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ON THE CONVERSES OF THE REDUCTION PRINCIPLE IN INNER PRODUCT SPACES

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Dedicated to Professor Ştefan Cobzaş at his 60^{th} *anniversary*

Abstract. Let H be an inner product space, X a complete subspace of H, and Y a closed subspace of X. The main result of this Note is the following converse of the Reduction Principle: if $x_0 \in X$, $h \in H \setminus X$ and $y_0 \in Y$ is the element of best approximation of both x_0 and h, $(x_0 - h, x_0 - y_0) = 0$ and $\operatorname{codim}_X Y = 1$, then x_0 is the element of best approximation of h in X.

1. **Introduction**

Let H be an inner product space, with real inner product $(\cdot, \cdot)$ and the norm $\|h\| = \sqrt{(h, h)}$, $h \in H$. For a subset M of H and $h \in H$, the distance of h to M is defined by

$$d(x, M) = \inf\{\|h - m\| : \ m \in M\}.$$

The set M is called **proximinal** if for every $h \in H$ there exists $m_0 \in M$ such that

$$\|h - m_0\| = d(h, M).$$

The set

$$P_M(h) := \{m \in M : \ \|h - m\| = d(h, M)\}, \quad h \in H$$

is called the set of **best approximation elements** of h by elements in M, and the application $P_M : H \to 2^M$ is called the metric projection of H on M.

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If card$P_M(h) = 1$ for every $h \in H$, then the set M is called a **Chebyshevian set** in H ([2], p.35).

The **existence** and the **uniqueness** of best approximation elements are treated in Chapter 3 of [2]: every complete convex set in an inner product space is a Chebyshev set ([2], Th.3.4).

Two elements $u, v \in H$ are called **orthogonal** if $(u, v) = 0$. The cosinus of the angle between the $u, v \in H \setminus \{0\}$ is defined by the formula

$$\cos \widehat{u,v} = \frac{(u, v)}{\|u\| \cdot \|v\|}.$$

Concerning the **characterization** of best approximation elements, the following result holds ([2], Th.4.9):

Let M be a subspace of H, $h \in H$ and $m_0 \in M$. Then $m_0 = P_M(h)$ iff

$$(h - m_0, m) = 0,$$

for all $m \in M$.

The geometric interpretation of this characterization result is that the element $h - P_M(h)$ is orthogonal to each element of M. This is the reason why $P_M(h)$ is often called the **orthogonal projection** of h on M.

The following result appears in [2], p.80 under the name "the Reduction Principle":

Let K be a convex subset of the inner product space H and let M be any Chebyshev subspace of H that contains K. Then

a) $P_K(P_M(h)) = P_K(h) = P_M(P_K(h)),\ h \in H$;

b) $d(h, K)^2 = d(h, M)^2 + d(P_M(h), K)^2$,

for every $h \in H$.

Obviously, if K is a closed and convex subset of a complete subspace M of the inner product space H, the properties a) and b) are also fulfilled (see Th.4.1 in [2], and Th. 2.2.6 in [3]).

2. **Results**

From now on, we consider the following particular case of the Reduction Principle:

Theorem 1. *Let H be an inner product space, X a complete subspace of H, and Y a closed subspace of X. Then*

a') $P_Y(h) = P_Y(P_X(h)) = P_X(P_Y(h)),\ h \in H$;

b') $d(h,Y)^2 = d(h,X)^2 + d(P_X(h),Y)^2$,

for every $h \in H$.

The proof of Theorem 1 is an immediate consequence of the characterization result ([2], Th.4.9) and the Pythagorean Law (see e.g. [1], Th.1, p.70).

A generalization of Theorem 1 is:

Theorem 2. *Let H be an inner product space and $M_1, M_2, \ldots, M_n$ $(n \geq 2)$ be subspaces of H with the following properties:*

1) M_1 is complete;

2) M_i, $i = 2, 3, \ldots, n$ are closed;

3) $M_1 \supset M_2 \supset \cdots \supset M_n$.

a) For every $h \in H$ the following equalities hold

$$P_{M_n}(h) = P_{M_n} P_{M_{n-1}} \ldots P_{M_1}(h) = P_{M_1} P_{M_2} \ldots P_{M_n}(h).$$

b) Let $P_{M_1}(h) = m_1,\ P_{M_k} P_{M_{k-1}}(h) = m_k,\ k = 2, 3, \ldots, n.$

The following equality holds:

$$d(h, M_n)^2 = \|h - m_1\|^2 + \sum_{k=2}^{h} \|m_k - m_{k-1}\|^2.$$

Proof. For every $y \in M_n$ we have

$$(h - P_{M_n} P_{M_{n-1}} \ldots P_{M_1}(P_1), y)$$

$$= (h - P_{M_1}(h) + P_{M_1}(h) - P_{M_2} P_{M_1}(h) + \ldots$$

$$+ P_{M_{n-1}} P_{M_{n-2}} \ldots P_{M_1}(h) - P_{M_n} P_{M_{n-1}} \ldots P_{M_1}(h), y)$$

$$= (h - P_{M_1}(h), y) + \sum_{k=2}^{n} (P_{M_{k-1}} \dots P_{M_1}(h) - P_{M_k} \dots P_{M_1}(h), y) = 0.$$

Using the characterization result ([2], Th.4.9) it follows that the element $P_{M_n} P_{M_{n-1}} \dots P_{M_1}(h)$ is the orthogonal projection of h on M_n.

On the other hand, $(h - P_{M_n}(h), y) = 0$ for every $y \in M_n$. Consequently

$$P_{M_n}(h) = P_{M_n} P_{M_{n-1}} \dots P_{M_1}(h).$$

The equality $P_{M_n}(h) = P_{M_1} P_{M_2} \dots P_{M_n}(h)$ is immediate.

For b) observe that

$$d(h, M_n)^2 = \|m_n - m_{n-1}\|^2 + \|h - m_{n-1}\|^2$$

$$= \|m_n - m_{n-1}\|^2 + \|m_{n-1} - m_{n-2}\|^2 + \|h - m_{n-2}\|^2 = \dots$$

$$= \|m_n - m_{n-1}\|^2 + \dots + \|m_2 - m_1\|^2 + \|h - m_1\|^2. \qquad \square$$

Remark. Obviously, Theorem 1 is also valid if H is a Hilbert space and X, Y are closed subspace of H, with $Y \subset X$. Also, Theorem 2 is valid if H is a Hilbert space and $M_1 \supset M_2 \supset \dots \supset M_n$ are closed subspaces of H.

A converse of the Reduction Principle is given in [3], Th.2.2.6:

Let H be an inner product space, X a complete subspace of H and K a closed and convex subset of X. If x is the orthogonal projection of $h \notin X$ on X, m is the metric projection of h on K, then m is the metric projection of x on K.

A **first** converse of Theorem 1 is:

Theorem 3. *Let H be an inner product space, X a complete subspace of H, and Y a closed subspace of X. Let $h \in H \setminus X$ and let $P_X(h)$ and $P_Y(h)$ be the orthogonal projections of h on X, respectively on Y. Then $P_Y(h)$ is the orthogonal projection of $P_X(h)$ on Y.*

Proof. Indeed, by hypothesis it follows:

$$(h - P_X(h), x) = 0, \ \forall\, x \in X,$$

$$(h - P_Y(h), y) = 0, \ \forall\, y \in Y,$$

so that for every $y \in Y$ one has:

$$(P_X(h) - P_Y(h), y) = (h - P_Y(h) - h + P_X(h), y)$$

$$= (h - P_Y(h), y) - (h - P_X(h), y) = 0.$$

It follows that $P_Y(h)$ is the orthogonal projection of $P_X(h)$ on Y. □

A second converse of Theorem 1 is:

Theorem 4. *Let H be an inner product space, X a complete subspace of H, and Y a closed subspace of X with* $\text{codim}_X Y = 1$. *Let $x_0 \in X \setminus Y$ and $P_Y(x_0)$ be the orthogonal projection of x_0 on Y. If $h \in H \setminus X$, $P_Y(h) = P_Y(x_0)$ and $(h - x_0, x_0 - P_Y(x_0)) = 0$, then $P_Y(h) = x_0$.*

Proof. If the equality $(h - x_0, x) = 0$ is fulfilled for every $x \in X$, then $P_X(h) = x_0$, i.e. the conclusion of the theorem.

For every $y \in Y$ we have

$$(h - x_0, y) = (h - P_Y(x_0) - (x_0 - P_Y(x_0)), y)$$

$$= (h - P_Y(x_0), y) - (x_0 - P_Y(x_0), y) = 0.$$

It follows that $h - x_0$ is orthogonal to Y.

Because, by hypothesis, $(h - x_0, x_0 - P_Y(x_0)) = 0$ it follows that $(h - x_0, u) = 0$ for every $u \in \text{span}\{x_0 - P_Y(x_0)\}$. Because $x_0 - P_Y(x_0)$ is orthogonal to Y and Y is a closed subspace of the Hilbert space X, it follows that $X = \text{span}\{x_0 - P_Y(x_0)\} \oplus Y$, i.e. X is the direct sum of the subspaces $\text{span}\{x_0 - P_Y(x_0)\}$ and Y (see [2], Th.5.9 p.77 and [1], Th.4, p.65). Consequently $(h - x_0, x) = 0$ for every $x \in X$. □

Remark. The condition $\text{codim}_X Y = 1$ in Theorem 4 is essential. Indeed, let $\{e_1, e_2, e_3\}$ be the orthonormal basis of the Hilbert space $\mathbb{R}^3$, $X = \text{span}\{e_1, e_2\}$, $Y = \text{span}\{0\}$ and $h = 3e_1 + e_2 + 5e_3$. Let $x_0 = e_1 + 2e_2$. Then $P_Y(x_0) = 0$ and $P_Y(h) = 3e_1 + e_2$, $P_Y(h) = 0$. The conditions $P_Y(x_0) = P_Y(h)$ and $(h - x_0, x_0 - P_Y(x_0)) = (2e_1 - e_2, e_1 + 2e_2) = 0$ are fulfilled, but $P_X(h) = 3e_1 + e_2 \neq x_0 = e_1 + 2e_2$. Observe that $\text{codim}_X Y = 2$.

Examples. 1° Let $l_2 = l_2(\mathbb{N})$ be the space of all sequences $x = (x(i))$ of real numbers such that $\sum_{i=1}^{\infty} x^2(i) < \infty$. It is known that l_2 is a Hilbert space with respect to the inner product $(x, y) = \sum_{i=1}^{\infty} x(i)y(i)$ and the norm $\|x\| = \left(\sum_{i=1}^{\infty} x^2(i)\right)^{1/2}$. Let $\{e_1, e_2, \dots\}$ be the canonical basis of l_2. The closed subspace $X = \overline{\text{span}\{e_{2n-1} \mid n = 1, 2, 3, \dots\}}$ is Chebyshevian in l_2 and the orthogonal projection of $h = (h(1), h(2), \dots) \in l_2$ is $P_X(h) = \sum_{i=1}^{\infty} h(2i - 1)e_{2i-1}$, because $h - P_X(h) = \sum_{j=1}^{\infty} h(2j)e_{2j}$ is orthogonal on X.

Let $Y = \text{span}\{e_1, e_3 + e_5\}$. Then Y is a Chebyshevian subspace of l_2 (and of X) and

$$P_Y(h) = h(1)e_1 + \frac{1}{2}[h(3) + h(5)](e_3 + e_5).$$

By Theorem 1 one obtains

$$P_Y(h) = P_Y P_X(h) = P_X P_Y(h).$$

By Theorem 3, the orthogonal projection of the element

$$x = \sum_{n=1}^{\infty} h(2n - 1)e_{2n-1}$$

on Y is

$$y_0 = h(1)e_1 + \frac{1}{2}[h(3) + h(5)](e_3 + e_5).$$

Indeed,

$$x - y_0 = \frac{1}{2}[h(3) - h(5)]e_3 + \frac{1}{2}[h(5) - h(3)]e_5 + \sum_{n=4}^{\infty} h(2n - 1)e_{2n-1}$$

is orthogonal to Y, so $y_0 = P_Y(x)$.

2° Let $l_2(4) = \text{span}\{e_1, e_2, e_3, e_4\}$ where $e_i(j) = \delta_{ij}$, $i, j = 1, 2, 3, 4$ (see 1°), and $X = \text{span}\{e_1, e_2, e_3\}$, $Y = \text{span}\{e_1, e_2\}$ and $Z = \text{span}\{e_1\}$.

If $x_0 = 2e_1 + e_2 + 2e_3$, then $P_Y(x_0) = 2e_1 + e_2$. For $\alpha, \beta \in \mathbb{R}$ let $h = 2e_1 + e_2 + \alpha e_3 + \beta e_4$. Then $P_Y(h) = 2e_1 + e_2$ and $(h - x_0, x_0 - P_Y(x_0)) = 2(\alpha - 2) = 0$ implies $\alpha = 2$.

Every element $h = 2e_1 + e_2 + 2e_3 + \beta e_4$, $\beta \in \mathbb{R}$ has as orthogonal projection on X

$$P_X(h) = 2e_1 + e_2 + 2e_3 = x_0.$$

Observe that $\operatorname{codim}_X Y = 1$.

Consider now the orthogonal projections on Z ($\operatorname{codim}_X Z = 2$). Then $P_Z(x_0) = 2e_1$, $P_Z(h) = 2e_1$ and $(h - x_0, x_0 - P_Z(x_0)) = \alpha + \beta - 3 = 0$ implies $\alpha + \beta = 3$.

Choosing the element $h = 2e_1 + 2e_2 + e_3 + 2e_4$ one obtains

$$P_X(h) = 2e_1 + 2e_2 + e_3 \neq 2e_1 + e_2 + 2e_3 = x_0.$$

3° Let $L_2[-1,1]$ be the Hilbert space of all (Lebesgue) measurable real-valued functions on $[-1,1]$ with the property that $\int_{-1}^{1} h^2(t)dt < \infty$. The inner product on $L_2[-1,1]$ is $(x,y) = \int_{-1}^{1} x(t)y(t)dt$ and the associated norm is $\|h\| = \left(\int_{-1}^{1} h^2(t)dt\right)^{1/2}$. Consider also the Legendre polynomials (see [2])

$$p_0(t) = \frac{1}{\sqrt{2}},\ p_1(t) = \frac{\sqrt{6}}{2}t,\ p_2(t) = \frac{\sqrt{10}}{4}(3t^2 - 1),\ p_3(t) = \frac{\sqrt{14}}{4}(5t^3 - 3t)$$

and in general

$$p_n(t) = \frac{(-1)^n\sqrt{2n+1}}{2^n \cdot \sqrt{2} \cdot n!} \cdot \frac{d^n}{dt^n}[(1-t^2)^n],$$

for $n \geq 0$.

The set $\{p_0, p_1, \ldots, p_n\}$, $n \geq 0$ is orthonormal in $L_2[-1,1]$. Consider the following subspaces of $L_2[-1,1]$:

$$X = \operatorname{span}\{p_0, p_1, p_2, p_3\}, \quad Y = \operatorname{span}\{p_0, p_1, p_2\} \quad \text{and}$$

$$Z = \operatorname{span}\{p_0, p_1\}.$$

For every $h \in L_2[-1,1]$ one obtains ([2], Th.4.14)

$$P_X(h) = (h,p_0)p_0 + (h,p_1)p_1 + (h,p_2)p_2 + (h,p_3)p_3,$$

$$P_Y(h) = (h,p_0)p_0 + (h,p_1)p_1 + (h,p_2)p_2 \quad \text{and}$$

$$P_Z(h) = (h, p_0)p_0 + (h, p_1)p_1.$$

Obviously, $Z \subset Y \subset X \subset L_2[-1, 1]$ and $P_Z(h) = P_Z P_Y P_X(h)$.

Let $x_0 = p_0 + 2p_1 + 2p_2 + p_3$. If $h \in L_2[-1, 1] \setminus X$ then $P_Y(h) = P_Y(x_0)$ iff $(h, p_0) = 1$, $(h, p_1) = 2$ and $(h, p_2) = 2$. The condition $(x_0 - P_Y(x_0), h - x_0) = 0$ implies $(p_3, h - x_0) = 0$ and, consequently, $(p_3, h) = (p_3, x_0) = 1$. It follows $P_X(h) = x_0$. Observe that $\text{codim}_X Y = 1$.

Now $P_Z(x_0) = p_0 + 2p_1$ and $P_Z(h) = P_Z(x_0)$ implies $(h, p_0) = 1$, $(h, p_1) = 2$. The condition $(x_0 - P_Z(x_0), h - x_0) = 0$ implies

$$(2p_2 + p_3, h - x_0) = 2(p_2, h) + (p_3, h) - 5 = 0.$$

Let $h_1 = p_0 + 2p_1 + p_2 + 3p_3 + p_4$ and $h_2 = p_0 + 2p_1 + \frac{1}{2}p_2 + 4p_3 + p_4$. Then $P_Z(h_i) = P_Z(x_0)$, $i = 1, 2$ and $(x_0 - P_Z(x_0), h_i - x_0) = 0$, $i = 1, 2$, but $P_X(h_1) \neq P_X(h_2) \neq x_0$. Observe that $\text{codim}_X Z = 2$.

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