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ON ANALOGS OF THE DUAL BRUNN-MINKOWSKI INEQUALITY FOR WIDTH-INTEGRALS OF CONVEX BODIES

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Abstract. In this paper we prove two new inequalities about width-integrals of centroid and projection bodies. Two analogs of the dual Brunn-Minkowski inequality for width-integral of convex bodies are established.

0. Definitions and preliminary results

The setting for this paper is n-dimensional Euclidean space $\mathbb{R}^n (n > 2)$. Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) in $\mathbb{R}^n$. Let φ^n denote the set of star bodies in $\mathbb{R}^n$. The subset of φ^n consisting of the centred star bodies will be denoted by φ_c^n. We reserve the letter u for unit vectors, and the letter B is reserved for the unit ball centered at the origin. The surface of B is S^{n-1}. For $u \in S^{n-1}$, let E_u denote the hyperplane, through the origin, that is orthogonal to u. We will use K^u to denote the image of K under an orthogonal projection onto the hyperplane E_u.

We use $V(K)$ for the n-dimensional volume of convex body K. Let $h(K, \cdot) : S^{n-1} \to R$, denote the support function of $K \in \mathcal{K}^n$; i.e.

$$h(K, u) = Max\{u \cdot x : x \in K\}, u \in S^{n-1}, \tag{1}$$

where $u \cdot x$ denotes the usual inner product u and x in $\mathbb{R}^n$.

Let δ denote the Hausdorff metric on $\mathcal{K}^n$; i.e., for $K, L \in \mathcal{K}^n$,

$$\delta(K, L) = |h_K - h_L|_\infty,$$

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where $|\cdot|_\infty$ denotes the sup-norm on the space of continuous functions, $C(S^{n-1})$.

For a convex body K and a nonnegative scalar $\lambda, \lambda K$, is used to denote $\{\lambda x : x \in K\}$. For $K_i \in \mathcal{K}^n, \lambda_i \geq 0, (i = 1, 2, \ldots, r)$,the Minkowski linear combination $\lambda_1 K_1 + \cdots + \lambda_r K_r \in \mathcal{K}^n$ is defined by

$$\lambda_1 K_1 + \cdots + \lambda_r K_r = \{\lambda_1 x_1 + \cdots + \lambda_r x_r \in \mathcal{K}^n : x_i \in K_i\}. \tag{2}$$

It is trivial to verify that

$$h(\lambda_1 K_1 + \cdots + \lambda_r K_r, \cdot) = \lambda_1 h(K_1, \cdot) + \cdots + \lambda_r h(K_r, \cdot). \tag{3}$$

1.1 Mixed volumes

If $K_i \in \mathcal{K}^n (i = 1, 2, \ldots, r)$ and $\lambda_i (i = 1, 2, \ldots, r)$are nonnegative real numbers, then of fundamental impotence is the fact that the volume of $\lambda_1 K_1 + \cdots + \lambda_r K_r$ is a homogeneous polynomial in λ_i given by [4,p.275]

$$V(\lambda_1 K_1 + \cdots + \lambda_r K_r) = \sum_{i_1, \ldots, i_n} \lambda_{i_1} \ldots \lambda_{i_n} V_{i_1 \ldots i_n}, \tag{4}$$

where the sum is taken over all n-tuples $(i_1, \ldots, i_n)$ of positive integers not exceeding r. The coefficient $V_{i_1 \ldots i_n}$ depends only on the bodies $K_{i_1}, \ldots, K_{i_n}$, and is uniquely determined by (8), it is called the mixed volume of $K_i, \ldots, K_{i_n}$, and is written as $V(K_{i_1}, \ldots, K_{i_n})$. Let $K_1 = \ldots = K_{n-i} = K$ and $K_{n-i+1} = \ldots = K_n = L$, then the mixed volume $V(K_1 \ldots K_n)$ is usually written $V_i(K, L)$. If $L = B$, then $V_i(K, B)$ is the ith projection measure(Quermassintegral) of K and is written as $W_i(K)$.
If $K_i (i = 1, 2, \ldots, n-1) \in \mathcal{K}^n$, then the mixed volume of the convex figures $K_i^u (i = 1, 2, \ldots, n-1)$ in the $(n-1)$-dimensional space E_u will be denoted by $v(K_1^u, \ldots, K_{n-1}^u)$. It is well known, and easily shown [5,p.45], that for $K_i \in \mathcal{K}^n (i = 1, 2, \ldots, n-1)$, and $u \in S^{n-1}$

$$v(K_1^u, \ldots, K_{n-1}^u) = nV(K_1, \ldots, K_{n-1}, [u]) \tag{5}$$

where $[u]$ denotes the line segment joining $u/2$ and $-u/2$.

1.2 Width-integrals of convex bodies

For $u \in S^{n-1}$, $b_K = \frac{1}{2}(h(K, u) + h(K, -u))$ is called as half the width of K in the direction u. Two convex bodies K and L are said to have similar width if there exists

a constant $\lambda > 0$ such that $b_K = \lambda b_L$ for all $u \in S^{n-1}$. The width-integral of index i is defined by: For $K \in \mathcal{K}^n, i \in \mathbb{R}$

$$B_i(K) = \frac{1}{n} \int_{S^{n-1}} b_K^{n-i} dS(u), \tag{6}$$

where dS is the $(n-1)$-dimensional volume element on S^{n-1}.
The width-integral of index i is a map

$$B_i : \mathcal{K}^n \to \mathbb{R}.$$

It is positive, continuous, homogeneous of degree $n-i$ and invariant under motion. In addition, for $i \leq n$ it is also bounded and monotone under set inclusion.

The following result easy is proved, for $K_j \in \mathcal{K}^n (j = 1, \ldots, m)$

$$b_{K_1 + \cdots + K_m} = b_{K_1} + \cdots + b_{K_m}, \tag{7}$$

1.3 The Blaschke linear combination and the harmonic Blaschke linear combination

A convex body K is said to have a positive continuous curvature function [8],

$$f(K, \cdot) : S^{n-1} \to [0, \infty),$$

if for each $L \in \varphi^n$, the mixed volume $V_1(K, L)$ has the integral representation

$$V_1(K, L) = \frac{1}{n} \int_{S^{n-1}} f(K, u) h(L, u) dS(u).$$

The subset of $\mathcal{K}^n$ consisting of bodies which have a positive continuous curvature function will be denoted by κ^n. Let κ_c^n denote the set of centrally symmetric member of κ^n.

The following result is true [9], for $K \in \kappa^n$

$$\int_{S^{n-1}} u f(K, u) dS(u) = 0.$$

Suppose $K, L \in \kappa^n$ and $\lambda, \mu \geq 0$(not both zero). From above it follows that the function $\lambda f(K, \cdot) + \mu f(L, \cdot)$ satisfies the hypothesis of Minkowski's existence theorem (see [5]). The solution of the Minkowski problem for this function is denoted by $\lambda \cdot K \ddot{+} \mu \cdot L$ that is

$$f(\lambda \cdot K \ddot{+} \mu \cdot L, \cdot) = \lambda f(K, \cdot) + \mu f(L, \cdot), \tag{8}$$

where the linear combination $\lambda \cdot K \ddot{+} \mu \cdot L$ is called a Blaschke linea combination.

The relationship between Blaschke and Minkowski scalar multiplication is given by

$$\lambda \cdot K = \lambda^{1/(n-1)} K. \tag{9}$$

A new addition, harmonic Blaschke addition, be defined by Lutwak [8]. Suppose $K, L \in \varphi^n$, and $\lambda, \mu \geq 0$(not both zero). To define the harmonic Blaschke linear combination, $\lambda K \hat{+} \mu L$, first define $\xi > 0$ by

$$\xi^{1/(n+1)} = \frac{1}{n} \int_{S^{n-1}} [\lambda V(K)^{-1} \rho(K, u)^{n+1} + \mu V(L)^{-1} \rho(L, u)^{n+1}]^{n/(n+1)} dS(u). \tag{10}$$

The body $\lambda K \hat{+} \mu L \in \varphi^n$ is defined as the body whose radial function is given by

$$\xi^{-1} \rho(\lambda K \hat{+} \mu L, \cdot)^{n+1} = \lambda V(K)^{-1} \rho(K, \cdot)^{n+1} + \mu V(L)^{-1} \rho(L, \cdot)^{n+1}. \tag{11}$$

It follows immediately that $\xi = V(\lambda K \hat{+} \mu L)$, and hence

$$V(\lambda K \hat{+} \mu L)^{-1} \rho(\lambda K \hat{+} \mu L, \cdot)^{n+1} = \lambda V(K)^{-1} \rho(K, \cdot)^{n+1} + \mu V(L)^{-1} \rho(L, \cdot)^{n+1}.$$

Lutwak [10] define a mapping:

$$\Lambda : \varphi_c^n \to \kappa_c^n$$

and point out that Λ tansforms harmonic Blaschke linear combination into Blaschke linear combinations, i.e.

If $K, L \in \varphi_c^n$ and $\lambda, \mu \geq 0$, then

$$\Lambda(\lambda K \hat{+} \mu L) = \lambda \cdot \Lambda K \ddot{+} \mu \cdot \Lambda L.$$

Further, We obtain that

If $K_j \in \varphi_c^n (j = 1, \ldots, m)$, and $\lambda_j \geq 0 (j = 1, \ldots, m)$, then

$$\Lambda(\lambda_1 K_1 \hat{+} \cdots \hat{+} \lambda_m K_m) = \lambda_1 \cdot \Lambda K_1 \ddot{+} \cdots \ddot{+} \lambda_m \cdot \Lambda K_m. \tag{12}$$

and

$$\Lambda(\lambda K) = \lambda \Lambda K \tag{13}$$

1.4 Projection bodies and Centroid bodies

The projection bodies, ΠK, of the body $K \in \mathcal{K}^n$ is defined as the convex figure whose support function is given, for $u \in S^{n-1}$, by [6]

$$h(\Pi K, u) = v(K^u) \tag{14}$$

It is easy to see, that a projection body is always centered(symmetric about the origin), and if K has interior points then ΠK will have interior point as well.

Here, we introduce the following property.

If $K, L \in \mathcal{K}^n$ and $\lambda, \mu \geq 0$, then [7]

$$\Pi(\lambda \cdot K \ddot{+} \mu \cdot L) = \lambda \Pi K + \mu \Pi L.$$

Further, we may prove that

If $K_j \in \mathcal{K}^n (j = 1, \ldots, m)$ and $\lambda_j \geq 0 (j = 1, \ldots, m)$, then

$$\Pi(\lambda_1 \cdot K_1 \ddot{+} \cdots \ddot{+} \lambda_m \cdot K_m) = \lambda_1 \Pi K_1 + \cdots + \lambda_m \Pi K_m. \tag{15}$$

The centroid body, ΓK, of $K \in \varphi^n$, is the convex body whose support function, at $x \in \mathbb{R}^n$, is given by [8]:

$$h(\Gamma K, x) = \frac{1}{V(K)} \int_K \mid x \cdot y \mid dy. \tag{16}$$

Here, we give the following property:

If $K_j \in \varphi^n (j = 1, \ldots, m)$, and $\lambda_j \geq 0 (j = 1, \ldots, m)$, then

$$\Gamma(\lambda_1 K_1 \hat{+} \cdots \hat{+} \lambda_m K_m) = \lambda_1 \Gamma K_1 + \cdots + \lambda_m \Gamma K_m. \tag{17}$$

If $K \in \varphi^n$, then from (20) it follows that ΓK is centered.

Please see the next section for above interrelated notations, definitions and their background material.

1. **Main results**

Width-integrals were first considered by Blaschke [1,p.85] and later by Hadwiger [2,p.266]. In [3], Lutwak also introduced the width-integral of index i and proved some important results, one of them is the following Theorem:

Theorem A. *If $K, L \in \mathcal{K}^n$ and $i < n-1$, then*

$$B_i(K+L)^{1/(n-i)} \leq B_i(K)^{1/(n-i)} + B_i(L)^{1/(n-i)} \tag{18}$$

with equality if and only if K and L have similar width.

Since inequality (1) is a new result similar to the following Brunn-Minkowski inequality for the cross-sectional measures [2, p.249].

Theorem B. *If $K, L \in \mathcal{K}^n$ and $i < n-1$, then*

$$W_i(K+L)^{1/(n-i)} \leq W_i(K)^{1/(n-i)} + W_i(L)^{1/(n-i)} \tag{19}$$

with equality if and only if K and L are homothetic.

Hence, inequality (1) is called as the dual Brunn-Minkowski inequality for width-integral of convex bodies.

The main purpose of this paper is to establish two analogs of inequality (1), them can be stated as:

Theorem C. If $K_1, \ldots, K_m \in \varphi^n$, $\lambda_1, \ldots, \lambda_m > 0$ *and* $i < n-1$, *then*

$$B_i(\Gamma(\lambda_1 K_1 \hat{+} \cdots \hat{+} \lambda_m K_m))^{1/(n-i)} \leq \lambda_1 B_i(\Gamma K_1))^{1/(n-i)} + \cdots + \lambda_m B_i(\Gamma K_m)^{1/(n-i)}, \tag{20}$$

with equality if and only if $\Gamma K_j (j = 1, 2, \ldots, m)$ have similar width.

Theorem D. *If $K_1, \ldots, K_m \in \varphi_c^n$, $\lambda_1, \ldots, \lambda_m > 0$ and $i < n-1$, then*

$$\begin{aligned} &B_i(\Pi(\Lambda(\lambda_1 K_1 \hat{+} \ldots \hat{+} \lambda_m K_m)))^{1/(n-i)} \\ &\leq \lambda_1 B_i(\Pi(\Lambda K_1))^{1/(n-i)} + \ldots + \lambda_m B_i(\Pi(\Lambda K_m))^{1/(n-i)}, \end{aligned} \tag{21}$$

with equality if and only if $\Pi(\Lambda K_j)(j = 1, 2, \ldots, m)$ have similar width.

2. A dual Brunn-Minkowski inequality about the width-integrals of centroid bodies for the harmonic Blaschke linear combination

The following dual Brunn-Minkowski inequality about the width-integrals of centroid bodies will be proved.

Theorem C. If $K_1, \ldots, K_m \in \varphi^n$, $\lambda_1, \ldots, \lambda_m > 0$ *and* $i < n-1$, *then*

$$B_i(\Gamma(\lambda_1 K_1 \hat{+} \cdots \hat{+} \lambda_m K_m))^{1/(n-i)} \leq \lambda_1 B_i(\Gamma K_1))^{1/(n-i)} + \cdots + \lambda_m B_i(\Gamma K_m)^{1/(n-i)}, \tag{22}$$

with equality if and only if $\Gamma K_j (j = 1, 2, \ldots, m)$ *have similar width.*

Proof. From (7),(10),(11),(21) and in view of Minkowski inequality for integral [11, p.147], we obtain that

$$\begin{aligned} B_i(\Gamma(\lambda_1 K_1 \hat{+} \cdots \hat{+} \lambda_m K_m))^{\frac{1}{n-i}} &= \left(\frac{1}{n}\int_{S^{n-1}} b^{n-i}_{\Gamma(\lambda_1 K_1 \hat{+} \cdots \hat{+} \lambda_m K_m)} dS(u)\right)^{\frac{1}{n-i}} \\ &= \left(\frac{1}{n}\int_{S^{n-1}} \left(b_{\lambda_1 \Gamma K_1 + \cdots + \lambda_m \Gamma K_m}\right)^{n-i} dS(u)\right)^{\frac{1}{n-i}} \\ &= \left(\frac{1}{n}\int_{S^{n-1}} \left(\lambda_1 b_{\Gamma K_1} + \cdots + \lambda_m b_{\Gamma K_m}\right)^{n-i} dS(u)\right)^{\frac{1}{n-i}} \\ &\leq \lambda_1 \left(\frac{1}{n}\int_{S^{n-1}} b^{n-i}_{\Gamma K_1} dS(u)\right)^{\frac{1}{n-i}} + \cdots + \lambda_m \left(\frac{1}{n}\int_{S^{n-1}} b^{n-i}_{\Gamma K_m} dS(u)\right)^{\frac{1}{n-i}} \\ &= \lambda_1 B_i(\Gamma K_1)^{\frac{1}{n-i}} + \cdots + \lambda_m B_i(\Gamma K_m)^{\frac{1}{n-i}}, \end{aligned}$$

with equality if and only if $\Gamma K_j (j = 1, \ldots, m)$ have similar width.

The proof is complete. □

Taking $m = 2$ to (23), we have

Corollary 1. *If* $K, L \in \varphi^n$, $\lambda, \mu > 0$ *and* $i < n - 1$, *then*

$$B_i(\Gamma(\lambda K \hat{+} \mu L))^{1/(n-i)} \leq \lambda B_i(\Gamma K)^{1/(n-i)} + \mu B_i(\Gamma L)^{1/(n-i)}, \tag{23}$$

with equality if and only if ΓK and ΓL *have similar width.*

Another important consequence is obtained when $\lambda = \mu = 1$.

Corollary 2. *If*$K, L \in \varphi^n$ and $i < n - 1$, then

$$B_i(\Gamma(K \hat{+} L))^{1/(n-i)} \leq B_i(\Gamma K)^{1/(n-i)} + B_i(\Gamma L)^{1/(n-i)}, \tag{24}$$

with equality if and only if ΓK and ΓL have similar width.

3. A dual Brunn-Minkowski inequality about the width-integrals of projection bodies for the harmonic Blaschke linear combination

The following dual Brunn-Minkowski inequality about the width-integrals of projection bodies will be proved.

Theorem D. *If* $K_1, \ldots, K_m \in \varphi^n_c$, $\lambda_1, \ldots, \lambda_m > 0$ *and* $i < n - 1$, *then*

$$B_i(\Pi(\Lambda(\lambda_1 K_1 \hat{+} \ldots \hat{+} \lambda_m K_m)))^{1/(n-i)}$$

$$\leq \lambda_1 B_i(\Pi(\Lambda K_1))^{1/(n-i)} + \ldots + \lambda_m B_i(\Pi(\Lambda K_m))^{1/(n-i)}, \tag{25}$$

with equality if and only if $\Pi(\Lambda K_j)(j = 1, 2, \ldots, m)$ *have similar width.*

Proof. From (7),(10),(11),(16),(19) and in view of Minkowski inequality for integral [11, p.147], we obtain that

$$B_i(\Pi(\Lambda(\lambda_1 K_1 \hat{+} \cdots \hat{+} \lambda_m K_m)))^{\frac{1}{n-i}} = \left(\frac{1}{n}\int_{S^{n-1}} b^{n-i}_{\Pi(\Lambda(K_1 \hat{+} \cdots \hat{+} K_m))} dS(u)\right)^{\frac{1}{n-i}}$$

$$= \left(\frac{1}{n}\int_{S^{n-1}} b^{n-i}_{\Pi(\lambda_1 \Lambda K_1 \ddot{+} \cdots \ddot{+} \lambda_m \Lambda K_m)} dS(u)\right)^{\frac{1}{n-i}}$$

$$= \left(\frac{1}{n}\int_{S^{n-1}} b^{n-i}_{\sum_{j=1}^m \lambda_j \Pi(\Lambda K_j)} dS(u)\right)^{\frac{1}{n-i}}$$

$$= \left(\frac{1}{n}\int_{S^{n-1}} \left(\sum_{j=1}^m \lambda_j b_{\Pi(\Lambda K_j)}\right)^{n-i} dS(u)\right)^{\frac{1}{n-i}}$$

$$\leq \sum_{j=1}^m \lambda_j \left(\frac{1}{n}\int_{S^{n-1}} b^{n-i}_{\Pi(\Lambda K_j)} dS(u)\right)^{\frac{1}{n-i}} = \sum_{j=1}^m \lambda_j B_i(\Pi(\Lambda K_j))^{\frac{1}{n-i}},$$

with equality if and only if $\Pi(\Lambda K_j)(j = 1, \ldots, m)$ have similar width.

The proof is complete. □

Taking $m = 2$ to (26), we have

Corollary 3. *If* $K, L \in \varphi_c^n$, $\lambda, \mu > 0$ *and* $i < n - 1$, *then*

$$B_i(\Pi(\Lambda(\lambda K \hat{+} \mu L)))^{1/(n-i)} \leq \lambda B_i(\Pi(\Lambda K))^{1/(n-i)} + \mu B_i(\Pi(\Lambda L))^{1/(n-i)}, \tag{26}$$

with equality if and only if $\Pi(\Lambda K)$ and $\Pi(\Lambda L)$ *have similar width.*

Another remarkable case is obtained for $\lambda = \mu = 1$.

Corollary 4. *If* $K, L \in \varphi_c^n$ and $i < n - 1$, then

$$B_i(\Pi(\Lambda(K \hat{+} L)))^{1/(n-i)} \leq B_i(\Pi(\Lambda K))^{1/(n-i)} + B_i(\Pi(\Lambda L))^{1/(n-i)}, \tag{27}$$

with equality if and only if $\Pi(\Lambda K)$ and $\Pi(\Lambda L)$ have similar width.

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