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ON THE HOMOGENIZATION OF A CLIMATIZATION PROBLEM

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Abstract. This paper deals with the homogenization of a nonlinear model for heat conduction through the exterior of a domain containing periodically distributed conductive grains. We assume that on the walls of the grains we have climatizators governing the heat flux through the boundary. The effective behavior of this nonlinear flow is described by a new elliptic boundary-value problem containing an extra zero-order term which captures the effect of the boundary climatization.

1. Introduction

The aim of this paper is to study the homogenization of some nonlinear thermal flows through periodically perforated domains. We will focus our attention on a nonlinear problem which describes the heat conduction through the exterior of a domain containing periodically distributed conductive grains (or conductive obstacles). We suppose that on the walls of the grains we have climatizators governing the heat flux through the boundary.

Let Ω be an open bounded set in $\mathbb{R}^n$ and let us perforate it by holes. As a result, we obtain an open set Ω^ε which will be referred to as being the *perforated domain*; ε represents a small parameter related to the characteristic size of the perforations.

The nonlinear problem studied in this paper concerns the stationary flow of a fluid confined in Ω^ε, of temperature u^ε, with a given heat flux on the boundary of

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the grains:

$$\begin{cases} -D_f \Delta u^\varepsilon = f \quad \text{in } \Omega^\varepsilon, \\ -D_f \dfrac{\partial u^\varepsilon}{\partial \nu} = a\varepsilon g(u^\varepsilon) \quad \text{on } S^\varepsilon, \\ u^\varepsilon = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here, ν is the exterior unit normal to Ω^ε, $a > 0$, $f \in L^2(\Omega)$ and S^ε is the boundary of our porous medium $\Omega \setminus \overline{\Omega^\varepsilon}$. Moreover, the fluid is assumed to be homogeneous and isotropic, with a constant diffusion coefficient $D_f > 0$.

In the semilinear boundary condition on S^ε in problem (1) the function g is assumed to be given. We shall address here the case of a single-valued maximal monotone graph with $g(0) = 0$, i.e. the case in which g is the derivative of a convex lower semicontinuous function G. This situation is well illustrated by the following example, which is of practical importance in climatization problems:

$$g(r) = \begin{cases} 1 & r \geq \dfrac{1}{k}, \\ kr & \mid r \mid < \dfrac{1}{k}, \\ -1 & r \leq -\dfrac{1}{k}, \end{cases}$$

for a given $k > 0$.

The existence and uniqueness of a weak solution of (1) can be settled by using the classical theory of semilinear monotone problems (see [1]). As a result, we know that there exists a unique weak solution $u^\varepsilon \in V^\varepsilon \bigcap H^2(\Omega^\varepsilon)$, where

$$V^\varepsilon = \{v \in H^1(\Omega^\varepsilon) \mid v = 0 \text{ on } \partial\Omega\}.$$

If with Ω^ε we associate the following nonempty convex subset of V^ε:

$$K^\varepsilon = \{v \in V^\varepsilon \mid G(v)|_{S^\varepsilon} \in L^1(S^\varepsilon)\}, \tag{2}$$

then u^ε is also known to be the unique solution of the following variational problem:

$$\begin{cases} \text{Find } \ u^\varepsilon \in K^\varepsilon \ \text{ such that} \\ D_f \displaystyle\int_{\Omega^\varepsilon} Du^\varepsilon D(v^\varepsilon - u^\varepsilon)dx - \int_{\Omega^\varepsilon} f(v^\varepsilon - u^\varepsilon)dx + a\left\langle \mu^\varepsilon, G(v^\varepsilon) - G(u^\varepsilon)\right\rangle \geq 0, \\ \forall v^\varepsilon \in K^\varepsilon, \end{cases} \tag{3}$$

where μ^ε is the linear form on $W_0^{1,1}(\Omega)$ defined by

$$\langle \mu^\varepsilon, \varphi \rangle = \varepsilon \int_{S^\varepsilon} \varphi d\sigma \quad \forall \varphi \in W_0^{1,1}(\Omega).$$

From a geometrical point of view, we shall just consider periodic structures obtained by removing periodically from Ω, with period εY (where Y is a given hyper-rectangle in $\mathbb{R}^n$), an elementary hole T which has been appropriated rescaled and which is strictly included in Y, i.e. $\overline{T} \subset Y$.

We shall prove that the solution u^ε, properly extended to the whole of Ω, converges to the unique solution of the following variational inequality:

$$\begin{cases} u \in H_0^1(\Omega) \\ \displaystyle\int_\Omega QDuD(v-u)dx \geq \int_\Omega f(v-u)dx - a\frac{|\partial T|}{|Y \setminus \overline{T}|}\int_\Omega (G(v)-G(u))dx, \\ \forall v \in H_0^1(\Omega). \end{cases} \tag{4}$$

Here, $Q = ((q_{ij}))$ is the homogenized matrix (symmetric and positive-definite):

$$q_{ij} = D_f \left(\delta_{ij} + \frac{1}{|Y \setminus \overline{T}|} \int_{Y \setminus \overline{T}} \frac{\partial \chi_j}{\partial y_i} dy \right), \tag{5}$$

defined in terms of the functions χ_i, $i = 1, ..., n$, solutions of the cell problems

$$\begin{cases} -\Delta \chi_i = 0 \ \text{ in } \ Y \setminus \overline{T}, \\ \dfrac{\partial(\chi_i + y_i)}{\partial \nu} = 0 \ \text{ on } \ \partial T, \\ \chi_i \ \ Y- \text{ periodic.} \end{cases} \tag{6}$$

We can treat in a similar manner the case of a multi-valued maximal monotone graph, which includes various semilinear boundary-value problems, such as Dirichlet or Neumann problems, Robin boundary conditions, Signorini's unilateral conditions, problems arising in chemistry (see [2], [4] and [5]).

The structure of the paper is as follows: first, let us mention that we shall just focus on the case $n \geq 3$. The case $n = 2$ is much more simpler and we shall omit to treat it. Section 2 is devoted to the setting of our problem and to the formulation of

the main result of this paper. Section 3 contains some necessary preliminary results. In the last section we give the proof of our main result.

2. **Setting of the problem and the main result**

Let Ω be a smooth bounded connected open subset of $\mathbb{R}^n$ $(n \geq 3)$ and let $Y = [0, l_1[\times ...[0, l_n[$ be the representative cell in $\mathbb{R}^n$. Denote by T an open subset of Y with smooth boundary ∂T such that $\overline{T} \subset Y$.

Let ε be a real parameter taking values in a sequence of positive numbers converging to zero. For each ε and for any integer vector $k \in \mathbb{Z}^n$, set $T_k^\varepsilon = \varepsilon(kl + T)$ the translated image of εT by the vector $kl = (k_1 l_1, ..., k_n l_n)$ and denote by T^ε the set of all the holes contained in Ω, i.e. $T^\varepsilon = \bigcup \left\{T_k^\varepsilon \ \mid \ \overline{T_k^\varepsilon} \subset \Omega, \ k \in \mathbb{Z}^n\right\}$. Set $\Omega^\varepsilon = \Omega \setminus \overline{T^\varepsilon}$ and $S^\varepsilon = \cup\{\partial T_k^\varepsilon \mid \overline{T_k^\varepsilon} \subset \Omega, \ k \in \mathbb{Z}^n\}$. Also, let $Y^* = Y \setminus \overline{T}$ and $\rho = \dfrac{|Y^*|}{|Y|}$. Moreover, for an arbitrary function $\psi \in L^2(\Omega^\varepsilon)$, we shall denote by $\widetilde{\psi}$ its extension by zero inside the holes.

As already mentioned, we are interested in studying the asymptotic behavior of the solution of problem (1). We shall treat the case in which the function g appearing in (1) has a single-valued maximal monotone graph in $\mathbb{R} \times \mathbb{R}$, with $g(0) = 0$. Also, if we denote by $D(g)$ the domain of g, i.e. $D(g) = \{\xi \in \mathbb{R} \mid g(\xi) \neq \emptyset\}$, then we suppose that $D(g) = \mathbb{R}$. Moreover, we assume that g is continuous and there exist $C \geq 0$ and an exponent q, with $0 \leq q < n/(n-2)$, such that

$$|g(v)| \leq C(1 + |v|^q). \tag{7}$$

We know that in this case there exists a lower semicontinuous convex function G from $\mathbb{R}$ to $]-\infty, +\infty]$, G proper, i.e. $G \not\equiv +\infty$ such that g is the subdifferential of G, $g = \partial G$ (G is an indefinite "integral" of g). Let $G(v) = \int\limits_0^v g(s)ds$.
If the convex set K^ε is defined by (2), then, for a given function $f \in L^2(\Omega)$, the weak solution of the problem (1) is also the unique solution of the variational inequality

(3). Also, notice that u^ε is the unique solution of the minimization problem:

$$\begin{cases} u^\varepsilon \in K^\varepsilon, \\ J^\varepsilon(u^\varepsilon) = \inf\limits_{v \in K^\varepsilon} J^\varepsilon(v), \end{cases}$$

where

$$J^\varepsilon(v) = \frac{1}{2} D_f \int\limits_{\Omega^\varepsilon} |Dv|^2 \, dx + a \langle \mu^\varepsilon, G(v) \rangle - \int\limits_{\Omega^\varepsilon} f v dx.$$

Let us introduce the following functional defined on $H_0^1(\Omega)$:

$$J^0(v) = \frac{1}{2} \int\limits_{\Omega} Q Dv Dv dx + a \frac{|\partial T|}{|Y^*|} \int\limits_{\Omega} G(v) dx - \int\limits_{\Omega} f v dx.$$

The main result of this paper is the following one:

Theorem 2.1. *One can construct an extension $P^\varepsilon u^\varepsilon$ of the solution u^ε of the variational inequality (3) such that*

$$P^\varepsilon u^\varepsilon \rightharpoonup u \quad \text{weakly in } \ H_0^1(\Omega),$$

where u is the unique solution of the minimization problem

$$\begin{cases} \text{Find } u \in H_0^1(\Omega) \text{ such that} \\ J^0(u) = \inf\limits_{v \in H_0^1(\Omega)} J^0(v). \end{cases} \tag{8}$$

Moreover, $G(u) \in L^1(\Omega)$. Here, $Q = ((q_{ij}))$ is the classical homogenized matrix, whose entries were defined by (5)-(6).

3. Preliminary results

In order to extend the solution u^ε of problem (1) to the whole of Ω, let us recall the following well-known result (see [3]):

Lemma 3.1. *There exists a linear continuous extension operator $P^\varepsilon \in \mathcal{L}(L^2(\Omega^\varepsilon); L^2(\Omega)) \cap \mathcal{L}(V^\varepsilon; H_0^1(\Omega))$ and a positive constant C, independent of ε, such that, for any $v \in V^\varepsilon$,*

$$\|P^\varepsilon v\|_{L^2(\Omega)} \leq C \|v\|_{L^2(\Omega^\varepsilon)}$$

and

$$\|\nabla P^\varepsilon v\|_{L^2(\Omega)} \leq C \|\nabla v\|_{L^2(\Omega^\varepsilon)}.$$

For getting the effective behavior of our solution u^ε, we have to pass to the limit in (3). In order to do this, let us introduce, for any $h \in L^{s'}(\partial T)$, $1 \le s' \le \infty$, the linear form μ_h^ε on $W_0^{1,s}(\Omega)$ defined by

$$\langle \mu_h^\varepsilon, \varphi \rangle = \varepsilon \int_{S^\varepsilon} h(\frac{x}{\varepsilon})\varphi d\sigma \quad \forall \varphi \in W_0^{1,s}(\Omega),$$

with $1/s + 1/s' = 1$. It is proved in [2] that

$$\mu_h^\varepsilon \to \mu_h \quad \text{strongly in } (W_0^{1,s}(\Omega))', \tag{9}$$

where $\langle \mu_h, \varphi \rangle = \mu_h \int_\Omega \varphi dx$, with $\mu_h = \dfrac{1}{|Y|} \displaystyle\int_{\partial T} h(y) d\sigma$.
In the particular case in which $h \in L^\infty(\partial T)$ or even h is constant, we have

$$\mu_h^\varepsilon \to \mu_h \quad \text{strongly in } W^{-1,\infty}(\Omega). \tag{10}$$

We shall denote by μ^ε the above introduced measure in the case in which $h = 1$.

Let F be a continuously differentiable function, monotonously non-decreasing and such that $F(v) = 0$ iff $v = 0$. We shall suppose that there exist a positive constant C and an exponent q, with $0 \le q < n/(n-2)$, such that $\left|\dfrac{\partial F}{\partial v}\right| \le C(1 + |v|^q)$. It is not difficult to prove (see [4]) that for any $\varphi \in \mathcal{D}(\Omega) = C_0^\infty(\Omega)$ and for any $z^\varepsilon \rightharpoonup z$ weakly in $H_0^1(\Omega)$, we get

$$\varphi F(z^\varepsilon) \rightharpoonup \varphi F(z) \quad \text{weakly in } W_0^{1,\overline{q}}(\Omega), \tag{11}$$

where $\overline{q} = \dfrac{2n}{q(n-2)+n}$.

4. **Proof of the main result**

Proof of Theorem 2.1. Let u^ε be the solution of the variational inequality (3) and let $P^\varepsilon u^\varepsilon$ be the extension of u^ε given by Lemma 3.1. It is not difficult to see that $P^\varepsilon u^\varepsilon$ is bounded in $H_0^1(\Omega)$. So by extracting a subsequence, one has

$$P^\varepsilon u^\varepsilon \rightharpoonup u \quad \text{weakly in } \ H_0^1(\Omega).$$

Let $\varphi \in \mathcal{D}(\Omega)$. By classical regularity results $\chi_i \in L^\infty$. Using the boundedness of χ_i and φ, there exists $M \geq 0$ such that $\left\| \frac{\partial \varphi}{\partial x_i} \right\|_{L^\infty} \left\| \chi_i \right\|_{L^\infty} < M$. Let

$$v^\varepsilon = \varphi + \sum_i \varepsilon \frac{\partial \varphi}{\partial x_i}(x) \chi_i(\frac{x}{\varepsilon}).$$

Then, $v^\varepsilon \in K^\varepsilon$, which will allow us to take it as a test function in (3). Moreover, $v^\varepsilon \to \varphi$ strongly in $L^2(\Omega)$. If we compute Dv^ε, we get:

$$Dv^\varepsilon = \sum_i \frac{\partial \varphi}{\partial x_i}(x)(\mathbf{e}_i + D\chi_i(\frac{x}{\varepsilon})) + \varepsilon \sum_i D\frac{\partial \varphi}{\partial x_i}(x)\chi_i(\frac{x}{\varepsilon}),$$

where $\mathbf{e}_i$, $1 \leq i \leq n$, are the elements of the canonical basis in $\mathbb{R}^n$.

Using v^ε as a test function in (3), we can write

$$D_f \int_\Omega DP^\varepsilon u^\varepsilon \widetilde{(Dv^\varepsilon)} dx \geq \int_{\Omega^\varepsilon} f(v^\varepsilon - u^\varepsilon) dx +$$

$$+ D_f \int_{\Omega^\varepsilon} Du^\varepsilon Du^\varepsilon dx - a \left\langle \mu^\varepsilon, G(v^\varepsilon) - G(u^\varepsilon) \right\rangle . \tag{12}$$

Denote

$$\rho Q \mathbf{e}_j = \frac{1}{|Y^*|} D_f \int_{Y^*} (D\chi_j + \mathbf{e}_j) dy, \tag{13}$$

where $\rho = |Y^*| / |Y|$. Neglecting the term $\varepsilon \sum_i D\frac{\partial \varphi}{\partial x_i}(x)\chi_i(\frac{x}{\varepsilon})$ which actually tends strongly to zero, we can pass easily to the limit in the left-hand side of (12). Hence

$$D_f \int_\Omega DP^\varepsilon u^\varepsilon \widetilde{Dv^\varepsilon} dx \to \int_\Omega \rho Q Du D\varphi dx. \tag{14}$$

For the first term of the right-hand side of (12) we get

$$\int_{\Omega^\varepsilon} f(v^\varepsilon - u^\varepsilon) dx = \int_\Omega f \chi_{\Omega^\varepsilon}(v^\varepsilon - P^\varepsilon u^\varepsilon) dx \to \int_\Omega f \rho(\varphi - u) dx. \tag{15}$$

For the third term of the right-hand side of (12), assuming (7) for the maximal monotone graph g and using (11) written for G and for $z^\varepsilon = P^\varepsilon u^\varepsilon$, we get

$$G(P^\varepsilon u^\varepsilon) \rightharpoonup G(u) \quad \text{weakly in } W_0^{1,\overline{q}}(\Omega).$$

Combining this with the convergence (10) written for $h = 1$, we have

$$\langle \mu^\varepsilon, G(P^\varepsilon u^\varepsilon)\rangle \to \frac{|\partial T|}{|Y|}\int\limits_\Omega G(u)dx.$$

Using a similar technique for the convergence of $\langle \mu^\varepsilon, G(v^\varepsilon)\rangle$, we obtain

$$a\,\langle \mu^\varepsilon, G(v^\varepsilon) - G(P^\varepsilon u^\varepsilon)\rangle \to a\frac{|\partial T|}{|Y|}\int\limits_\Omega (G(\varphi) - G(u))dx. \tag{16}$$

For passing to the limit in the second term of the right-hand side of (12) let us write down the subdifferential inequality

$$D_f\int\limits_{\Omega^\varepsilon} Du^\varepsilon Du^\varepsilon dx \geq D_f\int\limits_{\Omega^\varepsilon} Dw^\varepsilon Dw^\varepsilon dx + 2D_f\int\limits_{\Omega^\varepsilon} Dw^\varepsilon(Du^\varepsilon - Dw^\varepsilon)dx, \tag{17}$$

for any $w^\varepsilon \in H_0^1(\Omega)$. Reasoning as before and choosing $w^\varepsilon = \overline{\varphi} + \sum_i \varepsilon\frac{\partial\overline{\varphi}}{\partial x_i}(x)\chi_i(\frac{x}{\varepsilon})$, where $\overline{\varphi}$ enjoys similar properties as the corresponding φ, the right-hand side of the inequality (17) passes to the limit and one has

$$\liminf_{\varepsilon\to 0} D_f\int\limits_{\Omega^\varepsilon} Du^\varepsilon Du^\varepsilon dx \geq \int\limits_\Omega \rho Q D\overline{\varphi} D\overline{\varphi} dx + 2\int\limits_\Omega \rho Q D\overline{\varphi}(Du - D\overline{\varphi})dx,$$

for any $\overline{\varphi} \in \mathcal{D}(\Omega)$ and, by density, for any $\overline{\varphi} \in H_0^1(\Omega)$. So, for $u \in H_0^1(\Omega)$, we have

$$\liminf_{\varepsilon\to 0} D_f\int\limits_{\Omega^\varepsilon} Du^\varepsilon Du^\varepsilon dx \geq \int\limits_\Omega \rho Q Du Du dx. \tag{18}$$

Putting together (14)-(16) and (18), we get

$$\int\limits_\Omega \rho Q Du D\varphi dx \geq \int\limits_\Omega f\rho(\varphi - u)dx + \int\limits_\Omega \rho Q Du Du dx - a\frac{|\partial T|}{|Y|}\int\limits_\Omega (G(\varphi) - G(u))dx,$$

for any $\varphi \in \mathcal{D}(\Omega)$ and hence, by density, for any $v \in H_0^1(\Omega)$. So, finally, we obtain

$$\int\limits_\Omega Q Du D(v-u)dx \geq \int\limits_\Omega f(v-u)dx - a\frac{|\partial T|}{|Y^*|}\int\limits_\Omega (G(v) - G(u))dx,$$

which gives exactly the limit problem (4). This ends the proof of Theorem 2.1. □

References

[1] Brézis, H., *Problèmes unilatéraux*, J. Math. Pures et Appl., **51**(1972), 1-168.

[2] Cioranescu, D., Donato, P., *Homogénéisation du problème de Neumann non homogène dans des ouverts perforés*, Asymptotic Anal. **1**(1988), 115-138.

[3] Cioranescu, D., Saint Jean Paulin, J., *Homogenization in open sets with holes*, J. Math. Anal. Appl. **71**(1979), 590-607.

[4] Conca, C., Díaz, J.I., Timofte, C., *Effective chemical processes in porous media*, Math. Models Methods Appl. Sci. (M3AS) **13**(10)(2003), 1437-1462.

[5] Conca, C., Murat, F., Timofte, C., *A Generalized Strange Term in Signorini's Type Problems*, ESAIM: Modél. Math. Anal. Numér. (M2AN) **37**(5)(2003), 773-806.

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