon) > 1 - \epsilon.$$

By (4.6) it follows

$$\forall m \geq n_0 \; \forall q \in A_m \; \exists p \in A \text{ such that } F_{pq}(\epsilon) > 1 - \epsilon,$$

so that

$$\forall m \geq n_0 \; A_m \subset A_\epsilon,$$

or, equivalently,

$$\bigcup_{m\geq n_0} A_m \subset A_\epsilon.$$

But then

$$\bigcap_{n\geq 1} \operatorname{cl}\left(\bigcup_{m\geq n} A_m\right) \subset \operatorname{cl}\left(\bigcup_{m\geq n_0} A_m\right) \subset A_{2\epsilon}.$$

Since $0 < \epsilon < 1/2$ is arbitrary we have

$$\bigcap_{n\geq 1} \operatorname{cl}\left(\bigcup_{m\geq n} A_m\right) \subset \bigcap_{0<\epsilon<1/2} A_{2\epsilon} = \operatorname{cl} A = A.$$

It follows

$$A = \bigcap_{n\geq 1} \operatorname{cl}\left(\bigcup_{m\geq n} A_m\right). \tag{4.20}$$

Let's prove now that

$$A \subset \bigcap_{\epsilon>0} \bigcup_{n\geq 1} \bigcap_{m\geq n} (A_m)_\epsilon. \tag{4.21}$$

For $0 < \epsilon < 1/2$ choose $n_0 \in \mathbb{N}$ such that

$$\forall m \geq n_0 \; F_{AA_m}(\epsilon) > 1 - \epsilon.$$

By (4.5) we have

$$\forall m \geq n_0 \; \forall p \in A \; \exists q \in A_m \; F_{pq}(\epsilon) > 1 - \epsilon,$$

implying

$$A \subset \bigcap_{m \geq n_0} (A_m)_\epsilon \subset \bigcup_{n \geq 1} \bigcap_{m \geq n} (A_m)_\epsilon$$

Again, since $0 < \epsilon < 1/2$ was arbitrarily chosen, we get (4.21).

Finally, prove that

$$B := \bigcap_{\epsilon > 0} \bigcup_{n \geq 1} \bigcap_{m \geq n} (A_m)_\epsilon \subset \bigcap_{n \geq 1} \mathrm{cl} \left(\bigcup_{m \geq n} A_m \right) =: C. \tag{4.22}$$

If $p \in B$ then

$$\forall \epsilon, \; 0 < \epsilon < 1, \; \exists n_0(\epsilon) \; \forall m \geq n_0(\epsilon) \; p \in (A_m)_\epsilon.$$

For $n \geq 1$ letting $m = \max\{n, n_0(\epsilon)\}$ we have

$$p \in (A_m)_\epsilon \subset \left(\bigcup_{m' \geq n} A_{m'} \right)_\epsilon .$$

We have obtained

$$\forall n \geq 1 \; \forall \epsilon > 0 \; p \in \left(\bigcup_{m' \geq n} A_{m'} \right)_\epsilon ,$$

implying

$$\forall n \geq 1 \quad p \in \mathrm{cl} \left(\bigcup_{m' \geq n} A_{m'} \right),$$

so that

$$p \in \bigcap_{n \geq 1} \mathrm{cl} \left(\bigcup_{m' \geq n} A_{m'} \right).$$

Combining now (4.20), (4.21) and (4.22) we obtain (4.18). □

Now we shall prove that the family $P_{fc}(L)$ of all nonempty closed convex subsets of a complete Šerstnev random normed space L is complete with respect to the probabilistic Pompeiu-Hausdorff metric H.

A *Šerstnev random normed space* (RN space) is a triple (L, ν, τ) where L is a real linear space, τ is a continuous triangle function such that $\tau(D^+ \times D^+) \subset D^+$, and ν is a mapping $\nu : L \to D^+$ satisfying the following conditions:

(RN1) $\nu(p) = \epsilon_0 \iff p = \theta$;

(RN2) $\nu(ap)(x) = \nu(p)(\frac{x}{|a|})$ for $x \geq 0$ and $a \neq 0$;

(RN3) $\nu(p+q) \geq \tau(\nu(p), \nu(q)),\ p, q \in L$.

If (L, ν, τ) is a Šerstnev RN space then

$$\rho(p, q) = \nu(p - q),\ p, q \in L, \tag{4.23}$$

is a random metric on L. The topology of L is the strong topology corresponding to the random metric (4.23), and L is a metrizable topological vector space with respect to this topology. Random normed spaces were defined and studied by A. N. Šerstnev [21, 22, 24] (see also [19, Ch. 15, Sect. 1]).

The following result holds:

Theorem 4.8. *Let (L, ν, τ) be a Šestnev random normed space with sup-continuous triangle function satisfying the condition*

$$\tau(F, G)(x) \geq \sup_{t \in [0,1]} \min\{F(tx), G((1-t)x)\}, \tag{4.24}$$

for $x \geq 0$ and $F, G \in D^+$.

Then the family $P_{fc}(L)$ of all nonempty closed convex subsets of L is closed in $P_f(L)$ with respect to the probabilistic Pompeiu-Hausdorff metric H, hence complete if the random normed space L is complete.

If L is complete then the family $P_{kc}(L)$ of all nonempty compact convex subsets of L is complete with respect to the probabilistic Pompeiu-Hausdorff metric.

Proof. Observe first that if the set $A \subset L$ is convex then the set A_ϵ is convex too.

Indeed, let $q_1, q_2 \in A_\epsilon$ and $t_1, t_2 > 0, ; t_1 + t_2 = 1$. If $p_1, p_2 \in A$ are such that $\nu(p_i - q_i)(\epsilon) > 1 - \epsilon,\ i = 1, 2$, then $t_1 p_1 + t_2 p_2 \in A$ and, by (4.24) and (RN2),

$$\nu(t_1p_1 + t_2p_2 - (t_1q_1 + t_2q_2))(\epsilon) \geq$$
$$\geq \min\{\nu(t_1(p_1 - q_1))(t_1\epsilon), \nu(t_2(p_2 - q_2))(t_2\epsilon)\}$$
$$= \min\{\nu(p_1 - q_1)(\epsilon), \nu(p_2 - q_2)(\epsilon)\} > 1 - \epsilon,$$

showing that $t_1q_1 + t_2q_2 \in A_\epsilon$.

Let now (A_n) be a sequence of nonempty closed convex subsets of L converging to $A \in P_f(L)$ with respect to H. By Proposition 4.7

$$A = \bigcap_{\epsilon>0} \bigcup_{n\geq 1} \bigcap_{m\geq n} (A_m)_\epsilon.$$

Since each A_m is convex, the same is true for $(A_m)_\epsilon$, as well as for

$$B_{n,\epsilon} = \bigcap_{m\geq n} (A_m)_\epsilon,\ n = 1, 2, ...$$

The union of the increasing sequence $B_{1,\epsilon} \subset B_{2,\epsilon} \subset ...$ of convex sets will be convex too, so that their intersection for all $\epsilon > 0$ is a convex set.

The assertion concerning the family $P_{kc}(L)$ of all nonempty compact convex subsets of L follows from Theorem 4.5 and the first assertion of the theorem. □

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Babeş-Bolyai University,
Faculty of Mathematics and Computer Science,
Ro-3400 Cluj-Napoca, Romania
E-mail address: scobzas@math.ubbcluj.ro