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FUNCTIONAL-DIFFERENTIAL EQUATION WITH RETARDED ARGUMENT

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Dedicated to Professor Grigore Călugăreanu on his 60th birthday

Abstract. Sufficient conditions are obtained for all positive solutions of
$$\frac{dx(t)}{dt} = [f(x(t), x(t-\tau))]x(t)$$
to converges as $t \to \infty$ to a positive equilibrium solution.

1. **Introduction**

Trade cycles, business cycles, and fluctuations in the price and supply of various commodities have attracted the attention of economists for well over 100 years and possible more than thousands of years. Early authors often attribute these fluctuations to random factors, e.g. the weather for agricultural commodities, see for instance Slutzky [17] and Kalecki [11].

Other workers speculated that economic cycling of fluctuations might be an inherent endogenous dynamical behavior characteristic of instable economic systems, (Ezekiel [3] and the references therein). A number of business cycle models postulating the existence of nonlinearities to account for limit cycle behavior have played a fundamental role in sharpening the debate between the proponent of the exogenous versus endogenous (or stochastic versus deterministic) school (cf. Zarnowitz [19]).

The development of modern dynamical system theory (Guckenheimer and Holmes [10], Lasota and Mackey [12], Glass and Mackey [6]) have shed new light on this debate. The possibility that economic fluctuations may reflect underlying

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periodic or chaotic dynamics in nonlinear economics systems have been explored in various context, se for instance Goodwin at all [7] , Grandomnd and Malgrange [9], Gabisch and Lorenz [5] and references therein.

Mackey [13] developed a price adjustment model for a single commodity market with state dependent production and storage delays. Conditions for the equilibrium price to be stable are derived in terms of a variety of economic parameters. Also Blaire and Mackey [2] developed a model for the dynamics of price adjustment model in a single commodity market where nonlinearities in both supply and demand functions are considered explicitly. Farahani and Grove [4] have studied a special case of a general model studied of Blaire and Mackey [2] which it calls the case of naive consumer.

Our purpose here is to study the following model.

$$x'(t) = [f(x(t), x(t-\tau))]x(t), \quad t \in \mathbb{R}_+ \tag{1}$$

$$x(t) = \varphi(t), \quad t \in [-\tau, 0] \tag{2}$$

where $\tau > 0$, $f, g \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $\varphi \in C([-\tau, 0], \mathbb{R}_+^*)$. Sufficient conditions are obtained for all positive solutions of (1) to converges as $t \to \infty$ to a positive equilibrium solution. We say that the function $x^*(t)$ oscillate about r^* if $x^*(t) - r^*$ has arbitrarily large zeros. If is not the case that $x^*(t)$ oscillate about r^*, then we say that $x^*(t)$ is nonoscillatory about r^*.

2. **The main result**

Consider the problem (1)+(2). The following theorem establish sufficient conditions that $x^*(t)$ oscillate about r^* where x^* is the unique positive solution of problem (1)+(2) and r^* is the unique positive equilibrium solution of (1).

Theorem 1. *Suppose that*

(i) $f \in C^1(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$, $\quad \varphi \in C([-\tau, 0], \mathbb{R}_+^*)$,

(ii) $f(\,\cdot\,, y)$ *is locally Lipschitz,*

(iii) *There exists* $M_f > 0$ *such that* $|f(u, v)| \leq M_f$ *for all* $u, v \in \mathbb{R}_+$,

(iv)

$$\frac{\partial f(u,v)}{\partial x}M + M_f \leq 0, \frac{\partial f(u,v)}{\partial y} \leq 0 \quad \textit{for all} \quad (u,v) \in \mathbb{R}_+ \times \mathbb{R}_+$$

and $M > 0$ *such that* $x^*(t) \leq M$ *for all* $t \in \mathbb{R}_+$,

(v) $1 + \lambda \dfrac{\partial f(u,u)}{\partial x} + \lambda \dfrac{\partial f(u,u)}{\partial y} \leq M_1 < 1$ *for all* $u \in \mathbb{R}_+$.

Then

(a) *Problem* (1)+(2) *has an unique positive solution* $x^*(t)$.

(b) *There exists* $m, M \in \mathbb{R}_+, 0 < m < M$ *such that* $m \leq x^*(t) \leq M$ *for all* $t \in \mathbb{R}_+$.

(c) *Equation* (1) *has a unique positive equilibrium solution* r^*.

(d) *if* x^* *is* r^* *-nonoscillatory then*

$$\lim_{t\to\infty} x^*(t) = r^*.$$

Proof:

(a) Let $x^* \in C([-\tau, t_+), \mathbb{R}_+) \cap C^1([0, t_+), \mathbb{R}_+)$ be a maximal solution of (1)+(2). We can rewrite the equation (1) in the form

$$\frac{x'(t)}{x(t)} = f(x(t), x(t-\tau)). \tag{3}$$

Integrating the equation (3) from 0 to t we obtain

$$\ln x(t) - \ln x(0) = \int_0^t f(x(s), x(s-\tau))ds.$$

From (2) we have

$$\ln \frac{x(t)}{\varphi(0)} = \int_0^t f(x(s), x(s-\tau))ds,$$

and

$$x(t) = \varphi(0) \exp \int_0^t f(x(s), x(s-\tau))ds.$$

From (iii) we have that

$$x(t) \leq \varphi(0) e^{M_f t}, \quad \text{for all} \qquad t \in [0, t_+).$$

From steps method and the Theorem of the maximal solution (see [1] and [16]) we have that there exists a unique x^* and $t_+ = +\infty$

(b) Follows from (a).

(c) Let $F:\mathbb{R}_+^2\to\mathbb{R}_+^2,\ F(x,y)=(F_1(x),F_1(y))$

where

$$F_1(x)=x+\lambda f(x,x)$$

$$||F(x,y)-F(u,v)||_{\mathbb{R}^2}$$
$$\leq\left(\begin{array}{cc}1+\lambda\frac{\partial f}{\partial x}(u,u)+\lambda\frac{\partial f}{\partial y}(u,u) & 001+\lambda\frac{\partial f}{\partial x}(v,v)+\lambda\frac{\partial f}{\partial y}(v,v)\end{array}\right)||(x,y)-(u,v)||,$$

where $(\mathbb{R}_+^2,||\circ||)$, and $||x,y||=\left(\begin{array}{c}|x|\\|y|\end{array}\right)$.

From (v) we have that

$$||F(x,y)-F(u,v)||_{\mathbb{R}^2}\leq\left(\begin{array}{cc}M_1 & 0\\0 & M_1\end{array}\right)||(x,y)-(u,v)||,$$

and

$$S^n\to O_2,\ \text{ as }\ n\to\infty,$$

where

$$S=\left(\begin{array}{cc}M_1 & 0\\0 & M_1\end{array}\right).$$

It follows that F is a contraction and from Contraction Principle of Perov we have that F has a unique fixed point, i.e.

$$x=x+\lambda f(x,x).$$

This implies that the equation

$$f(x,x)=0$$

has a unique solution and consequently that equation (1) has a positive equilibrium solution r^*.

(d) We rewrite equation (1) in the form

$$\frac{dy}{dt}=G(y(t),y(t-\tau))-G(0,0), \tag{4}$$

where

$$G(y(t),y(t-\tau))=[f(y(t+r^*,y(t-\tau+r^*)](y(t)+r^*),$$

and

$$y(t) = x(t) - r^*.$$

It is now sufficient to show that $y(t) \to 0$ as $t \to \infty$. An application of the mean-value Theorem to (4) leads to

$$\frac{dy}{dt} = -a(t)y(t) - b(t)y(t-\tau), \tag{5}$$

where

$$-a(t) = \frac{\partial G}{\partial y}(u(t), v(t)),$$
$$-b(t) = \frac{\partial G}{\partial y}(u(t), v(t)),$$

and $(u(t), v(t))$ lies on the line segment joining $(0,0)$ and $(y(t), y(t-\tau))$. It is found that

$$-a(t) = \frac{\partial f(u,v)}{\partial y} \cdot (y(t) + r^*) + f(y(t) + r^*, y(t-\tau) + r^*),$$

and

$$-b(t) = \frac{\partial f(u,v)}{\partial y} \cdot (y(t) + r^*).$$

Note that $a(t)$, and $b(t)$ are positive and bounded away from zero. The existence of solution of (5) for all $t \geq 0$ is a consequence of boundedness of $x(t)$ for all $t \geq 0$. If y is nonoscillatory then $|y(t)| > 0$, for all $t > 0$.

If $y(t) > 0$ for all $t > T$ then we have from (5) that $y'(t) < 0$ and so $\lim_{t\to\infty} \lim y(t)$ exists.

Since $y(t) > 0$ eventually, $\lim_{t\to\infty} \lim y(t) = l \geq 0$. We claim that $l = 0$; suppose that $l > 0$. Then there exists $t_0 > 0$ such that

$$y(t) \geq \frac{l}{2}, \text{ for } t \geq t_0.$$

We have directly from (5) that

$$\frac{dy(t)}{dt} \leq -a(t)\frac{l}{2}$$

leading to

$$y(t) - y(t_0) \leq -\frac{l}{2}\int_{t_0}^{t} a(s)ds,$$

which implies that

$$y(t) \to -\infty \text{ as } t \to \infty.$$

But this contradicts the eventual positivity of y.

Thus

$$\lim_{t\to\infty} y(t) = l = 0.$$

If $y(t) < 0$ for $t > T$, the arguments are similar.

Thus the result follows from

$$\lim_{t\to\infty} y(t) = 0.$$

References

[1] Altman, M., *Dilating mappings, implicit functions and fixed point theorems in finite dimensional spaces*, Fund. Math. (1970), 129-141.

[2] Bélair, J., Mackey, M.C., *Consumer memory and price fluctuations in commodity markets: An integrodifferential model*, J. Dynamics Diff. Equations **3**, (1980), 299-325.

[3] Ezekiel, M., *The cobweb theorem*, Q. J. Econ. **52**, (1938), 255-280.

[4] Farahani, A.M., Grove, E.A., *A simple model for price fluctuations in a single commodity market*, Contemporary Mathematics **129** (1992), 97-103.

[5] Gabisch, G., Lorenz, H.W., *Business cycle theory*, Lecture Notes in Economisc and Mathematical Systems, **283**, Springer-Verlag, Berlin, 1987.

[6] Glass, L., Mackey, M.C., From clocks to chaos: The rythm of life, Princeton University Press, Princeton. N. J., 1988.

[7] Goodwin, R.M., Kruger, M., Varcelli, A., *Nonlinear models of fluctuating growth*, Lecture Notes in Economisc and Mathematical Systems, **228**, Springer-Verlag, Berlin, 1984.

[8] Gopalsamy, K., *On an equation modelling haematopoiesis*, Automedica **15**(1993), 251-257.

[9] Grandmont, J.M., Malagrange, P., *Nonlinear economics dynamics: Introduction*, J. Ecom Theory **40**, (1986), 3-12.

[10] Guckenheimer, J., Holmes, P., *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer-Verlag, New Yorke, 1983.

[11] Kalecki, M., *Theory and economic dynamics*, Unwin University Books, London, 1952.

[12] Lasota, A., Mackey, M.C., *Probabilistic properties of deterministic systems*, Cambridge University Press, New York, 1985.

[13] Mackey, M.C., *Commodity price fluctuations: price dependent delays and nonlinearities as explanatory factors*, Journal of Economic Theory **48** (1989), 497-509.

[14] Mureşan, A.S., *On some models of price fluctuation in a market economy*, Studia Univ. Babeş-Bolyai, Mathematica, XXXVIII, 2, 1993.

[15] Rus, I.A., Iancu, C., *A functional-differential model for price fluctuations in single commodity market*, Studia Univ. Babeş-Bolyai, Mathematica, XXXVIII, 2, 1993.

[16] Rus, I.A., *Some remarks on coincidence theory*, Pure Math. Manuscript **9**(1990-1991), 137-148.

[17] Slutzky, E., *The summation of random causes as the source of cyclic processes*, Econometrica **5** (1937), 105-146.

[18] Tarţa, A.A., *Functional-differential equations that appear in price theory*, Semin. Fixed Point Theory Cluj-Napoca **3**(2002), 375-380.

[19] Zarnowitz, V., *Recent work on business cycle in historical perspective: A review of theories and evidence*, J. Econ. Lit. **23**(1985), 523-580.

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