ON ANALOGS OF THE DUAL BRUNN-MINKOWSKI INEQUALITY FOR WIDTH-INTEGRALS OF CONVEX BODIES

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Abstract. In this paper we prove two new inequalities about width-integrals of centroid and projection bodies. Two analogs of the dual Brunn-Minkowski inequality for width-integral of convex bodies are established.

0. Definitions and preliminary results

The setting for this paper is \( n \)-dimensional Euclidean space \( \mathbb{R}^n (n > 2) \). Let \( K^n \) denote the set of convex bodies (compact, convex subsets with non-empty interiors) in \( \mathbb{R}^n \). Let \( \varphi^n \) denote the set of star bodies in \( \mathbb{R}^n \). The subset of \( \varphi^n \) consisting of the centred star bodies will be denoted by \( \varphi_c^n \). We reserve the letter \( u \) for unit vectors, and the letter \( B \) is reserved for the unit ball centered at the origin. The surface of \( B \) is \( S^{n-1} \). For \( u \in S^{n-1} \), let \( E_u \) denote the hyperplane, through the origin, that is orthogonal to \( u \). We will use \( K^n_u \) to denote the image of \( K \) under an orthogonal projection onto the hyperplane \( E_u \).

We use \( V(K) \) for the \( n \)-dimensional volume of convex body \( K \). Let \( h(K, \cdot) : S^{n-1} \rightarrow \mathbb{R} \), denote the support function of \( K \in K^n \); i.e.

\[
 h(K, u) = \text{Max} \{ u \cdot x : x \in K \}, \quad u \in S^{n-1},
\]

where \( u \cdot x \) denotes the usual inner product \( u \) and \( x \) in \( \mathbb{R}^n \).

Let \( \delta \) denote the Hausdorff metric on \( K^n \); i.e., for \( K, L \in K^n \),

\[
 \delta(K, L) = |h_K - h_L|_{\infty},
\]
where $| \cdot |_\infty$ denotes the sup-norm on the space of continuous functions, $C(S^{n-1})$.

For a convex body $K$ and a nonnegative scalar $\lambda, \lambda K$, is used to denote \{\lambda x : x \in K\}. For $K_i \in \mathcal{K}^n$, $\lambda_i \geq 0, (i = 1, 2, \ldots, r)$, the Minkowski linear combination $\lambda_1 K_1 + \cdots + \lambda_r K_r \in \mathcal{K}^n$ is defined by

$$\lambda_1 K_1 + \cdots + \lambda_r K_r = \{\lambda_1 x_1 + \cdots + \lambda_r x_r \in \mathcal{K}^n : x_i \in K_i\}. \quad (2)$$

It is trivial to verify that

$$h(\lambda_1 K_1 + \cdots + \lambda_r K_r, \cdot) = \lambda_1 h(K_1, \cdot) + \cdots + \lambda_r h(K_r, \cdot). \quad (3)$$

### 1.1 Mixed volumes

If $K_i \in \mathcal{K}^n (i = 1, 2, \ldots, r)$ and $\lambda_i (i = 1, 2, \ldots, r)$ are nonnegative real numbers, then of fundamental impotence is the fact that the volume of $\lambda_1 K_1 + \cdots + \lambda_r K_r$ is a homogeneous polynomial in $\lambda_i$ given by $[4,p.275]

$$V(\lambda_1 K_1 + \cdots + \lambda_r K_r) = \sum_{i_1, \ldots, i_n} \lambda_{i_1} \ldots \lambda_{i_n} V_{i_1 \ldots i_n}, \quad (4)$$

where the sum is taken over all $n$-tuples $(i_1, \ldots, i_n)$ of positive integers not exceeding $r$. The coefficient $V_{i_1 \ldots i_n}$ depends only on the bodies $K_{i_1}, \ldots, K_{i_n}$, and is uniquely determined by (8), it is called the mixed volume of $K_{i_1}, \ldots, K_{i_n}$, and is written as $V(K_{i_1}, \ldots, K_{i_n})$. Let $K_1 = \cdots = K_{n-r} = K$ and $K_{n-i+1} = \cdots = K_n = L$, then the mixed volume $V(K_1 \ldots K_n)$ is usually written $V_i(K, L)$. If $L = B$, then $V_i(K, B)$ is the $i$th projection measure (Quermassintegral) of $K$ and is written as $W_i(K)$.

If $K_i (i = 1, 2, \ldots, n-1) \in \mathcal{K}^n$, then the mixed volume of the convex figures $K_i^u (i = 1, 2, \ldots, n-1)$ in the $(n-1)$-dimensional space $E_n$ will be denoted by $v(K_1^u, \ldots, K_{n-1}^u)$. It is well known, and easily shown $[5,p.45]$, that for $K_i \in \mathcal{K}^n (i = 1, 2, \ldots, n-1)$, and $u \in S^{n-1}$

$$v(K_1^u, \ldots, K_{n-1}^u) = n V(K_1, \ldots, K_{n-1}, [u]) \quad (5)$$

where $[u]$ denotes the line segment joining $u/2$ and $-u/2$.

### 1.2 Width-integrals of convex bodies

For $u \in S^{n-1}$, $b_K = \frac{1}{2}(h(K, u) + h(K, -u))$ is called as half the width of $K$ in the direction $u$. Two convex bodies $K$ and $L$ are said to have similar width if there exists

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a constant $\lambda > 0$ such that $b_K = \lambda b_L$ for all $u \in S^{n-1}$. The width-integral of index $i$ is defined by: For $K \in \mathcal{K}^n$, $i \in \mathbb{R}$

$$B_i(K) = \frac{1}{n} \int_{S^{n-1}} b_K^{n-i} dS(u),$$

(6)

where $dS$ is the $(n-1)$-dimensional volume element on $S^{n-1}$. The width-integral of index $i$ is a map

$$B_i : \mathcal{K}^n \to \mathbb{R}.$$ 

It is positive, continuous, homogeneous of degree $n-i$ and invariant under motion. In addition, for $i \leq n$ it is also bounded and monotone under set inclusion.

The following result easy is proved, for $K_j \in \mathcal{K}^n (j = 1, \ldots, m)$

$$b_{K_1 + \cdots + K_m} = b_{K_1} + \cdots + b_{K_m},$$

(7)

1.3 The Blaschke linear combination and the harmonic Blaschke linear combination

A convex body $K$ is said to have a positive continuous curvature function [8],

$$f(K, \cdot) : S^{n-1} \to [0, \infty),$$

if for each $L \in \varphi^n$, the mixed volume $V_1(K, L)$ has the integral representation

$$V_1(K, L) = \frac{1}{n} \int_{S^{n-1}} f(K, u) h(L, u) dS(u).$$

The subset of $\mathcal{K}^n$ consisting of bodies which have a positive continuous curvature function will be denoted by $\kappa^n$. Let $\kappa^n_\sigma$ denote the set of centrally symmetric member of $\kappa^n$.

The following result is true [9], for $K \in \kappa^n$

$$\int_{S^{n-1}} u f(K, u) dS(u) = 0.$$ 

Suppose $K, L \in \kappa^n$ and $\lambda, \mu \geq 0$(not both zero). From above it follows that the function $\lambda f(K, \cdot) + \mu f(L, \cdot)$ satisfies the hypothesis of Minkowski’s existence theorem (see [5]). The solution of the Minkowski problem for this function is denoted by

$$\lambda \cdot K + \mu \cdot L$$

that is

$$f(\lambda \cdot K + \mu \cdot L, \cdot) = \lambda f(K, \cdot) + \mu f(L, \cdot),$$

(8)
where the linear combination \( \lambda \cdot K + \mu \cdot L \) is called a Blaschke linear combination.

The relationship between Blaschke and Minkowski scalar multiplication is given by

\[
\lambda \cdot K = \lambda^{1/(n-1)} K. \tag{9}
\]

A new addition, harmonic Blaschke addition, be defined by Lutwak [8]. Suppose \( K, L \in \varphi^n \), and \( \lambda, \mu \geq 0 \)(not both zero). To define the harmonic Blaschke linear combination, \( \lambda K + \mu L \), first define \( \xi > 0 \) by

\[
\xi^{1/(n+1)} = \frac{1}{n} \int_{S^{n-1}} \left[ \lambda V(K)^{-1} \rho(K, u)^{n+1} + \mu V(L)^{-1} \rho(L, u)^{n+1} \right]^{n/(n+1)} dS(u). \tag{10}
\]

The body \( \lambda K + \mu L \in \varphi^n \) is defined as the body whose radial function is given by

\[
\xi^{-1} \rho(\lambda K + \mu L, \cdot)^{n+1} = \lambda V(K)^{-1} \rho(K, \cdot)^{n+1} + \mu V(L)^{-1} \rho(L, \cdot)^{n+1}. \tag{11}
\]

It follows immediately that \( \xi = V(\lambda K + \mu L) \), and hence

\[
V(\lambda K + \mu L)^{-1} \rho(\lambda K + \mu L, \cdot)^{n+1} = \lambda V(K)^{-1} \rho(K, \cdot)^{n+1} + \mu V(L)^{-1} \rho(L, \cdot)^{n+1}.
\]

Lutwak [10] define a mapping:

\[
\Lambda : \varphi^n \rightarrow \kappa^n
\]

and point out that \( \Lambda \) transforms harmonic Blaschke linear combination into Blaschke linear combinations, i.e.

If \( K, L \in \varphi^n \) and \( \lambda, \mu \geq 0 \), then

\[
\Lambda(\lambda K + \mu L) = \lambda \cdot \Lambda K + \mu \cdot \Lambda L.
\]

Further, We obtain that

If \( K_j \in \varphi^n (j = 1, \ldots, m) \), and \( \lambda_j \geq 0(j = 1, \ldots, m) \), then

\[
\Lambda(\lambda_1 K_1 + \cdots + \lambda_m K_m) = \lambda_1 \cdot \Lambda K_1 + \cdots + \lambda_m \cdot \Lambda K_m. \tag{12}
\]

and

\[
\Lambda(\lambda K) = \lambda \Lambda K \tag{13}
\]
1.4 Projection bodies and Centroid bodies

The projection bodies, $\Pi K$, of the body $K \in \mathcal{K}^n$ is defined as the convex figure whose support function is given, for $u \in S^{n-1}$, by

$$h(\Pi K, u) = v(K^n)$$

(14)

It is easy to see, that a projection body is always centered (symmetric about the origin), and if $K$ has interior points then $\Pi K$ will have interior point as well. Here, we introduce the following property.

If $K, L \in \mathcal{K}^n$ and $\lambda, \mu \geq 0$, then

$$\Pi(\lambda \cdot K + \mu \cdot L) = \lambda \Pi K + \mu \Pi L.$$  
(15)

Further, we may prove that

If $K_j \in \mathcal{K}^n (j = 1, \ldots, m)$ and $\lambda_j \geq 0 (j = 1, \ldots, m)$, then

$$\Pi(\lambda_1 \cdot K_1 + \cdots + \lambda_m \cdot K_m) = \lambda_1 \Pi K_1 + \cdots + \lambda_m \Pi K_m.$$  
(16)

The centroid body, $\Gamma K$, of $K \in \mathcal{K}^n$, is the convex body whose support function, at $x \in \mathbb{R}^n$, is given by

$$h(\Gamma K, x) = \frac{1}{V(K)} \int_K |x \cdot y| \, dy.$$  
(17)

Here, we give the following property:

If $K_j \in \mathcal{K}^n (j = 1, \ldots, m)$, and $\lambda_j \geq 0 (j = 1, \ldots, m)$, then

$$\Gamma(\lambda_1 \cdot K_1 + \cdots + \lambda_m \cdot K_m) = \lambda_1 \Gamma K_1 + \cdots + \lambda_m \Gamma K_m.$$  
(18)

If $K \in \mathcal{K}^n$, then from (20) it follows that $\Gamma K$ is centered.

Please see the next section for above interrelated notations, definitions and their background material.

1. Main results

Width-integrals were first considered by Blaschke [1,p.85] and later by Hadwiger [2,p.266]. In [3], Lutwak also introduced the width-integral of index $i$ and proved some important results, one of them is the following Theorem:
**Theorem A.** If $K, L \in \mathcal{K}^n$ and $i < n - 1$, then
\[ B_i(K + L)^{1/(n-i)} \leq B_i(K)^{1/(n-i)} + B_i(L)^{1/(n-i)} \] (18)
with equality if and only if $K$ and $L$ have similar width.

Since inequality (1) is a new result similar to the following Brunn-Minkowski inequality for the cross-sectional measures [2, p.249].

**Theorem B.** If $K, L \in \mathcal{K}^n$ and $i < n - 1$, then
\[ W_i(K + L)^{1/(n-i)} \leq W_i(K)^{1/(n-i)} + W_i(L)^{1/(n-i)} \] (19)
with equality if and only if $K$ and $L$ are homothetic.

Hence, inequality (1) is called as the dual Brunn-Minkowski inequality for width-integrals of convex bodies.

The main purpose of this paper is to establish two analogs of inequality (1), them can be stated as:

**Theorem C.** If $K_1, \ldots, K_m \in \varphi^n$, $\lambda_1, \ldots, \lambda_m > 0$ and $i < n - 1$, then
\[ B_i(\Gamma(\lambda_1 K_1 + \cdots + \lambda_m K_m))^{1/(n-i)} \leq \lambda_1 B_i(\Gamma K_1)^{1/(n-i)} + \cdots + \lambda_m B_i(\Gamma K_m)^{1/(n-i)}, \] (20)
with equality if and only if $\Gamma K_j (j = 1, 2, \ldots, m)$ have similar width.

**Theorem D.** If $K_1, \ldots, K_m \in \varphi^n_\subset$, $\lambda_1, \ldots, \lambda_m > 0$ and $i < n - 1$, then
\[ B_i(\Pi(\Lambda(\lambda_1 K_1 + \cdots + \lambda_m K_m)))^{1/(n-i)} \leq \lambda_1 B_i(\Pi(\Lambda K_1))^{1/(n-i)} + \cdots + \lambda_m B_i(\Pi(\Lambda K_m))^{1/(n-i)}, \] (21)
with equality if and only if $\Pi(\Lambda K_j) (j = 1, 2, \ldots, m)$ have similar width.

2. A dual Brunn-Minkowski inequality about the width-integrals of centroid bodies for the harmonic Blaschke linear combination

The following dual Brunn-Minkowski inequality about the width-integrals of centroid bodies will be proved.

**Theorem C.** If $K_1, \ldots, K_m \in \varphi^n$, $\lambda_1, \ldots, \lambda_m > 0$ and $i < n - 1$, then
\[ B_i(\Gamma(\lambda_1 K_1 + \cdots + \lambda_m K_m))^{1/(n-i)} \leq \lambda_1 B_i(\Gamma K_1)^{1/(n-i)} + \cdots + \lambda_m B_i(\Gamma K_m)^{1/(n-i)}, \] (22)
with equality if and only if $\Gamma K_j (j = 1, 2, \ldots, m)$ have similar width.

Proof. From (7), (10), (11), (21) and in view of Minkowski inequality for integral [11, p.147], we obtain that

$$B_i(\Gamma(\lambda_1 K_1 + \cdots + \lambda_m K_m))^{1/(n-i)} = \left( \frac{1}{n} \int_{S^{n-1}} b_{\Gamma(\lambda_1 K_1 + \cdots + \lambda_m K_m)}^{n-i} dS(u) \right)^{1/(n-i)}$$

$$= \left( \frac{1}{n} \int_{S^{n-1}} (\lambda_1 b_{\Gamma K_1} + \cdots + \lambda_m b_{\Gamma K_m})^{n-i} dS(u) \right)^{1/(n-i)}$$

$$\leq \lambda_1 \left( \frac{1}{n} \int_{S^{n-1}} b_{\Gamma K_1}^{n-i} dS(u) \right)^{1/(n-i)} + \cdots + \lambda_m \left( \frac{1}{n} \int_{S^{n-1}} b_{\Gamma K_m}^{n-i} dS(u) \right)^{1/(n-i)}$$

$$= \lambda_1 B_i(\Gamma K_1) + \cdots + \lambda_m B_i(\Gamma K_m),$$

with equality if and only if $\Gamma K_j (j = 1, \ldots, m)$ have similar width.

The proof is complete. □

Taking $m = 2$ to (23), we have

Corollary 1. If $K, L \in \varphi^n$, $\lambda, \mu > 0$ and $i < n - 1$, then

$$B_i(\Gamma(\lambda K + \mu L))^{1/(n-i)} \leq \lambda B_i(\Gamma K)^{1/(n-i)} + \mu B_i(\Gamma L)^{1/(n-i)},$$

(23)

with equality if and only if $\Gamma K$ and $\Gamma L$ have similar width.

Another important consequence is obtained when $\lambda = \mu = 1$.

Corollary 2. If $K, L \in \varphi^n$ and $i < n - 1$, then

$$B_i(\Gamma(\lambda K + \mu L))^{1/(n-i)} \leq B_i(\Gamma K)^{1/(n-i)} + B_i(\Gamma L)^{1/(n-i)},$$

(24)

with equality if and only if $\Gamma K$ and $\Gamma L$ have similar width.

3. A dual Brunn-Minkowski inequality about the width-integrals of projection bodies for the harmonic Blaschke linear combination

The following dual Brunn-Minkowski inequality about the width-integrals of projection bodies will be proved.

Theorem D. If $K_1, \ldots, K_m \in \varphi^n$, $\lambda_1, \ldots, \lambda_m > 0$ and $i < n - 1$, then

$$B_i(\Pi(\Lambda(\lambda_1 K_1 + \cdots + \lambda_m K_m)))^{1/(n-i)}$$
\[ \leq \lambda_1 B_i(\Pi(\Lambda K_1))^{1/(n-i)} + \ldots + \lambda_m B_i(\Pi(\Lambda K_m))^{1/(n-i)}, \quad (25) \]

with equality if and only if \( \Pi(\Lambda K_j)(j = 1, 2, \ldots, m) \) have similar width.

**Proof.** From (7), (10), (11), (16), (19) and in view of Minkowski inequality for integral [11, p.147], we obtain that

\[
B_i(\Pi(\Lambda(\lambda_1 K_1 + \cdots + \lambda_m K_m)))^{\frac{1}{n-i}} = \left( \frac{1}{n} \int_{S_{n-1}} b^{n-i}_{\Pi(\Lambda(\lambda_1 K_1 + \cdots + \lambda_m K_m))} dS(u) \right)^{\frac{1}{n-i}} \\
= \left( \frac{1}{n} \int_{S_{n-1}} b^{n-i}_{\Pi(\lambda_1 \Lambda K_1 + \cdots + \lambda_m \Lambda K_m)} dS(u) \right)^{\frac{1}{n-i}} \\
= \left( \frac{1}{n} \int_{S_{n-1}} b^{n-i}_{\sum_{j=1}^{m} \lambda_j \Pi(\Lambda K_j)} dS(u) \right)^{\frac{1}{n-i}} \\
\leq \sum_{j=1}^{m} \lambda_j \left( \frac{1}{n} \int_{S_{n-1}} b^{n-i}_{\Pi(\Lambda K_j)} dS(u) \right)^{\frac{1}{n-i}} = \sum_{j=1}^{m} \lambda_j B_i(\Pi(\Lambda K_j))^{\frac{1}{n-i}},
\]

with equality if and only if \( \Pi(\Lambda K_j)(j = 1, \ldots, m) \) have similar width.

The proof is complete. \( \square \)

Taking \( m = 2 \) to (26), we have

**Corollary 3.** If \( K, L \in \varphi_n^0, \lambda, \mu > 0 \) and \( \lambda < n - 1 \), then

\[
B_i(\Pi(\Lambda(\lambda K_1 + \lambda_2 K_2)))^{1/(n-i)} \leq AB_i(\Pi(\Lambda K))^{1/(n-i)} + \mu B_i(\Pi(\Lambda L))^{1/(n-i)},
\]

with equality if and only if \( \Pi(\Lambda K) \) and \( \Pi(\Lambda L) \) have similar width.

Another remarkable case is obtained for \( \lambda = \mu = 1 \).

**Corollary 4.** If \( K, L \in \varphi_n^0 \) and \( \lambda < n - 1 \), then

\[
B_i(\Pi(\Lambda(\lambda K + \lambda_2 L)))^{1/(n-i)} \leq B_i(\Pi(\Lambda K))^{1/(n-i)} + B_i(\Pi(\Lambda L))^{1/(n-i)},
\]

with equality if and only if \( \Pi(\Lambda K) \) and \( \Pi(\Lambda L) \) have similar width.

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