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PROJECTORS IN FINITE π-SOLVABLE GROUPS AND ASSOCIATED PRIMITIVE CLASSES

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Dedicated to Professor Grigore Călugăreanu on his 60th birthday

Abstract. In [7], W. Gaschütz introduced the notion of *primitive class* and studied its connection with projectors in finite solvable groups. Let π be a set of primes. Introduced by S.A. Čunihin in [6], the π-solvable groups are more general than the soluble groups. It is the aim of this paper to give conditions under which we can establish a connection between projectors in finite π-solvable groups and their associated primitive classes. The main results establishing this connection are based on some new properties of projectors in finite π-solvable groups which are also proved in this paper.

1. Preliminaries

All groups considered in this paper are finite. Let π be a set of primes and π' the complement to π in the set of all primes.

We first remind some useful definitions and theorems.

Definition 1.1. a) A positive integer n is said to be a *π-number* if for any prime divisor p of n we have $p \in \pi$.

b) A finite group G is a *π-group* if the order of G is a π-number.

c) A subgroup H of a group G is said to be a *π-subgroup* if H is a π-group.

Notation 1.2. Let G be a group. We denote by $O_{\pi'}(G)$ the largest normal π'-subgroup of G.

Definition 1.3. Let G be a group.

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a) M is a *minimal normal subgroup* of G if:

(i) $M \lhd G$ and $M \neq 1$;

(ii) if $M^* \lhd G$, $M^* \neq 1$ and $M^* \leq M$ then $M^* = M$.

b) Let M and N be normal subgroups of G such that $N \subseteq M$. The factor M/N is called a *chief factor* of G if M/N is a minimal normal subgroup of G/N.

c) Let

$$C : G = G_0 > G_1 > G_2 > \cdots > G_r = 1$$

be a chain of normal subgroups of G. C is called a *chief chain* of G if G_i/G_{i+1} is a chief factor of G, for any i.

The following theorem given by R. Baer in [1] is of special interest for our considerations.

Theorem 1.4. ([1]) *A solvable minimal normal subgroup of a finite group is abelian.*

Introduced by S.A. Čunihin in [6], the π-solvable groups are more general than the solvable groups.

Definition 1.5. ([6]) A group G is *π-solvable* if every chief factor of G is either a solvable π-group or a π'-group. For π the set of all primes, we obtain the notion of solvable group.

Remark 1.6. a) A finite group G is π-solvable if and only if G is a chief chain

$$C : G = G_0 > G_1 > G_2 > \cdots > G_r = 1$$

such that any factor G_i/G_{i+1} is either a solvable π-group or a π'-group. Moreover, if G is a finite π-solvable group then any chief chain of G has the above property.

b) We use in the paper the following basic properties of π-solvable groups:

(i) If G is a π-solvable group and H is a subgroup of G, then H is also a π-solvable group.

(ii) If G is a π-solvable group and N is a normal subgroup of G, then G/N is a π-solvable group.

We define below the special classes of groups and subgroups which will appear in the paper.

Definition 1.7. a) A class X of groups is a *homomorph* if X is epimorphically closed, i.e. if $G \in X$ and N is a normal subgroup of G, then $G/N \in X$.

b) A group G is *primitive* if G has a *stabilizer*, i.e. a maximal subgroup H of G with $core_G H = 1$, where $core_G H = \cap\{H^g / g \in G\}$. We remind here that $core_G H$ is a normal subgroup of G.

c) ([7]) A homomorph X is called a *Schunck class* if X is *primitively closed*, i.e. if any group G, all of whose primitive factor groups are in X, is itself in X.

d) A class X of groups is said to be π-*closed* if:

$$G/O_{\pi'}(G) \in X \;\Rightarrow\; G \in X.$$

A π-closed homomorph, respectively a π-closed Schunck class will be called π-*homomorph*, respectively π-*Schunck class*.

Definition 1.8. ([7]) Let X be a class of groups, G a group and H a subgroup of G.

a) H is an X-*maximal subgroup* of G if:

(i) $H \in X$;

(ii) $H \leq H^* \leq G$, $H^* \in X \;\Rightarrow\; H = H^*$.

b) H is an X-*projector* of G if, for any normal subgroup N of G, HN/N is X-maximal in G/N.

Remark 1.9. Let X be a class of groups.

a) Any X-projector of a group G is X-maximal in G.

b) If H is an X-maximal subgroup of G and $H \leq K \leq G$, then H is also an X-maximal subgroup of K.

Theorem 1.10. ([7]) *Let X be a class of groups, G a group and H a subgroup of G. H is an X-projector of G if and only if:*

(a) H is an X-maximal subgroup of G;

(b) HM/M is an X-projector of G/M, for all minimal normal subgroups M of G.

Theorem 1.11. ([7]) *Let X be a homomorph, G a group and H a subgroup of G. If H is an X-projector of G and N is a normal subgroup of G, then HN/N is an X-projector of G/N.*

In [3], we introduced the P property, which is important for the present paper.

Definition 1.12. Let X be a class of groups and G a π-solvable group. We say that G has the *property* (α) with respect to X if the following implication is true:

$$M \text{ minimal normal subgroup of } G,\ M\ \pi' - \text{subgroup} \ \Rightarrow\ G/M \in X. \qquad (\alpha)$$

Definition 1.13. ([3]) Let X be a class of groups and π a set of primes. We say that X has the P *property* with respect to π if any π-solvable group G has the property (α) with respect to X.

The following two results on properties (α) and P will be used in our considerations.

Theorem 1.14. *Let X be a π-homomorph and G a π-solvable group having property (α) with respect to X and suppose that there is a minimal normal subgroup M of G such that M is a π'-group. Then $G \in X$.*

Proof. From $M \lhd G$ and M π'-group follows that $M \leq O_{\pi'}(G)$ and so

$$G/O_{\pi'}(G) \cong (G/M)/(O_{\pi'}(G)/M). \qquad (1)$$

By property (α), $G/M \in X$. Hence, by (1) and X being a homomorph, $G/O_{\pi'}(G) \in X$. This implies by the π-closure of X that $G \in X$. $\square$

Theorem 1.15. *Let X be a π-homomorph with the P property with respect to π and G a π-solvable group such that there is a minimal normal subgroup M of G, M π'-group. Then $G \in X$.*

Proof. Since X has the P property with respect to π and G is a π-solvable group, it follows that G has property (α) with respect to X. Applying now 1.14, we obtain that $G \in X$. $\square$

2. Some new properties of projectors in finite π-solvable groups

We first give some sufficient conditions on a subgroup of a π-solvable group to be an X-projector, where X is a π-Schunck class.

Theorem 2.1. ([5]) *Let X be a π-Schunck class, G a finite π-solvable group such that for any minimal normal subgroup M of G which is a π'-group we have $G/M \in X$ and let B be a normal abelian subgroup of G and S a subgroup of G such that:*

(i) S is X-maximal in BS;

(ii) BS/B is an X-projector of G/B.

Then S is an X-projector of G.

Theorem 2.2. *Let X be a π-Schunck class with the P property with respect to π, G a finite π-solvable group and let B be a normal abelian subgroup of G and S a subgroup of G such that:*

(i) S is X-maximal in BS;

(ii) BS/B is an X-projector of G/B.

Then S is an X-projector of G.

Proof. X having the P property with respect to π and G being a π-solvable group, we deduce that G has the property (α) with respect to X. Applying now 2.1, it follows that S is an X-projector of G. $\square$

Theorem 2.3. *Let X be a π-Schunck class and G a finite π-solvable group with $O_{\pi'}(G) = 1$. If B is a normal abelian subgroup of G and S is a subgroup of G such that*

(i) S is X-maximal in BS;

(ii) BS/B is an X-projector of G/B,

then S is an X-projector of G.

Proof. The proof is similar with that of 2.1 given in [5]. Two cases are considered:

1) $B = 1$. From (ii) follows that S is an X-projector of G.

2) $B \neq 1$. We prove that S is an X-projector of G by using 1.10.

(a) S is X-maximal in G. This follows like in the proof of 2.1 given in [5].

(b) Let M be a minimal normal subgroup of G. Since G is π-solvable, it follows that M is either a solvable π-group or a π'-group. If M is a solvable π-group, we prove like in the proof of 2.1 given in [5] that SM/M is an X-projector of G/M. If M is a π'-group, we deduce that $M \leq O_{\pi'}(G) = 1$, which leads to the contradiction $M = 1$. So this case cannot happen. $\square$

We continue by giving two new characterizations of projectors in finite π-solvable groups.

Theorem 2.4. *Let X be a π-Schunck class with the P property with respect to π, G a finite π-solvable group and S a subgroup of G. Let*

$$C : G = G_0 > G_1 > \cdots > G_{r-1} > G_r = 1$$

be a chief chain of G. Then S is an X-projector of G if and only if G_iS/G_i is X-maximal in G/G_i, for any i.

Proof. We consider two cases: 1) $S = G$; 2) $S \neq G$.

1) $S = G$. If S is an X-projector of G, then G is an X-projector of G, and so $G \in X$. But X being a homomorph, we have also $G/G_i \in X$, for any i. It follows that $G/G_i = G_iS/G_i$ is X-maximal in G/G_i, for any i. Conversely, let G_iS/G_i be X-maximal in G/G_i, for any i. Then G/G_i is X-maximal in G/G_i, for any i. In particular, for $i = r$ we obtain that G is X-maximal in G, hence $S = G \in X$ and so $S = G$ is an X-projector of G.

2) $S \neq G$. Let S be an X-projector of G. For any i, G_i is normal in G. Hence, by 1.11, G_iS/G_i is an X-projector of G/G_i, for any i. Then, by 1.9.a), G_iS/G_i is X-maximal in G/G_i, for any i. The converse is proved by induction on $|G|$. Suppose that G_iS/G_i is X-maximal in G/G_i, for any i. In particular, for $i = r$, we obtain that S is X-maximal in G. By 1.6.a), G_i/G_{i+1} is either a solvable π-group or a π'-group, for any i. In particular, for $i = r-1$, $G_{r-1} \cong G_{r-1}/G_r$ is either a solvable π-group or a π'-group. We consider the two cases and prove in each of them that S is an X-projector of G.

a) *G_{r-1} is a solvable π-group.* Then, by 1.4, G_{r-1} is abelian. In this case, we are in the hypotheses of theorem 2.2, taking G_{r-1} as a normal abelian subgroup of G and S as a subgroup of G. Indeed, we will prove below that (i) and (ii) from 2.2 are true.

(i) S is X-maximal in $G_{r-1}S$, since S is X-maximal in G and we apply 1.9.b).

(ii) $G_{r-1}S/G_{r-1}$ is an X-projector of G/G_{r-1}. Indeed, this can be obtained if we observe that the group G/G_{r-1} is also π-solvable (by (1.6.b)) and apply for G/G_{r-1} the induction. It follows, by Theorem 2.2, that S is an X-projector of G.

b) *G_{r-1} is a π'-group.* In this case, we are in the hypotheses of theorem 1.15, where $G_{r-1} \cong G_{r-1}/G_r$ is the minimal normal subgroup of $G \cong G/G_r$ which is a π'-group. By 1.15, $G \in X$. From this and S being X-maximal in G, it follows that $S = G$. But $G \in X$ implies that G is its own X-projector. So $S = G$ is an X-projector of G. $\square$

In order to prove the second characterization of projectors in finite π-solvable groups, we first give a lemma.

Lemma 2.5. ([2]) *If X is a π-homomorph, G a π-solvable group, H a subgroup of G, $H \neq G$, H X-maximal in G and N is a minimal normal subgroup of G with $HN = G$, then N is abelian.*

Theorem 2.6. *Let X be a π-Schunck class, G a finite π-solvable group with the property (α) with respect to X and S a subgroup of G. Let*

$$C : G = G_0 > G_1 > \cdots > G_{r-1} > G_r = 1$$

be a chief chain of G such that $G/G_{r-1} \in X$. Then S is an X-projector of G if and only if G_iS/G_i is X-maximal in G/G_i, for any i.

Proof. We consider two cases: 1) $S = G$; 2) $S \neq G$.

1) $S = G$. This case has the same proof as the proof of case 1) for theorem 2.4.

2) $S \neq G$. If S is an X-projector of G, then we prove like in the proof of case 2) for theorem 2.4 that G_iS/G_i is X-maximal in G/G_i, for any i. Conversely, suppose that G_iS/G_i is X-maximal in G/G_i, for any i. In particular, for $i = r$, we obtain that

S is X-maximal in G. Obviously, $G_{r-1} \cong G_{r-1}/G_r$ is a minimal normal subgroup of $G/G_r \cong G$. Putting $i = r-1$ in our hypothesis, we have that $G_{r-1}S/G_{r-1}$ is X-maximal in G/G_{r-1}. But $G/G_{r-1} \in X$. It follows that $G_{r-1}S/G_{r-1} = G/G_{r-1}$, and so $G_{r-1}S = G$. We are now in the hypotheses of lemma 2.5, for $H = S$ and $N = G_{r-1}$. Applying 2.5, we obtain that G_{r-1} is abelian. In order to prove that S is an X-projector of G we use theorem 2.1, in our case the abelian normal subgroup of G being G_{r-1} and S verifying the conditions (i) and (ii) from 2.1. Indeed, we have:

(i) S is X-maximal in $G_{r-1}S$, since S is X-maximal in G and we apply 1.9.b).

(ii) $G_{r-1}S/G_{r-1}$ is an X-projector of G/G_{r-1}, since $G_{r-1}S = G$ implies that $G_{r-1}S/G_{r-1} = G/G_{r-1} \in X$.

By applying 2.1, it follows that S is an X-projector of G. $\square$

We finally give conditions under which the property of being projector in a finite π-solvable group is hereditary to subgroups. In preparation for the main result, we first give a lemma.

Lemma 2.7. *Let G be a finite group and let*

$$C : G = G_0 > G_1 > \cdots > G_{r-1} > G_r = 1$$

be a chief chain of G. Let H be a characteristic subgroup of G. We put $H_i = H \cap G_i$, for any i. Then

$$C^* : H = H_0 > H_1 > \cdots > H_{r-1} > H_r = 1$$

is a chief chain of H.

Proof. We have to prove that H_i/H_{i+1} is a minimal normal subgroup of H/H_{i+1}, for any i. Obviously H_i/H_{i+1} is a normal $\neq 1$ subgroup of H/H_{i+1}. Let N/H_{i+1} be a normal $\neq 1$ subgroup of G/H_{i+1} such thar $N/H_{i+1} \subseteq H_i/H_{i+1}$. We will prove that $N/H_{i+1} = H_i/H_{i+1}$. From $H_i \lhd H$ and H characteristic in G follows that $H_i \lhd G$, for any i. Furthermore,

$$H_i/H_{i+1} = (H \cap G_i)/(H \cap G_{i+1})$$

$$= (H \cap G_i)/(H \cap G_i) \cap G_{i+1} \cong (H \cap G_i)G_{i+1}/G_{i+1} \leq G_i/G_{i+1}.$$

This means that H_i/H_{i+1} is isomorphic with a normal $\neq 1$ subgroup of G/G_{i+1} included in G_i/G_{i+1}. Since G_i/G_{i+1} is a minimal normal subgroup of G/G_{i+1}, this subgroup coincides with G_i/G_{i+1}. So, for any i, we have:

$$H_i/H_{i+1} \cong G_i/G_{i+1}. \tag{2}$$

Similarly, since

$$H/H_{i+1} = (H \cap G)/(H \cap G_{i+1})$$

$$= (H \cap G)/(H \cap G) \cap G_{i+1} \cong (H \cap G)G_{i+1}/G_{i+1} \leq G/G_{i+1}$$

we deduce that H/H_{i+1} is isomorphic with a subgroup G^*/G_{i+1} of G/G_{i+1}. But from $N \lhd H$ and H characteristic in G follows that $N \lhd G$. From this and from $N/H_{i+1} \subseteq H_i/H_{i+1}$, it follows by (2) that N/H_{i+1} is isomorphic with a normal $\neq 1$ subgroup of G/G_{i+1} included in G_i/G_{i+1}. Since G_i/G_{i+1} is a minimal normal subgroup of G/G_{i+1}, this subgroup coincides with G_i/G_{i+1}, which means that

$$N/H_{i+1} \cong G_i/G_{i+1}. \tag{3}$$

From (2) and (3) follows that $N/H_{i+1} \cong H_i/H_{i+1}$. But $N/H_{i+1} \subseteq H_i/H_{i+1}$. So $N/H_{i+1} = H_i/H_{i+1}$. $\square$

Theorem 2.8. *Let X be a π-Schunck class with the P property with respect to π, G a finite π-solvable group and $S \leq H \leq G$ such that H is characteristic in G. If S is an X-projector of G, then S is an X-projector of H.*

Proof. By 1.6.a), G has a chief chain

$$C : G = G_0 > G_1 > \cdots > G_{r-1} > G_r = 1$$

such that G_i/G_{i+1} is either a solvable π-group or a π'-group, for any i. Since S is an X-projector of G, by applying theorem 2.4, we deduce that G_iS/G_i is X-maximal in G/G_i, for any i. We put $H_i = H \cap G_i$, for any i. Applying lemma 2.7 for the group G, its chief chain C and its characteristic subgroup H, it follows that

$$C^* : H = H_0 > H_1 > \cdots > H_{r-1} > H_r = 1$$

is a chief chain of H. Since G is π-solvable, by 1.6.b) follows that H is also π-solvable. In order to prove that S is an X-projector of H we use theorem 2.4 for the π-solvable group H and its chief chain C^*. It remains only to show that H_iS/H_i is X-maximal in H/H_i, for any i. Indeed, we have that

$$H_iS/H_i \cong S/S \cap H_i = S/S \cap (H \cap G_i) = S/(S \cap H) \cap G_i = S/S \cap G_i \cong G_iS/G_i$$

and so

$$H_iS/H_i \cong G_iS/G_i. \tag{4}$$

We also have that

$$H/H_i = H/H \cap G_i \cong HG_i/G_i \leq G/G_i.$$

From G_iS/G_i X-maximal in G/G_i and $G_iS/G_i \leq G_iH/G_i \leq G/G_i$ follows by 1.9.b) that

$$G_iS/G_i \text{ is } X-\text{maximal in } HG_i/G_i. \tag{5}$$

We also have

$$HG_i/G_i \cong H/H \cap G_i = H/H_i \tag{6}$$

and

$$H_iS/H_i \leq H/H_i. \tag{7}$$

(5) becomes by using (4), (6) and (7):

$$H_iS/H_i \text{ is } X-\text{maximal in } H/H_i, \text{ for any } i.$$

Finally, applying theorem 2.4, it follows that S is an X-projector of H. $\square$

3. Primitive classes. Their connection with projectors in finite π-solvable groups

In 1.7.b), we defined the notion of *primitive group*. If G is a group and N is a normal subgroup of G, we will call a primitive quotient group G/N simply by a *primitive factor* of G.

Proposition 3.1. ([7]) Let G be a finite group.

a) If N is a normal subgroup of G, then: G/N is primitive if and only if there is a maximal subgroup W of G such that $N = core_G W$.

b) If W is a maximal subgroup of G, then $G/core_G W$ is primitive.

Notation 3.2. We put Pr for the class of all finite primitive groups.

Remark 3.3. If G is a primitive group and N is a normal subgroup of G, it doesn't generally follow that G/N is primitive. So the class Pr is neither a homomorph nor a Schunck class.

In [7], W. Gaschütz introduced the notion of *primitive class* and gave some theorems which reveal a connection between primitive classes and projectors in the universe of finite solvable groups. It is the main aim of this paper to give conditions under which such a connection also holds in the more general case of π-solvable groups.

Definition 3.4. ([7]) A class P of finite groups is said to be a *primitive class* if P satisfies the following two conditions:

(i) if $G \in P$, then G is primitive;

(ii) if $G \in P$ and G/N is a primitive factor of G, then $G/N \in P$.

In other words, a primitive class P is a class of finite groups such that $P \subset Pr$ and which together with a group G contains all primitive factors of G.

Notation 3.5. ([7]) Let G be a finite group. We put

$$P(G) = \{G/N \ / \ G/N \text{ is primitive}\}.$$

Proposition 3.6. If G is a finite group, then $P(G)$ is a primitive class.

Proof. Obviously $P(G)$ satisfies (i). In order to verify (ii), let $G/N \in P(G)$ and let $(G/N)/(M/N)$ be a primitive factor of G/N. Since

$$(G/N)/(M/N) \cong G/M, \tag{8}$$

we deduce that G/M is a primitive factor of G, and so $G/M \in P(G)$. It follows by (8) that $(G/N)/(M/N) \in P(G)$. So (ii) is also satisfied. $\square$

Definition 3.7. Let G be a finite group. We call $P(G)$ given by

$$P(G) = \{G/N \ / \ G/N \text{ is primitive}\}$$

the primitive class associated to the group G.

Notation 3.8. ([7]) Let X be an arbitrary class of groups. We denote by

$$P(X) = \cup\{P(G)/G \in X\}$$

the union of the primitive classes associated to all groups belonging to the class X.

b) Let P be a primitive class. We denote by

$$X(P) = \{G \ / \ G/N \in P, \text{ for all primitive factors } G/N \text{ of } G\},$$

i.e. $X(P)$ is the class of those finite groups G whose primitive factors are in P.

Proposition 3.9. ([7]) If X is a Schunck class and P is a primitive class, then:

a) $X = X(P(X))$;

b) $P = P(X(P))$.

We give below the main results of this paper.

Theorem 3.10. *Let X be a π-Schunck class with the P property with respect to π, G a finite π-solvable group and S a subgroup of G. If S is an X-projector of G, then:*

(i) $P(S) \subseteq X$;

(ii) $S \leq W < H \leq G$, with W maximal in H and H characteristic in G imply that $H/core_H W \notin X$.

Proof. (i) S being an X-projector of G, by 1.9.a) S is X-maximal in G, hence $S \in X$. Let $S/N \in P(S)$. Since $S \in X$ and X being a homomorph, we obtain that $S/N \in X$. So $P(S) \subseteq X$.

(ii) Let $S \leq W < H \leq G$, with W maximal in H and H characteristic in G. Suppose that $H/core_H W \in X$. By theorem 2.8, it follows that S is an X-projector of H. Hence by theorem 1.11 we obtain that $S\ core_H W/core_H W$ is an X-projector in $H/core_H W$, and so by 1.9.a) $S\ core_H W/core_H W$ is X-maximal in $H/core_H W$. But we supposed that $H/core_H W \in X$. Hence $S\ core_H W/core_H W = H/core_H W$ and so $S\ core_H W = H$. It follows that

$$H = S\ core_H W \leq W,$$

in contradiction with the hypothesis that W is a maximal subgroup of H. So we conclude that $H/core_H W \notin X$. $\square$

Theorem 3.11. *Let X be a π-Schunck class with the P property with respect to π, G a finite π-solvable group and S a subgroup of G. If the following two conditions are verified:*

(i) $P(S) \subseteq X$;

(ii) $S \leq W < H \leq G$, with W maximal in $H \Rightarrow H/core_H W \notin X$,

then S is an X-projector of G.

Proof. We prove by induction on $|G|$.

We first remark that $S \in X$. Indeed, from (i) follows that any primitive factor S/N of S is in X, which, by the primitive closure of the Schunck class X, leads to $S \in X$.

Let M be a minimal normal subgroup of G. We put $S^* = SM$. Applying the induction for the π-solvable group G/M (see 1.6.b)) and its subgroup S^*/M, we deduce that S^*/M is an X-projector of G/M.

We now prove that S is X-maximal in S^*. We saw that $S \in X$. Let $S \leq T \leq S^*$ and $T \in X$. We will prove that $S = T$. Suppose that $S \neq T$. Let then W be a maximal subgroup of T such that $S \leq V < T \leq S^* \leq G$. Applying (ii), we obtain that $T/core_T W \notin X$, in contradiction with $T/core_T W \in X$, which comes from $T \in X$ by applying the hypothesis that X is a homomorph. So $S = T$.

G being a π-solvable group and M being a minimal normal subgroup of G, it follows that M is either a solvable π-group or a π'-group. We continue the proof by considering two cases:

1) M is a solvable π-group. Then, by theorem 1.4, M is abelian. So we are in the hypotheses of theorem 2.2, where we take M as normal abelian subgroup of G. Applying theorem 2.2, we obtain that S is an X-projector of G.

2) M is a π'-group. Since X has the P property with respect to π, it follows that $G/M \in X$. From the fact that S^*/M is an X-projector of G/M we deduce, by 1.9.a), that S^*/M is X-maximal in G/M, which gives, using $G/M \in X$, that $S^*/M = G/M$. So $S^* = G$. We saw that S is X-maximal in S^*. It follows that

S is X-maximal in G. On the other side, by applying theorem 1.15, we obtain that $G \in X$. This leads to $S = G \in X$. So $S = G$ is its own X-projector. $\square$

In our last considerations, we generalize the definition of the notion of *projector* given by W. Gaschütz in [7] and give some connection of projectors with primitive classes in the universe of finite π-solvable groups.

Definition 3.12. Let G be a finite π-solvable group and S a subgroup of G. We say that S is a *projector* of G if there is some π-Schunck class X with the P property with respect to π, such that S is an X-projector of G.

Theorem 3.13. *Let G be a finite π-solvable group and S a subgroup of G. If S is a projector of G, then the following implication is true: $S \leq W < H \leq G$, with W maximal in H and H characteristic in G $\Rightarrow$ $H/core_H W \notin P(S)$.*

Proof. Let S be a projector of G and let $S \leq W < H \leq G$, with W maximal in H and H characteristic in G. It follows that there is a π-Schunck class X with the P property with respect to π, such that S is an X-projector of G. Applying now theorem 3.10, it follows that $P(S) \subseteq X$ and also that $S \leq W < H \leq G$, with W maximal in H and H characteristic in G imply $H/core_H W \notin X$. Then obviously $H/core_H W \notin P(S)$. $\square$

Theorem 3.14. *Let G be a finite π-solvable group and S a subgroup of G. If*

(i) $X = X(P(S))$ is a π-Schunck class with the P property with respect to π;

(ii) $S \leq V < H \leq G$, with W maximal in H $\Rightarrow$ $H/core_H W \notin P(S)$,

then S is a projector of G.

Proof. It is easy to see that $P(S) \subseteq X(P(S)) = X$. Let $S \leq W < H \leq G$, with W maximal in H. By 3.1.b), $H/core_H W$ is primitive. By (ii), we have that $H/core_H W \notin P(S)$. From this and from $P(S) \subseteq X(P(S)) = X$ and $H/core_H W$ primitive, we deduce that $H/core_H W \notin X = X(P(S))$. So we proved that conditions (i) and (ii) from theorem 3.11 are verified. Applying now theorem 3.11, we obtain that S is an X-projector of G, where $X = X(P(S))$ is a π-Schunck class with the P property with respect to π. Hence S is a projector of G. $\square$

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