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REPRESENTATION THEOREMS AND ALMOST UNIMODAL SEQUENCES

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Dedicated to Professor Gheorghe Coman at his 70^{th} *anniversary*

Abstract. We define the almost unimodal sequences and we show that under some conditions the polynomial $P(X^k + n)$ is almost unimodal (Theorem 1.7). A nontrivial example of almost unimodality shows that the sequence $A_k^{(1)}(n)$, $k = -\frac{n(n+1)}{2}, \dots, -1, 0, 1, \dots, \frac{n(n+1)}{2}$ is symmetric and almost unimodal (Theorem 3.1). This result is connected to some representation properties of integers.

1. Almost unimodal sequences and polynomials

A finite sequence of real numbers $\{d_0, d_1, \dots, d_m\}$ is said to be unimodal if there exists an index $0 \leq m^* \leq m$, called the mode of the sequence, such that d_j increases up to $j = m^*$ and decreases from then on, that is, $d_0 \leq d_1 \leq \cdots \leq d_{m^*}$ and $d_{m^*} \geq d_{m^*+1} \geq \cdots \geq d_m$. A polynomial is said to be unimodal if its sequence of coefficients is unimodal.

Unimodal polynomials arise often in combinatorics, geometry and algebra. The reader is referred to [BoMo] and [AlAmBoKaMoRo] for surveys of the diverse techniques employed to prove that specific families of polynomials are unimodal.

We recall few basic results concerning the unimodality.

Theorem 1.1. *If P is a polynomial with positive nondecreasing coefficients, then $P(X + 1)$ is unimodal.*

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Theorem 1.2. *Let $b_k > 0$ be a nondecreasing sequence. Then the sequence*

$$c_j = \sum_{k=j}^{m} b_k \binom{k}{j}, \quad 0 \le j \le m \tag{1.1}$$

is unimodal with mode $m^* = \left\lfloor \frac{m-1}{2} \right\rfloor$.

Theorem 1.3. *Let $0 \le a_0 \le a_1 \le \cdots \le a_m$ be a sequence of real numbers and $n \in \mathbb{N}$, and consider the polynomial*

$$P = a_0 + a_1 X + a_2 X^2 + \cdots + a_m X^m. \tag{1.2}$$

Then the polynomial $P(X+n)$ is unimodal with mode $m^* = \left\lfloor \frac{m}{n+1} \right\rfloor$.

We can reformulate Theorem 1.3 in terms of the coefficients of polynomial P.

Theorem 1.4. *Let $0 \le a_0 \le a_1 \le \cdots \le a_m$ be a sequence of real numbers and $n \in \mathbb{N}$. Then the sequence*

$$q_j = q_j(m,n) = \sum_{k=j}^{m} a_k \binom{k}{j} n^{k-j}, \quad 0 \le j \le m \tag{1.3}$$

is unimodal with mode $m^* = \left\lfloor \frac{m}{n+1} \right\rfloor$.

In order to introduce the almost unimodality of a sequence we need the following notion.

Definition 1.5. A finite sequence of real numbers $\{c_0, c_1, \ldots, c_n\}$ is called **almost nondecreasing** if it is nondecreasing excepting a subsequence which is zero.

It is clear that, if the sequence $\{c_0, c_1, \ldots, c_n\}$ is nondecreasing, then it is almost nondecreasing. The converse is not true, as we can see from the following example. The sequence $\{0, 1, 0, 2, 0, 3, \ldots, 0, m\}$ is almost nondecreasing but it is not nondecreasing.

Definition 1.6. A finite sequence of real numbers $\{d_0, d_1, \ldots, d_m\}$ is called **almost unimodal** if there exists an index $0 \le m^* \le m$, such that d_j almost increases up to $j = m^*$ and d_j almost decreases from then on.

As in the situation of unimodality, the index m^* is called the mode of the sequence. Also, a polynomial is said to be almost unimodal, if its sequence of coefficients is almost unimodal.

For instance, the polynomial

$$(X^k+1)^m = \binom{m}{0} + \binom{m}{1}X^k + \binom{m}{2}X^{2k} + \cdots + \binom{m}{m}X^{mk}$$

is almost unimodal for $k \geq 2$, but it is not unimodal.

The following result is useful in the study of almost unimodality.

Theorem 1.7. *Let $0 \leq a_0 \leq a_1 \leq \cdots \leq a_m$ be a sequence of real numbers, let n be a positive integer and consider the polynomial*

$$P = a_0 + a_1X + a_2X^2 + \cdots + a_mX^m.$$

Then for any integer $k \geq 2$, the polynomial $P(X^k+n)$ is almost unimodal.

Proof. We note that if Q is a unimodal polynomial, then for any $k \geq 2$ the polynomial $Q(X^k)$ is almost unimodal. Applying Theorem 1.3 we get that $P(X+n)$ is unimodal and now using the remark above it follows that $P(X^k+n)$ is almost unimodal with mode $m^* = k\left\lfloor \dfrac{m}{n+1} \right\rfloor$. □

Remark 1.8. If $n \geq m$, then $m^* = 0$, hence the sequence of coefficients of $P(X^k+n)$ is almost nonincreasing. For example, the sequence of coefficients of $(X^k+3)^3$ is

$$27, \underbrace{0,\ldots,0}_{k-1}, 27, \underbrace{0,\ldots,0}_{k-1}, 9, \underbrace{0,\ldots,0}_{k-1}, 1.$$

2. **Some representation results for integers**

In 1960, P. Erdös and J. Surányi ([ErSu], Problem 5, pp.200) have proved the following result: Any integer k can be written in infinitely many ways in the form

$$k = \pm 1^2 \pm 2^2 \pm \cdots \pm n^2 \tag{2.1}$$

for some positive integer n and for some choices of signs $+$ and $-$.

In 1979, J. Mitek [Mi] has extended the above result as follows: For any fixed positive integer $s \geq 2$ the result in (2.1) holds in the form

$$k = \pm 1^s \pm 2^s \pm \cdots \pm n^s \tag{2.2}$$

The following notion has been introduced in [Dr] by M.O. Drimbe:

Definition 2.1. A sequence $(a_n)_{n\geq 1}$ of positive integers is an **Erdös-Surányi sequence** if any integer k can be represented in infinitely many ways in the form

$$k = \pm a_1 \pm a_2 \pm \cdots \pm a_n \tag{2.3}$$

for some positive integer n and for some choices of signs $+$ and $-$.

The main result in [Dr] is contained in

Theorem 2.2. *Any sequence $(a_n)_{n\geq 1}$ of positive integers satisfying:*

i) $a_1 = 1$,

ii) $a_{n+1} \leq 1 + a_1 + \cdots + a_n$, for any positive integer n,

iii) $(a_n)_{n\geq 1}$ contains infinitely many odd integers,

is an Erdös-Suranyi sequence.

As direct consequences of Theorem 2.1, in the paper [Dr], the following examples of Erdös-Suranyi sequences are pointed out:

1) The Fibonacci's sequence $(F_n)_{n\geq 0}$, where $F_0 = 1$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$, for $n \geq 1$;

2) The sequence of primes $(p_n)_{n\geq 1}$.

We can see that the sequence $(n^s)_{n\geq 1}$ does not satisfy condition ii) in Theorem 2.2 but it is an Erdös Suranyi sequences, according to the result of J. Mitek [Mi] contained in (2.2). Following the paper [Ba] one can extend Theorem 2.2 in such way to include sequences $(n^s)_{n\geq 1}$. The following notion has been introduced in [Kl] by T. Klove:

Definition 2.3. A sequence $(a_n)_{n\geq 1}$ of positive integers is **complete** if any sufficiently great integer can be expressed as a sum of distinct terms of $(a_n)_{n\geq 1}$.

The above property is equivalent to the fact that for any sufficient great integer k there exists a positive integer $t = t(k)$ such that

$$k = u_1 a_1 + u_2 a_2 + \cdots + u_t a_t, \tag{2.4}$$

where $u_i \in \{0, 1\}$, $i = 1, 2, \ldots, t$.

The main result in [Ba] is contained in

Theorem 2.4. *Any complete sequence $(a_n)_{n\geq 1}$ of positive integers, containing infinitely many odd integers, is an Erdös-Surányi sequence.*

Proof. Let q can be represented as in (2.4). Let $S_n = a_1 + \cdots + a_n$, $n \geq 1$. The sequence $(S_n)_{n\geq 1}$ is increasing and it contains infinitely many odd integers but also infinitely many even integers. Let k be a fixed positive integer. One can find infinitely many integers S_p, having the same parity as k, such that $S_p > k + 2q$. Consider S_n a such integer and let $m = \frac{1}{2}(S_n - k)$. Because $q < m$, it follows that m can be represented as in (2.4). Taking into account that $m < S_n$, we have $m = u_1 a_1 + \cdots + u_n a_n$, where $u_i \in \{0, 1\}$, $i = 1, 2, \ldots, n$. Then, we have

$$k = S_n - 2m = (1 - 2u_1)a_1 + \cdots + (1 - 2u_n)a_n.$$

From $u_i \in \{0, 1\}$ we get $1 - 2u_i \in \{-1, 1\}$, $i = 1, 2, \ldots, n$. □

Remark 2.5. The result of J. Mitek [Mi] follows from Theorem 2.4 and from the property that the sequence $(n^s)_{n\geq 1}$ is complete, for any positive integer s. The completeness of $(n^s)_{n\geq 1}$ is a result of P. Erdös (see [Si], pp.395).

3. **Integral formulae and almost unimodality**

Consider an Erdös-Surányi sequence $(a_m)_{m\geq 1}$. If we fix n, then there are 2^n integers of the form $\pm a_1 \pm \cdots \pm a_n$. In this section we explore the number of ways to express an integer k in the form (2.3). Denote $A_k(n)$ to be this value. Using the method in [AnTo] let us consider the function

$$f_n(z) = \left(z^{a_1} + \frac{1}{z^{a_1}}\right)\left(z^{a_2} + \frac{1}{z^{a_2}}\right)\cdots\left(z^{a_n} + \frac{1}{z^{a_n}}\right) \tag{3.1}$$

It is clear that this is the generating function for the sequence $A_k(n)$, i.e. we may write

$$f_n(z) = \sum_{j=-S_n}^{S_n} A_j(n) z^j, \tag{3.2}$$

where $S_n = a_1 + \cdots + a_n$. It is interesting to note the symmetry of the coefficients in (3.2), i.e. $A_j(n) = A_{-j}(n)$. If we write $z = \cos t + i \sin t$, then by using DeMoivre's formula we may rewrite (3.1) as

$$f_n(z) = 2^n \cos a_1 t \cdot \cos a_2 t \dots \cos a_n t \tag{3.3}$$

By noting that $A_k(n)$ is the constant term in the expansion $z^{-k} f_n(z)$, we obtain

$$\begin{aligned} z^{-k} f_n(z) &= 2^n (\cos kt - i \sin kt) \cos a_1 t \dots \cos a_n t \\ &= A_k(n) + \sum_{j \neq k} A_j(n) (\cos(j-k)t + i \sin(j-k)t) \end{aligned} \tag{3.4}$$

Finally, making use of the fact that $\int_0^{2\pi} \cos mt dt = \int_0^{2\pi} \sin mt dt = 0$, we integrate (3.4) on the interval $[0, 2\pi]$ to find an elegant integral formula for $A_k(n)$:

$$A_k(n) = \frac{2^n}{2\pi} \int_0^{2\pi} \cos a_1 t \dots \cos a_n t \cos kt dt \tag{3.5}$$

After integrating, we find that the imaginary part of $A_k(n)$ is 0, which implies the relation

$$\int_0^{2\pi} \cos a_1 t \dots \cos a_n t \sin kt dt = 0 \tag{3.6}$$

for each k between $-S_n$ and S_n.

Applying formula (3.5) for Erdös-Surányi sequence $(m^s)_{m \geq 1}$, we get

$$A_k^{(s)}(n) = \frac{2^n}{2\pi} \int_0^{2\pi} \cos 1^s t \cos 2^s t \dots \cos n^s t \cos kt dt,$$

where $A_k^{(s)}(n)$ denote the integer $A_k(n)$ for this sequence.

The following result gives a nontrivial example of almost unimodality.

Theorem 3.1. *The sequence* $A_k^{(1)}(n)$, $k = 0, 1, \ldots, \dfrac{n(n-1)}{2}$, *is almost nonincreasing and consequently, the sequence* $A_j^{(1)}(n)$, $j = -\dfrac{n(n+1)}{2}, \ldots, -1, 0, 1, \ldots, \dfrac{n(n+1)}{2}$ *is symmetric and almost unimodal.*

Proof. First of all we show that $A_k^{(1)}(n)$ is the number of representations of $\dfrac{1}{2}\left(\dfrac{n(n+1)}{2} - k\right)$ as $\displaystyle\sum_{i=1}^{n} \varepsilon_i i$, where $\varepsilon_i \in \{0, 1\}$. Indeed, we note that if $\varepsilon \in \{0, 1\}$, then $1 - 2\varepsilon \in \{-1, 1\}$ and we have $\displaystyle\sum_{i=1}^{n}(1 - 2\varepsilon_i)i = k$ if and only if

$$\frac{n(n+1)}{2} - 2\sum_{i=1}^{n} \varepsilon_i i = k,$$

hence

$$\sum_{i=1}^{n} \varepsilon_i i = \frac{1}{2}\left(\frac{n(n+1)}{2} - k\right). \tag{3.7}$$

Denote $B_k^{(1)}(n)$ the number of representations of $\dfrac{1}{2}\left(\dfrac{n(n+1)}{2} - k\right)$ in the form (3.7). It is clear that $B_k^{(1)}(n) = 0$ if and only if k and $\dfrac{n(n+1)}{2}$ have different parities. Also, we have $\dfrac{n(n+1)}{4} \leq j \leq \dfrac{n(n+1)}{2}$ for any integer j of the form $\dfrac{1}{2}\left(\dfrac{n(n+1)}{2} - k\right)$, $k = 0, 1, \ldots, \dfrac{n(n+1)}{2}$. Assume that we can write j as $\varepsilon_1 \cdot 1 + \varepsilon_2 \cdot 2 + \cdots + \varepsilon_n \cdot n$ and $\varepsilon_1 = 1$. Then, we have $j - 1 = \varepsilon_2 \cdot 2 + \cdots + \varepsilon_n \cdot n$, where $\varepsilon_2, \ldots, \varepsilon_n \in \{0, 1\}$. If we have in this sum three consecutive terms of the form $i - 1, 0, i + 1$, we can move 1 at the first position and obtain three consecutive terms of the form $i - 1, i, 0$. After another such step for other three consecutive terms $s - 1, 0, s + 1$, taking into account that a such map is injective it follows that $B_j^{(1)}(n) \leq B_{j-2}^{(1)}(n)$, hence $A_j^{(1)}(n) \leq A_{j-2}^{(1)}(n)$ if both $A_{j-2}^{(1)}(n)$ and $A_j^{(1)}(n)$ are not zero.

Remark 3.2. The conclusion of Theorem 3.1 is not generally true for $A_k^{(s)}(n)$, where $s \geq 2$ (see the values of $A_k^{(2)}(6)$ in the table below).

□

4. **Numerical results**

Numerical values for $A_k^{(1)}$ for n up to 9

$n =$	1
k	A_k
0	0
1	1

$n =$	2
k	A_k
0	0
1	1
2	0
3	1

$n =$	3
k	A_k
0	2
1	0
2	1
3	0
4	1
5	0
6	1

$n =$	4
k	A_k
0	2
1	0
2	2
3	0
4	2
5	0
6	1
7	0
8	1
9	0
10	1

$n =$	5
k	A_k
0	0
1	3
2	0
3	3
4	0
5	3
6	0
7	2
8	0
9	2
10	0
11	1
12	0
13	1
14	0
15	1

$n =$	6
k	A_k
0	0
1	5
2	0
3	5
4	0
5	4
6	0
7	4
8	0
9	4
10	0
11	3
12	0
13	2
14	0
15	2
16	0
17	1
18	0
19	1
20	0
21	1

$n =$	7
k	A_k
0	8
1	0
2	8
3	0
4	8
5	0
6	7
7	0
8	7
9	0
10	6
11	0
12	5
13	0
14	5
15	0
16	4
17	0
18	3
19	0
20	2
21	0
22	2
23	0
24	1
25	0
26	1
27	0
28	1

$n =$	8
k	A_k
0	14
1	0
2	13
3	0
4	13
5	0
6	13
7	0
8	12
9	0
10	11
11	0
12	10
13	0
14	9
15	0
16	8
17	0
18	7
19	0
20	6
21	0
22	5
23	0
24	4
25	0
26	3
27	0
28	2
29	0
30	2
31	0
32	1
33	0
34	1
35	0
36	1

$n =$	9
k	A_k
0	0
1	23
2	0
3	23
4	0
5	22
6	0
7	21
8	0
9	21
10	0
11	19
12	0
13	18
14	0
15	17
16	0
17	15
18	0
19	13
20	0
21	12
22	0
23	10
24	0
25	9
26	0
27	8
28	0
29	6
30	0
31	5
32	0
33	4
34	0
35	3
36	0
37	2
38	0
39	2
40	0
41	1
42	0
43	1
44	0
45	1

Numerical values for $A_k^{(2)}$ for n up to 6

$n =$	1
k	A_k
0	0
1	1

$n =$	2
k	A_k
1	0
2	0
3	1
4	0
5	1

$n =$	3
k	A_k
0	0
1	0
2	0
3	0
4	1
5	0
6	1
7	0
8	0
9	0
10	0
11	0
12	1
13	0
14	1

$n =$	4
k	A_k
0	0
1	0
2	1
3	0
4	1
5	0
6	0
7	0
8	0
9	0
10	1
11	0
12	1
13	0
14	0
15	0
16	0
17	0
18	0
19	0
20	1
21	0
22	1
23	0
24	0
25	0
26	0
27	0
28	1
29	0
30	1

$n =$	5
k	A_k
0	0
1	0
2	0
3	2
4	0
5	2
6	0
7	0
8	0
9	0
10	0
11	0
12	0
13	1
14	0
15	1
16	0
17	0
18	0
19	0
20	0
21	1
22	0
23	1
24	0
25	0
26	0
27	1

$n =$	5
k	A_k
28	0
29	1
30	0
31	0
32	0
33	0
34	0
35	1
36	0
37	1
38	0
39	0
40	0
41	0
42	0
43	0
44	0
45	1
46	0
47	1
48	0
49	0
50	0
51	0
52	0
53	1
54	0
55	1

$n =$	6
k	A_k
0	0
1	2
2	0
3	0
4	0
5	0
6	0
7	1
8	0
9	2
10	0
11	1
12	0
13	1
14	0
15	1
16	0
17	1
18	0
19	1
20	0
21	1
22	0
23	1
24	0
25	0
26	0
27	0
28	0
29	0
30	0
31	2
32	0
33	2
34	0
35	0
36	0
37	0
38	0
39	2
40	0
41	2
42	0
43	0
44	0

$n =$	6
k	A_k
45	0
46	0
47	0
48	0
49	1
50	0
51	1
52	0
53	0
54	0
55	0
56	0
57	1
58	0
59	1
60	0
61	0
62	0
63	1
64	0
65	1
66	0
67	0
68	0
69	0
70	0
71	1
72	0
73	1
74	0
75	0
76	0
77	0
78	0
79	0
80	0
81	1
82	0
83	1
84	0
85	0
86	0
87	0
88	0
89	1
90	0
91	1

Numerical values for $A_0^{(1)}(n)$ and $A_0^{(2)}(n)$

(1)

n	A_0
1	0
2	0
3	2
4	2
5	0
6	0
7	8
8	14
9	0
10	0
11	70
12	124
13	0
14	0
15	722
16	1314
17	0
18	0
19	8220
20	15272
21	0
22	0
23	99820
24	187692
25	0
26	0
27	1265204
28	2399784
29	0
30	0
31	16547220
32	31592878
33	0
34	0
35	221653776
36	425363952
37	0
38	0
39	3025553180
40	5830034720
41	0
42	0
43	41931984034
44	81072032060
45	0
46	0
47	588431482334
48	1140994231458
49	0
50	0

n	A_0
51	8346638665718
52	16221323177468
53	0
54	0
55	119447839104366
56	232615054822964
57	0
58	0
59	1722663727780132
60	3360682669655028
61	0
62	0
63	25011714460877474
64	48870013251334676
65	0
66	0
67	365301750223042066
68	714733339229024336
69	0
70	0
71	5363288299585278800
72	10506331021814142340
73	0
74	0
75	79110709437891746598
76	155141342711178904962
77	0
78	0
79	1171806326862876802144
80	2300241216389780443900
81	0
82	0
83	17422684839627191647442
84	34230838910489146400266
85	0
86	0
87	259932234752908992679732
88	511107966282059114105424
89	0
90	0
91	3890080539905554395312172
92	7654746470466776636508150
93	0
94	0
95	58384150201994432824279356
96	114963593898159699687805154
97	0
98	0
99	878552973096352358805720000
100	1731024005948725016633786324

(2)

n	A_0
1	0
2	0
3	0
4	0
5	0
6	0
7	2
8	2
9	0
10	0
11	2
12	10
13	0
14	0
15	86
16	114
17	0
18	0
19	478
20	860
21	0
22	0
23	5808
24	10838
25	0
26	0
27	55626
28	100426
29	0
30	0
31	696164
32	1298600
33	0
34	0
35	7826992
36	14574366
37	0
38	0
39	100061106
40	187392994
41	0
42	0
43	1223587084
44	2322159814
45	0
46	0
47	16019866270
48	30353305134
49	0
50	0

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