BOUNDARY VALUE PROBLEMS FOR SYSTEMS OF SECOND ORDER DIFFERENTIAL EQUATIONS OF MIXED TYPE

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Abstract. Applying the Perov’s fixed point theorem is approached boundary value problems for systems of second order differential equations of mixed type.

1. Introduction

The aim of this paper is to present some results about the existence and uniqueness, subsolutions and suprasolutions, continuity, monotony and data dependence of the solution of (1)+(2). We apply the W.P.O’s technique as in [12].

Consider the problem

\[-x''(t) = f(t, x(t), x(g(t)), x(h(t))), \quad t \in [a, b] \]

\[
\begin{cases}
  l_1(x(t)) = \alpha(t), \quad \text{for } t \in [a_1, a] \\
  l_2(x(t)) = \beta(t), \quad \text{for } t \in [b, b_1]
\end{cases}
\]

where \( \alpha \in C([a_1, a], \mathbb{R}^m) \), \( \beta \in C([b, b_1], \mathbb{R}^m) \), \( g, h \in C([a, b], [a_1, b_1]) \) and \( a_1 \leq a < b \leq b_1 \).

Here,

\( l_1 : C^1([a_1, a], \mathbb{R}^m) \longrightarrow C([a_1, a], \mathbb{R}^m) \)

and

\( l_2 : C^1([b, b_1], \mathbb{R}^m) \longrightarrow C([b, b_1], \mathbb{R}^m) \)

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are linear functions.

We suppose that the boundary value problem

\[
\begin{aligned}
-u'' &= \chi, \quad t \in [a, b] \\
l_1(u(t)) &= \alpha(t), \quad \text{for } t \in [a_1, a] \\
l_2(u(t)) &= \beta(t), \quad \text{for } t \in [b, b_1]
\end{aligned}
\]

has a unique solution \( u \in C([a_1, b_1], \mathbb{R}^m) \cap C^2([a, b], \mathbb{R}^m) \),

\[
u(t) = \begin{cases} 
\varphi(t), & t \in [a_1, a] \\
w(\alpha, \beta)(t) + \int_a^b G(t, s) \chi(s) \, ds, & t \in [a, b] \\
\psi(t), & t \in [b, b_1]
\end{cases}
\]

and there exist

\[ G \in C([a, b] \times [a, b], M_{m,m}(\mathbb{R})) , \]

where \( G \) is the corresponding Green function and

\[
w(\alpha, \beta)(t) = \frac{t-a}{b-a} \psi(b) + \frac{b-t}{b-a} \varphi(a), \quad t \in [a, b].
\]

Consider the following conditions:

(C1) \( g, h \in C([a, b], [a_1, b_1]); \alpha \in C([a_1, a], \mathbb{R}^m); \beta \in C([b, b_1], \mathbb{R}^m) \);

(C2) \( f = (f_1, f_2, \ldots, f_m) \in C([a, b] \times \mathbb{R}^{3m}, \mathbb{R}^m) \);

(C3) There exist a matrix \( L_f \in M_{m,m}(\mathbb{R}_+) \) such that

\[
\| f(t, u^i, u^j, u^k) - f(t, v^i, v^j, v^k) \|_{\mathbb{R}^m} \leq L_f (\| u^i - v^i \|_{\mathbb{R}^m} + \| u^j - v^j \|_{\mathbb{R}^m} + \| u^k - v^k \|_{\mathbb{R}^m})
\]

\[
\forall t \in [a, b], \quad u^i, v^i \in \mathbb{R}^m, \quad i = 1, 3,
\]

where

\[
\| u \|_{\mathbb{R}^m} := \begin{pmatrix} |u_1| \\ \vdots \\ |u_m| \end{pmatrix}
\]

is the vectorial norm on \( \mathbb{R}^m \).
In this study we will use the Weakly Picard Operator’s technique and the Perov’s fixed point theorem (see [1] and [8]). A variant of this theorem on vector valued normed spaces and the above vectorial norm is used in [1].

We need the following notions and notations:

Let \((X,d)\) be a generalized metric space, \(d(x,y) \in \mathbb{R}^m\) and \(A : X \rightarrow X\) an operator.

We shall use:

\[ F_A := \{ x \in X | A(x) = x \} \] - the fixed point set of the operator \(A\);

\[ I(A) := \{ Y \subset X | A(Y) \subset Y, Y \neq \emptyset \} \] - the family of the nonempty invariant subsets of \(A\);

\[ A^{n+1} := A \circ A^n; \ A^0 = 1_X; \ A^1 = A; \ n \in \mathbb{N}. \]

**Definition 1.** ([10]) An operator \(A\) is Weakly Picard Operator (W.P.O.) if the sequence \((A^n(x))_{n \in \mathbb{N}}\) converges, for all \(x \in X\) and the limit (which may depend on \(x\)) is a fixed point of \(A\).

**Definition 2.** ([10]) If the operator \(A\) is W.P.O. and \(F_A = \{x^*\}\), then by definition, the operator \(A\) is Picard Operator (P.O.).

**Definition 3.** ([10]) If \(A\) is W.P.O., then we consider the operator \(A^\infty\) defined by

\[ A^\infty : X \rightarrow X, \ A^\infty(x) := \lim_{n \to \infty} A^n(x). \]

2. Existence and uniqueness

The problem (1)+(2) is equivalent in \(C([a_1,b_1], \mathbb{R}^m) \cap C^2([a,b], \mathbb{R}^m)\) with the fixed point equation

\[
  x(t) = \begin{cases} 
    \varphi(t), & t \in [a_1,a] \\
    w(\alpha, \beta)(t) + \int_a^b G(t,s)f(s,x(s),x(g(s)),x(h(s)))ds, & t \in [a,b] \\
    \psi(t), & t \in [b,b_1] 
  \end{cases} 
\]

where

\[
  w(\alpha, \beta)(t) = \frac{t-a}{b-a} \psi(b) + \frac{b-t}{b-a} \varphi(a), \ t \in [a,b].
\]
Then, in $C([a_1, b_1], \mathbb{R}^m) \cap C^2([a, b], \mathbb{R}^m)$, the equation (1) is equivalent with
\[
x(t) = \begin{cases} 
  x(t), & t \in [a_1, a] \\
  w(x |_{[a_1, a]}, x |_{[b, b_1]})(t) + \int_a^b G(t, s)f(s, x(s), x(g(s)), x(h(s))) ds, & t \in [a, b] \\
  x(t), & t \in [b, b_1] 
\end{cases}
\] (5)

Consider the following operators:
\[
B_f, E_f : C([a_1, b_1], \mathbb{R}^m) \rightarrow C([a_1, b_1], \mathbb{R}^m)
\]
where
\[
B_f(x)(t) := \text{second part of (4)}
\]
\[
E_f(x)(t) := \text{second part of (5)}.
\]

Consider the functional spaces
\[
X := C([a_1, b_1], \mathbb{R}^m),
\]
\[
X_{\varphi, \psi} := \{x \in X : x |_{[a_1, a]} = \varphi, x |_{[b, b_1]} = \psi\}.
\]

Then
\[
X = \bigcup_{\varphi \in C([a_1, a], \mathbb{R}^m)} \bigcup_{\psi \in C([b, b_1], \mathbb{R}^m)} X_{\varphi, \psi}
\] is a partition of $X$.

**Lemma 4.** (see [12]) We suppose that the conditions (C1), (C2), (C3) are satisfied. Then:
(a) $B_f(X) \subset X_{\varphi, \psi}$ and $B_f(X_{\varphi, \psi}) \subset X_{\varphi, \psi}$;
(b) $B_f |_{X_{\varphi, \psi}} = E_f |_{X_{\varphi, \psi}}$.

**Proof.** Is similar as in [12] taking $X = C([a_1, b_1], \mathbb{R}^m)$.

Let
\[
M_G = (\|G_{ij}\|)_{i,j=1}^{m} \in M_{m,m} (\mathbb{R}_+),
\]
where $\|G_{ij}\| = \max \{|G_{ij}(x, s)| : (x, s) \in [a, b] \times [a, b]\}$, $\forall i, j = 1, m$. and
\[
Q = 3(b - a) M_G \cdot L_f \in M_{m,m} (\mathbb{R}_+).
\]
We have the following existence and uniqueness theorem:

**Theorem 5.** We suppose that:

(i) the conditions $(C1) - (C3)$ are satisfied;

(ii) $Q^n \to 0$ as $n \to \infty$.

Then the problem $(1)+(2)$ has a unique solution

$$x^*_f = (x^*_f, \ldots, x^*_{m_f}) \in C([a_1, b_1], \mathbb{R}^m) \cap C^2([a, b], \mathbb{R}^m).$$

**Proof.** Consider the Banach space $X = C([a_1, b_1], \mathbb{R}^m)$ with generalized Chebyshev's norm

$$\|u\|_C := \left(\begin{array}{c}
\|u_1\|_C \\
\vdots \\
\|u_m\|_C
\end{array}\right), \text{ where } \|u_i\|_C := \max_{a \leq t \leq b} |u_i(t)|, \forall i = 1, m.$$

The problem $(1) + (2)$ is equivalent on $X$ with the fixed point equation:

$$B_f(x) = x.$$

We prove now that the operator $B_f = (B_{f_1}, \ldots, B_{f_m})$ is Picard Operator. For $y, z \in X$ we have:

$$\|B_f(y)(t) - B_f(z)(t)\|_{\mathbb{R}^m} \leq$$

$$\leq \int_a^b \|G(t, s) [f(s, y(s), g(s)) - f(s, z(s), z(g(s)))) - f(s, z(s), z(g(s))))\|_{\mathbb{R}^m} ds \leq$$

$$\leq \int_a^b M_G \cdot L_f \cdot \|y(s) - z(s)\|_{\mathbb{R}^m} + \|y(g(s)) - z(g(s))\|_{\mathbb{R}^m} +$$

$$+ \|y(h(s)) - z(h(s))\|_{\mathbb{R}^m} ds \leq$$

$$\leq 3(b - a) M_G L_f \cdot \|y - z\|_C = Q \cdot \|y - z\|_C, \forall t \in [a, b].$$

Then,

$$\|B_f(y) - B_f(z)\|_C \leq Q \|y - z\|_C$$
and by (ii), the operator $B_f$ is $Q$-contraction. From the Perov’s fixed point theorem we infer that the operator $B_f$ is P.O. and has a unique fixed point

$$x_f^* = (x_f^{*1}, ..., x_f^{*m}) \in X.$$  

Since $f$ is continuous, deriving (4) two times by $t$, we infer that

$$x_f^* \in C^2([a, b], \mathbb{R}^m)$$

is the unique solution of (1)+(2).

**Remark 6.** Since from Theorem 5 we have that the operator $B_f$ is P.O. and because

$$B_f |_{X_{\varphi,\psi}} = E_f |_{X_{\varphi,\psi}},$$

$$X := C([a_1, b_1], \mathbb{R}^m) = \bigcup_{\varphi, \psi} X_{\varphi, \psi}, \quad X_{\varphi, \psi} \in I(E_f)$$

we infer that the operator $E_f$ is W.P.O. and

$$F_{E_f} \cap X_{\varphi,\psi} = \{x_{\varphi,\psi}^*\}, \quad \forall \varphi \in C([a_1, a], \mathbb{R}^m), \quad \forall \psi \in C([b, b_1], \mathbb{R}^m)$$

where $x_{\varphi,\psi}^*$ is the unique solution of the problem (1) + (2).

3. Data dependence

In this paragraph we shall study the subsolutions and suprasolutions of equation (1).

For the problem (1) + (2) we have:

**Theorem 7.** We suppose that:

(a) the conditions $(C1) - (C3)$ are satisfied;

(b) $Q^n \to 0$ as $n \to \infty$;

(c) the operator $f(t, \bullet, \bullet): \mathbb{R}^{3m} \to \mathbb{R}^m$ is increasing, where on $\mathbb{R}^m$ we have the order relation:

$$x \leq y \iff x_i \leq y_i, \forall i = 1, m.$$
Let $x^* = (x_{1}^*, ..., x_{m}^*)$ be a solution of (1) and $y^* = (y_{1}^*, ..., y_{n}^*)$ a solution of the inequality:
\[-y''(t) \leq f(t, y(t), y(g(t)), y(h(t))), \forall t \in [a, b].\]
Then
\[y^*(t) \leq x^*(t), \forall t \in [a_1, a] \cup [b, b_1] \implies y^* \leq x^*. \quad (**)\]

Proof. In terms of the operator $E_f$ we have
\[x = E_f(x), y \leq E_f(y)\]
and
\[w(y|_{a_1,a}, y|_{b,b_1}) \leq w(x|_{a_1,a}, x|_{b,b_1}).\]
From (c) we have that the operator $E_f^\infty$ is increasing and
\[y \leq E_f^\infty(y) = E_f^\infty(\tilde{w}(y)) \leq E_f^\infty(\tilde{w}(x)) = x \implies y \leq x,\]
where
\[\tilde{w}(z)(t) := \begin{cases} 
  z(t), & t \in [a_1, a] \\
  w(z|_{a_1,a}, z|_{b,b_1})(t), & t \in [a, b] \\
  z(t), & t \in [b, b_1] \end{cases} \]
for $z \in C([a_1, b_1], \mathbb{R}^m)$. According to Theorem 5 this lead to the inequality $(**).$ 

Now we shall study the monotony of the solution of the problem (1) + (2) with respect to $\varphi$, $\psi$ and $f$. In this aim we need the following abstract result:

Lemma 8. (see [10]) Let $(X, d, \leq)$ be an ordered generalized metric space with $d(x, y) \in \mathbb{R}^m$ and $A, B, C : X \rightarrow X$ be such that:
(i) $A \leq B \leq C$;
(ii) the operators $A, B, C$ are W.P.O.’s;
(iii) the operator $B$ is increasing.
Then $x \leq y \leq z \implies A^\infty(x) \leq B^\infty(y) \leq C^\infty(z).$
Theorem 9. Let $f^i \in C([a,b] \times \mathbb{R}^{3m}, \mathbb{R}^m)$, $i = \overline{1,3}$, $g$ and $h$ as in the Theorem 5.

We suppose that:

(a) $f^2(t, \bullet, \bullet, \bullet) : \mathbb{R}^{3m} \rightarrow \mathbb{R}^m$ is increasing;

(b) $f^1 \leq f^2 \leq f^3$.

Let $x^i$ be a solution of the system:

$$(-x^i)^\prime\prime(t) = f^i(t, x(t), x(g(t)), x(h(t))), \ t \in [a, b], \ i = \overline{1,3}.$$  

If $x^1(t) \leq x^2(t) \leq x^3(t)$, $\forall t \in [a_1, a] \cup [b_1, b]$ then $x^1 \leq x^2 \leq x^3$.

Proof. From Remark 6 we have that the operators $E_{f^1}, E_{f^2}, E_{f^3}$ are W.P.O.'s.

From condition (a) we infer that the operator $E_{f^2}$ is increasing.

From (b) it follows that $E_{f^1} \leq E_{f^2} \leq E_{f^3}$.

But $x^1 = E_{f^1}^2(\tilde{w}(x^1))$, $x^2 = E_{f^2}^2(\tilde{w}(x^2))$, $x^3 = E_{f^3}^2(\tilde{w}(x^3))$.

Using Lemma 9 we have that

$$x^1 \leq x^2 \leq x^3.$$  

Now, let $f^1, f^2 \in C([a,b] \times \mathbb{R}^{3m}, \mathbb{R}^m)$ and $L_{f^1}, L_{f^2} \in M_{m,m}(\mathbb{R}_+)$ as in the condition (C3). Consider $L_f \in M_{m,m}(\mathbb{R}_+)$ with

$$L_f(i, j) = \max(L_{f^1}(i, j), L_{f^2}(i, j)), \ \forall i, j = \overline{1,m}. $$

According to the result of Theorem 5, let $x(\bullet; \phi, \psi, f)$ the notation for the unique solution of (4). We investigate now, the dependence of $x(\bullet; \phi, \psi, f)$ by $\phi, \psi, f$.

Let $Q_1 = 3(b-a)M_G \cdot L_{f^1}$, $Q_2 = 3(b-a)M_G \cdot L_{f^2}$ and $Q = 3(b-a)M_G \cdot L_f$ being in $M_{m,m}(\mathbb{R}_+)$. We will denote $Q = \max\{Q_1, Q_2\}$ (only formally).

Theorem 10. Let $\alpha_1, \alpha_2 \in C([a_1,a], \mathbb{R}^m)$, $\beta_1, \beta_2 \in C([b_1,b], \mathbb{R}^m), \phi_i, \psi_i$, $i = \overline{1,2}$, and $f^1, f^2$ as in the Theorem 5. We suppose that:

(i) there exists $\eta_1 \in \mathbb{R}^m$ such that

$$\|\phi^1(t) - \phi^2(t)\|_{\mathbb{R}^m} \leq \eta_1, \ \forall t \in [a_1,a]$$

and

$$\|\psi^1(t) - \psi^2(t)\|_{\mathbb{R}^m} \leq \eta_1, \ \forall t \in [b_1,b].$$
Then follows that

\[ \| f^1(t, u_1, u_2, u_3) - f^2(t, u_1, u_2, u_3) \|_{\mathbb{R}^m} \leq \eta_2, \ \forall t \in [a, b], \ \forall u_1, u_2, u_3 \in \mathbb{R}^m. \]

Moreover, we have

\[ \| x(\bullet; \varphi^1, \psi^1, f^1) - x(\bullet; \varphi^2, \psi^2, f^2) \|_C \leq (I_m - Q)^{-1} \cdot (2\eta_1 + M_G (b - a) \cdot \eta_2). \]

**Proof.** Consider the operators \( B_{\varphi^1, \psi^1, f^1} \) and \( B_{\varphi^2, \psi^2, f^2} \) as in the Theorem 5. It follows that

\[ \| B_{\varphi^1, \psi^1, f^1}(x) - B_{\varphi^2, \psi^2, f^2}(y) \|_C \leq Q \| x - y \|_C, \ \forall x, y, \ \ i = 1, 2. \]

Moreover, we have

\[ \| B_{\varphi^1, \psi^1, f^1}(x)(t) - B_{\varphi^2, \psi^2, f^2}(x)(t) \|_{\mathbb{R}^m} \leq \| \varphi^1(a) - \varphi^2(a) \|_{\mathbb{R}^m} + \| \psi^1(b) - \psi^2(b) \|_{\mathbb{R}^m} + \]

\[ + \int_a^b \| G(t, s) \cdot [ f^1(s, x(s), x(g(s)), x(h(s))) - f^2(s, x(s), x(g(s)), x(h(s))) ] \|_{\mathbb{R}^m} ds \leq \]

\[ \leq 2\eta_1 + M_G (b - a) \cdot \eta_2, \ \forall t \in [a, b]. \]

Since

\[ \| x^\ast(\bullet; \varphi^1, \psi^1, f^1) - x^\ast(\bullet; \varphi^2, \psi^2, f^2) \|_C = \]

\[ = \| B_{\varphi^1, \psi^1, f^1}(x^\ast(\bullet; \varphi^1, \psi^1, f^1)) - B_{\varphi^2, \psi^2, f^2}(x^\ast(\bullet; \varphi^2, \psi^2, f^2)) \|_C \leq \]

\[ \leq \| B_{\varphi^1, \psi^1, f^1}(x^\ast(\bullet; \varphi^1, \psi^1, f^1)) - B_{\varphi^1, \psi^1, f^1}(x^\ast(\bullet; \varphi^2, \psi^2, f^2)) \|_C + \]

\[ + \| B_{\varphi^1, \psi^1, f^1}(x^\ast(\bullet; \varphi^2, \psi^2, f^2)) - B_{\varphi^2, \psi^2, f^2}(x^\ast(\bullet; \varphi^2, \psi^2, f^2)) \|_C \leq \]

\[ \leq Q \cdot \| x^\ast(\bullet; \varphi^1, \psi^1, f^1) - x^\ast(\bullet; \varphi^2, \psi^2, f^2) \|_C + 2\eta_1 + M_G (b - a) \cdot \eta_2, \]

and because \( Q^n \rightarrow 0 \) as \( n \rightarrow \infty \) imply that

\[ (I_m - Q)^{-1} \in M_{m,m} (\mathbb{R}^+), \]

we obtain,

\[ \| x(\bullet; \varphi^1, \psi^1, f^1) - x(\bullet; \varphi^2, \psi^2, f^2) \|_C \leq (I_m - Q)^{-1} \cdot (2\eta_1 + M_G (b - a) \cdot \eta_2). \]
Corollary 11. Let $\varphi^i, \psi^i, f^i, i \in \mathbb{N}$ and $\varphi, \psi, f$ be such in the Theorem 10. Let $Q_i, Q \in M_{m,m}(\mathbb{R}^+), i \in \mathbb{N}$ as above such that exist $Q_i = \max\{Q, Q_i\}, \forall i \in \mathbb{N}$. We suppose that:

$Q_i \rightarrow 0$ as $n \rightarrow \infty, \forall i \in \mathbb{N}$ and 

$\varphi^i \longrightarrow \varphi$ as $i \rightarrow \infty$;

$\psi^i \longrightarrow \psi$ as $i \rightarrow \infty$;

$f^i \longrightarrow f$ as $i \rightarrow \infty$.

Then $x(\bullet; \varphi^i, \psi^i, f^i) \longrightarrow x(\bullet; \varphi, \psi, f)$ as $i \rightarrow \infty$.

4. Smooth dependence by parameter

In this section we present the dependence by parameter $\lambda$ of the solution of problem (6) + (7).

Consider the following boundary value problem with parameter:

$$-x''(t; \lambda) = f(t, x(t; \lambda), x(g(t); \lambda), x(h(t); \lambda); \lambda), t \in [a, b], \lambda \in [c, d] \subset \mathbb{R}$$

$$l_1(x(t)) = \alpha(t), \text{ for } t \in [a_1, a]$$

$$l_2(x(t)) = \beta(t), \text{ for } t \in [b, b_1]$$

We suppose that:

$$(D1) g, h \in C([a, b], [a_1, b_1]);$$

$$(D2) f = (f_1, f_2, ..., f_m) \in C^1([a, b] \times \mathbb{R}^m \times [c, d], \mathbb{R}^m);$$

$$(D3) \text{There exist } L_f \in M_{m,m}(\mathbb{R}^+) \text{ such that:}$$

$$\left[\left|\frac{\partial f_i(t, u, v; \lambda)}{\partial u_j}\right|\right]_{i, j = 1, m} \leq L_f,$$

$$\left[\left|\frac{\partial f_i(t, u, v; \lambda)}{\partial v_j}\right|\right]_{i, j = 1, m} \leq L_f,$$

$$\left[\left|\frac{\partial f_i(t, u, v; \lambda)}{\partial w_j}\right|\right]_{i, j = 1, m} \leq L_f,$$

$\forall t \in [a, b], \forall u, v, w \in \mathbb{R}^m, i = 1, m, j = 1, m, \text{ and } \lambda \in [c, d];$ in respect by the componentwise order on $M_{m,m}(\mathbb{R}^+)$.  

$$(D4) \alpha \in C([a_1, a], \mathbb{R}^m), \beta \in C([b_1, b], \mathbb{R}^m).$$
For \( Q = 3(b-a)MG \cdot Lf \in M_{m,m}(\mathbb{R}_+) \) we have \( Q^n \to 0 \) as \( n \to \infty \).

In the above conditions, from Theorem 5 we have that the problem (6) + (7) has a unique solution \( x^*(\bullet; \lambda) \), for any \( \lambda \in [c,d] \).

Now we prove that \( x^*(t; \bullet) \in C^1([c,d]; \mathbb{R}^m) \), for all \( t \in [a,b] \).

For this we consider the equation

\[ -x''(t; \lambda) = f(t, x(t; \lambda), x(g(t); \lambda), x(h(t); \lambda), t \in [a,b], \lambda \in [c,d] \] \tag{8}

The equation (8) is equivalent with the following system:

\[ x(t; \lambda) = \begin{cases} 
  x(t), & t \in [a_1, a] \\
  w(\varphi, \psi)(t) + \int_a^b G(t,s)f(s, x(s; \lambda), x(g(s); \lambda), x(h(s); \lambda), t \in [a,b], \lambda \in [c,d] \\
  x(t), & t \in [b, b_1] \end{cases} \tag{9} \]

where

\[ w(\alpha, \beta)(t) = \frac{t-a}{b-a} \psi(b) + \frac{b-t}{b-a} \varphi(a), \quad t \in [a,b]. \]

Let \( X := C([a_1, b_1] \times [c,d], \mathbb{R}^m) \) with the Chebyshev norm

\[ \|x\|_C := \left( \begin{array}{c} \|x_1\|_C \\ \vdots \\ \|x_m\|_C \end{array} \right) \in \mathbb{R}^m. \]

Now we consider the operator

\[ B : C([a_1, b_1] \times [c,d], \mathbb{R}^m) \to C([a_1, b_1] \times [c,d], \mathbb{R}^m) \]

where

\[ B(x)(t; \lambda) := \text{second part of (9)} \]

Analogous as in Theorem 5 it probes that in the conditions \((D1)-(D5)\) the operator \( B \) is P.O., since

\[ \|B(y) - B(z)\|_C \leq Q \cdot \|y - z\|_C. \]
This implies that $B$ has a unique fixed point $x^*$. We suppose that there exists $\frac{\partial x^*}{\partial x}$.

From relation (9) and condition (D3) we have:

$$
\frac{\partial x^*(t; \lambda)}{\partial \lambda} = \begin{cases} 
0, & \text{for } t \in [a_1, a] \\
\int_a^b G(t, s) \left( \frac{\partial f_i(s, x^*(s; \lambda), x^*(g(s); \lambda), x^*(h(s); \lambda); \lambda)}{\partial u_j} \right)_{i,j} \cdot \frac{\partial x^*(s; \lambda)}{\partial \lambda} ds \\
+ \int_a^b G(t, s) \left( \frac{\partial f_i(s, x^*(s; \lambda), x^*(g(s); \lambda), x^*(h(s); \lambda); \lambda)}{\partial v_j} \right)_{i,j} \cdot \frac{\partial x^*(g(s); \lambda)}{\partial \lambda} ds \\
+ \int_a^b G(t, s) \left( \frac{\partial f_i(s, x^*(s; \lambda), x^*(g(s); \lambda), x^*(h(s); \lambda); \lambda)}{\partial w_j} \right)_{i,j} \cdot \frac{\partial x^*(h(s); \lambda)}{\partial \lambda} ds, & \text{for } t \in [a, b], \\
\end{cases}
$$

This relation suggest us to consider the following operator

$$
C : X \times X \rightarrow X
$$

$$
C(x, y)(t; \lambda) := \int_a^b G(t, s) \left( \frac{\partial f_i(s, x(s; \lambda), x(g(s); \lambda), x(h(s); \lambda); \lambda)}{\partial u_j} \right)_{i,j} \cdot y(s; \lambda) ds + \\
+ \int_a^b G(t, s) \left( \frac{\partial f_i(s, x(s; \lambda), x(g(s); \lambda), x(h(s); \lambda); \lambda)}{\partial v_j} \right)_{i,j} \cdot y(g(s); \lambda) ds + \\
+ \int_a^b G(t, s) \left( \frac{\partial f_i(s, x(s; \lambda), x(g(s); \lambda), x(h(s); \lambda); \lambda)}{\partial w_j} \right)_{i,j} \cdot y(h(s); \lambda) ds + \\
+ \int_a^b G(t, s) \left( \frac{\partial f_i(s, x(s; \lambda), x(g(s); \lambda), x(h(s); \lambda); \lambda)}{\partial \lambda} \right)_{i,j} ds
$$

$$
\forall t \in [a, b], \lambda \in [c, d].
$$
Systems of Second Order Differential Equations of Mixed Type

\[ C(x, y)(t; \lambda) = 0 \text{ for } t \in [a_1, a] \cup [b, b_1], \lambda \in [c, d]. \]

In this way we have the triangular operator

\[ A: X \times X \longrightarrow X \times X, \]

\[ A(x, y) = (B(x), C(x, y)) \]

where \( B \) is a Picard operator and

\[ C(x^*, \bullet): X \longrightarrow X \]

is \( Q - \)contraction.

Indeed, we have

\[ \|C(x^*, u)(t; \lambda) - C(x^*, v)(t; \lambda)\|_{\mathbb{R}^m} \leq Q \cdot \|u - v\|_C, \quad \forall t \in [a, b], \forall \lambda \in [c, d], \]

which implies that

\[ \|C(x^*, u) - C(x^*, v)\|_C \leq Q \cdot \|u - v\|_C, \quad \forall u, v \in X. \]

Since \( Q^n \longrightarrow 0 \) as \( n \longrightarrow \infty \), applying the Fiber Generalized Contraction Theorem (see [16]), follows that \( A \) is P.O. and has a unique fixed point \((x^*, y^*) \in X \times X\).

So the sequences

\[ (x^{n+1}, y^{n+1}) = (B(x^n), C(x^n, y^n)), \quad n \in \mathbb{N} \]

converges uniformly (with respect to \( t \in [a_1, b_1], \lambda \in [c, d] \)) to \((x^*, y^*)\)

for any \( x^0, y^0 \in C([a_1, b_1] \times [c, d], \mathbb{R}^m)\).

If we take \( x^0 = 0 \) and \( y^0 = \frac{\partial x^0}{\partial \lambda} = 0 \) then \( y^1 = \frac{\partial x^1}{\partial \lambda} \).

By induction we prove that

\[ y^n = \frac{\partial x^n}{\partial \lambda}, \quad \forall n \in \mathbb{N}. \]

Thus \( x^n \underset{\text{unif}}{\longrightarrow} x^* \), as \( n \longrightarrow \infty \)

and \( \frac{\partial x^n}{\partial \lambda} \underset{\text{unif}}{\longrightarrow} y^* \), as \( n \longrightarrow \infty \).

These imply that there exists \( \frac{\partial x^*}{\partial \lambda} \) and \( \frac{\partial x^*}{\partial \lambda} = y^* \).
Theorem 12. We consider the problem \((6) + (7)\) in the conditions \((D1) - (D5)\).

Then:
(i) the problem \((6) + (7)\) has a unique solution

\[ x^* = (x_1^*, ..., x_m^*) \in C([a_1, b_1] \times [c, d], \mathbb{R}^m); \]

(ii) \(x^*(t, \bullet) \in C^1([c, d], \mathbb{R}^m), \forall t \in [a_1, b_1].\)

Remark 13. If we consider \(l_1(x) = x, l_2(x) = x, \alpha = \varphi \) and \(\beta = \psi,\) we obtain the vectorial variant of the boundary value problem from [12]. In this case,

\[
G = \begin{pmatrix}
g & 0 & 0 & \cdots & 0 & 0 \\
0 & g & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & g & 0
\end{pmatrix} \in M_{m, m} (\mathbb{R}^+),
\]

where \(g\) is the Green function of the problem

\[
\begin{cases}
-x'' = \chi \\
x(a) = 0, \ x(b) = 0.
\end{cases}
\]

If \(l_1(x) = \alpha_{11}x + \alpha_{12}x', \ l_2(x) = \alpha_{21}x + \alpha_{22}x', \ \alpha(t) = \alpha \in \mathbb{R}^m, \ \forall t \in [a_1, a], \ \beta(t) = \beta \in \mathbb{R}^m, \ \forall t \in [b, b_1],\) we obtain the vectorial variant of the boundary value problem from [3] and [4]. Here,

\[
G = \begin{pmatrix}
g & \cdots & 0 \\
\vdots & \vdots & \vdots \\
0 & \cdots & g
\end{pmatrix},
\]

where

\[
g(t, s) = \begin{cases}
\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} + (b-a)\alpha_{11}\alpha_{21} & \left( t - a - \frac{\alpha_{12}}{\alpha_{11}} \right) \left( b - s + \frac{\alpha_{22}}{\alpha_{21}} \right), \ a \leq t < s \leq b \\
\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} + (b-a)\alpha_{11}\alpha_{21} & \left( s - a - \frac{\alpha_{12}}{\alpha_{11}} \right) \left( b - t + \frac{\alpha_{22}}{\alpha_{21}} \right), \ a \leq s < t \leq b.
\end{cases}
\]

References


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