ON EDGE-CONNECTIVITY OF INSERTED GRAPHS

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Abstract. The aim of this paper is to estimate the edge-connectivity of the inserted graph with the help of the degree of vertices of the inserted graph and the edge-connectivity of the original graph.

1. Introduction

Throughout the paper we consider ordinary graphs (finite, undirected, with no loops or multiple edges) and \( G \) denotes a graph with vertex set \( V_G \) and edge set \( E_G \). Each member of \( V_G \cup E_G \) will be called an element of \( G \). A graph \( G \) is called a trivial graph if it has a vertex set with single vertex and a null edge set. If \( e \) be an edge of a graph \( G \) with end vertices \( x \) and \( y \), then we denote the edge \( e \), by \( e = xy \).

We introduce the notions of box graph \( B(G) \) and inserted graph \( I(G) \) of a non-trivial graph \( G \) in [3]. It is an elementary basic fact that the inserted graph \( I(G) \) of a non-trivial connected graph \( G \) in connected. The edge-connectivity \( \lambda(G) \) of a graph \( G \) is the least number of edges whose removal disconnects \( G \); and a set of \( \lambda(G) \) edges satisfying this condition is called a minimal separating edge set of \( G \). Clearly, \( G \) is \( m \)-edge-connected if and only if \( \lambda(G) \geq m \).

In §2, we recall some definitions and results which will be used in §3 and also give an example of edge-connectivity of a graph \( G \) and its inserted graph \( I(G) \).

In [1], we investigate the relations between the connectivity and edge-connectivity of a graph and its inserted graph. In §3 of this paper we obtain more
results about edge-connectivity and give an alternative proof of some corollaries stated in [1].

2. Preliminaries

Definition 2.1. [3] A graph can be constructed by inserting a new vertex on each edge of \( G \), the resulting graph is called Box graph of \( G \), denoted by \( B(G) \).

Definition 2.2. [3] Let \( I_G \) be the set of all inserted vertices in \( B(G) \). A graph \( I(G) \) with vertex set \( I_G \) is called the inserted graph in which any two vertices are adjacent if they are joined by a path of length two in \( B(G) \).

\[
\lambda(G) = 1
\]

\[
\lambda(I(G)) = 2
\]

Figure 1: The edge-connectivity of a graph and its inserted graph

These concepts are illustrated for a graph \( G \) and its inserted graph \( I(G) \) in the Fig.1. Here \( \bigotimes \) marked vertices are the newly inserted vertices.

Now we recall the following theorems:

Theorem 2.3. [4] A graph \( G \) is \( m \)-edge-connected if and only if for every non-empty proper subset \( A \) of the vertex set \( V_G \) of the graph \( G \), the number of edges joining \( A \) and \( V_G - A \) is at least \( m \).

The next observation is due to Whitney [5].

Theorem 2.4. For any graph \( G \), \( \lambda(G) \leq \min \deg G \).

The order of a graph is the cardinality of its vertex set. If \( G' \) is a subgraph of \( G \) and \( V_{G'} \), \( V_G \) are the vertex sets of \( G' \) and \( G \) respectively, then the degree of
$G'$ in $G$ is the number of all edges of $G$ joining vertices in $V_{G'}$ with the vertices in $V_G - V_{G'}$.

3. **Edge-connectivity of $I(G)$**

To begin with let us prove the following lemma.

**Lemma 3.1.** If

$$\lambda(I(G)) < \lambda(G)\left\lceil \frac{\lambda(G) + 1}{2} \right\rceil,$$

then there exists a connected subgraph of $G$ of order 2 and degree $\lambda(I(G))$ in $G$.

**Proof:** Let $Y$ denote any nonempty proper subset of the edge set $E_G$ of $G$. Thus $Y$ induces a nonempty proper subset $\overline{Y}$ of the vertex set $V_{I(G)}$. For each vertex $u$ in $G$, denote the number of edges of $Y$ incident with $u$ by $\delta(u)$ and the number of edges of $E_G - Y$ incident with $u$ by $\delta'(u)$; and set $W = \{u; \delta(u) > 0, \delta'(u) > 0\}$. Suppose that each connected subgraph of $G$ with two vertices has degree at least $\lambda(I(G)) + 1$ in $G$. We shall show that

$$\sum_{u \in W} \delta(u)\delta'(u) \geq \lambda(I(G)) + 1.$$  

First, suppose that no two vertices of $W$ are adjacent. Now from the Theorem 2.4, $\deg u \geq \lambda(G)$ for every vertex $u \in W$. Thus one of the numbers $\delta(u)$ and $\delta'(u)$ must be $\left\lceil \frac{\lambda(G) + 1}{2} \right\rceil$. Consequently,

$$\sum_{u \in W} \delta(u)\delta'(u) \geq \left\lceil \frac{\lambda(G) + 1}{2} \right\rceil \sum_{u \in W} \delta_u(u),$$

where $\delta_u$ means $\delta$ or $\delta'$. From the $\lambda(G)$-edge-connectivity of $G$ it follows that

$$\sum_{u \in W} \delta_u(u) \geq \lambda(G),$$

and hence

$$\sum_{u \in W} \delta(u)\delta'(u) \geq \lambda(G)\left\lceil \frac{\lambda(G) + 1}{2} \right\rceil > \lambda(I(G)).$$

Suppose now that two adjacent vertices, say $v$ and $w$, belonging to $W$. We assume that the degree of the subgraph generated by $v$ and $w$ is at least $\lambda(I(G)) + 1$.  

in $G$, i.e.
\[ \delta(v) + \delta'(v) + \delta(w) + \delta'(w) \geq \lambda(I(G)) + 3. \]

Since for any natural numbers $N_1$ and $N_2$, $N_1N_2 \geq N_1 + N_2 - 1$, we may write
\[ \sum_{u \in W} \delta(u)\delta'(u) \geq \delta(v)\delta'(v) + \delta(w)\delta'(w) \geq \delta(v) + \delta'(v) - 1 + \delta(w) + \delta'(w) - 1 \geq \lambda(I(G)) + 1. \]

By application of Theorem 2.3, the inequality
\[ \sum_{u \in W} \delta(u)\delta'(u) \geq \lambda(I(G)) + 1 \]
proved above for a set $W$ derived from an arbitrary proper subset $\gamma$ of $V_I(G)$ shows that $I(G)$ is $(\lambda(I(G)) + 1)$-edge-connected, which is by definition impossible.

Therefore there exist a connected subgraph $G'$ of $G$ of order 2 and of degree at most $\lambda(I(G))$; if this degree becomes smaller than $\lambda(I(G))$, then the corresponding vertex of $I(G)$ have degree smaller than $\lambda(I(G))$, contradicting the Theorem 2.4. Hence $G'$ has precisely the degree $\lambda(I(G))$ in $G$.

We now show that Corollaries 3.5 and 3.6 of [1] follows from the above Lemma.

**Corollary 3.2.** [1] $\lambda(I(G)) \geq 2\lambda(G) - 2$.

**Proof:** We prove the corollary by the method of contradiction. Suppose that $\lambda(I(G)) < 2\lambda(G) - 2$. Since
\[ 2\lambda(G) - 2 \leq \lambda(G)\left\lfloor \frac{\lambda(G)}{2} \right\rfloor, \]
Lemma 3.1 implies the existence of a connected subgraph $G'$ of $G$ with two vertices of degree $\lambda(I(G))$ in $G$; since this degree is smaller than $2\lambda(G) - 2$, the degree of one of the vertices of $G'$ is at most $\lambda(G) - 1$, contradicting Theorem 2.4.

**Corollary 3.3.** [1] If $\lambda(G) \neq 2$, then $\lambda(I(G)) = 2\lambda(G) - 2$ if and only if there exist two adjacent vertices in $G$ with degree $\lambda(G)$.

**Proof:** For $\lambda(G) \neq 2$,
\[ 2\lambda(G) - 2 < \lambda(G)\left\lfloor \frac{\lambda(G)}{2} \right\rfloor. \]
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Hence by using Lemma 3.1, it follows that if

\[ \lambda(I(G)) = 2\lambda(G) - 2, \]

then there exist two adjacent vertices \( v, w \) in \( G \) so that

\[ \deg v + \deg w = \lambda(I(G)) + 2. \]

Since both \( v \) and \( w \) have degree \( \geq \lambda(G) \) and

\[ \deg v + \deg w = 2\lambda(G), \]

it follows immediately by Theorem 2.4 that

\[ \deg v = \deg w = \lambda(G). \]

Conversely, if \( v \) and \( w \) are adjacent vertices of \( G \) and

\[ \deg v = \deg w = \lambda(G), \]

then the vertex in \( I(G) \) corresponding to the edge joining \( v \) and \( w \) has degree \( 2\lambda(G) - 2 \).

Hence by Theorem 2.4,

\[ \lambda(I(G)) \leq 2\lambda(G) - 2. \]

Now by Corollary 3.2, it follows that

\[ \lambda(I(G)) = 2\lambda(G) - 2. \]

**Corollary 3.4.** If \( \lambda(G) \geq 3 \), then \( \lambda(I(G)) = 2\lambda(G) - 1 \) only if there exist two adjacent vertices in \( G \), one of degree \( \lambda(G) \) and the other of degree \( \lambda(G) + 1 \).

**Proof:** Proof is similar to that of Corollary 3.3

This procedure can be continued finitely as the graph is finite. Now we prove the following significant theorem.

**Theorem 3.5.** If

\[ \min \deg I(G) \leq \lambda(G)\left\lfloor \frac{\lambda(G) + 1}{2} \right\rfloor, \]

then \( \lambda(I(G)) = \min \deg I(G) \). Also if

\[ \min \deg I(G) \geq \lambda(G)\left\lceil \frac{\lambda(G) + 1}{2} \right\rceil, \]
then
\[
\lambda(G)[\frac{\lambda(G) + 1}{2}] \leq \lambda(I(G)) \leq \min \deg I(G).
\]

**Proof:** Theorem 2.4 implies \(\lambda(I(G)) \leq \min \deg I(G)\). Now for the case

\[
\min \deg I(G) \leq \lambda(G)[\frac{\lambda(G) + 1}{2}]
\]
suppose that \(\lambda(I(G)) < \min \deg I(G)\). Then Lemma 3.1 asserts that there exists a connected subgraph of order 2 and degree \(\lambda(I(G))\) in \(G\); this means that there is a vertex in \(I(G)\) of degree \(\lambda(I(G))\), violating the assumed inequality. Consequently,

\[
\lambda(I(G)) = \min \deg I(G).
\]

For the case

\[
\min \deg I(G) \geq \lambda(G)[\frac{\lambda(G) + 1}{2}],
\]
it remains to be shown that

\[
\lambda(G)[\frac{\lambda(G) + 1}{2}] \leq \lambda(I(G)).
\]

Suppose on the contrary that

\[
\lambda(G)[\frac{\lambda(G) + 1}{2}] > \lambda(I(G)).
\]

Then by Lemma 3.1 some vertex in \(I(G)\) has degree \(\lambda(I(G))\). Hence

\[
\min \deg I(G) \leq \lambda(I(G)).
\]

Thus it follows that

\[
\lambda(G)[\frac{\lambda(G) + 1}{2}] \leq \lambda(I(G)),
\]
contradicting the inequality assumed above.
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References


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