ON SOME INTEGRAL EQUATIONS WITH DEVIATING ARGUMENT

OLARU ION MARIAN

Abstract. The purpose of this paper is to study the following functional equation with modified argument:

\[ x(t) = g(t, hx(t), x(t), x(0)) + \int_{-\theta t}^{\theta t} K(t, s, x(s))ds, \]

where \( \theta \in (0, 1) \), \( t \in [-T, T] \), \( T > 0 \).

1. Introduction

Let \( (X, d) \) be a metric space and \( A : X \rightarrow X \) an operator. We shall use the following notations:

\( F_A := \{ x \in X \mid Ax = x \} \) the fixed points set of \( A \).
\( I(A) := \{ Y \in P(X) \mid A(Y) \subset Y \} \) the family of the nonempty invariant subsets of \( A \).
\( A^{n+1} = A \circ A^n, A^0 = 1_X, A^1 = A, n \in \mathbb{N} \).

Definition 1.1. [4] An operator \( A \) is weakly Picard operator (WPO) if the sequence 
\( (A^n x)_{n \in \mathbb{N}} \)

converges, for all \( x \in X \) and the limit (which depend on \( x \)) is a fixed point of \( A \).

Definition 1.2. [4], [1] If the operator \( A \) is WPO and \( F_A = \{ x^* \} \) then by definition \( A \) is Picard operator.
Definition 1.3. [4] If $A$ is WPO, then we consider the operator

$$A^\infty : X \to X, A^\infty(x) = \lim_{n \to \infty} A^n x.$$ 

We remark that $A^\infty(X) = F_A$.

Definition 1.4. [1] Let $A$ be an WPO and $c > 0$. The operator $A$ is $c$-WPO if 

$$d(x, A^\infty x) \leq d(x, Ax).$$

We have the following characterization of the WPOs.

Theorem 1.1. [4] Let $(X, d)$ be a metric space and $A : X \to X$ an operator. The operator $A$ is WPO ($c$-WPO) if and only if there exists a partition of $X$,

$$X = \bigcup_{\lambda \in \Lambda} X_\lambda$$

such that 

(a) $X_\lambda \in I(A)$

(b) $A : X_\lambda \to X_\lambda$ is a Picard ($c$-Picard) operator, for all $\lambda \in \Lambda$.

For the class of $c$-WPOs we have the following data dependence result.

Theorem 1.2. [4] Let $(X, d)$ be a metric space and $A_i : X \to X, i = 1, 2$ an operator. We suppose that:

(i) the operator $A_i$ is $c_i$-WPO $i = 1, 2$.

(ii) there exists $\eta > 0$ such that 

$$d(A_1 x, A_2 x) \leq \eta, (\forall) x \in X.$$ 

Then

$$H(F_{A_1}, F_{A_2}) \leq \eta \max\{c_1, c_2\}.$$ 

Here stands for Hausdorff-Pompeiu functional.

We have
Lemma 1.1. [4],[1] Let \((X,d,\leq)\) be an ordered metric space and \(A : X \rightarrow X\) an operator such that:

a) \(A\) is monotone increasing.

b) \(A\) is WPO.

Then the operator \(A^\infty\) is monotone increasing.

2. Main results

Data dependence for functional-integral equations was studied in [2],[3],[4],[1].

Let \((X,\| \cdot \|)\) a Banach space and the space \(C([-T,T],X)\) endowed with the Bielecki norm \(\| \cdot \|_\tau\) defined by

\[
\|x\|_\tau = \max_{t\in[-T,T]} \|x(t)\|e^{-\tau(t+T)}.
\]

In [1] Viorica Muresan studied the following functional integral equation:

\[
x(t) = g(t, h(x)(t), x(t), x(0)) + \int_0^t K(t, s, x(\theta s)) ds, t \in [0, b], \theta \in [0, 1]
\]

by the weakly Picard operators technique.

We consider the following functional-integral equations with modified argument:

\[
x(t) = g(t, hx(t), x(t), x(0)) + \int_{-\theta t}^{\theta t} K(t, s, x(s)) ds,
\]

where:

i) \(t \in [-T,T], T > 0\).

ii) \(h : C([-T,T],X) \rightarrow C([-T,T],X), g \in C([-T,T] \times X^3, X), K \in C([-T,T] \times [-T,T] \times X^2, X)\).

We suppose that the following conditions are satisfied:

(c1) there exists \(l > 0\) such that

\[
\|hx(t) - hy(t)\| \leq \|x(t) - y(t)\|,
\]
for all \(x, y \in C([-T, T], X), t \in [-T, T]\).

\((c_2)\) There exists \(l_1 > 0, l_2 > 0\) such that
\[
\|g(t, u_1, v_1, w) - g(t, u_2, v_2, w)\| \leq l_1 \|u_1 - u_2\| + l_2 \|v_1 - v_2\|.
\]
for all \(t \in [-T, T], u_i, v_i, w \in X, i = 1, 2.

\((c_3)\) There exists \(l_3 > 0\) such that
\[
\|K(t, s, u) - K(t, s, u_1)\| \leq l_3 \|u - u_1\|,
\]
for all \(t, s \in [-T, T], u, u_1 \in X.

\((c_4)\) \(l_1 l_2 < 1\).

\((c_5)\) \(g(0, h(x)(0), x(0), x(0)) = x(0)\) for any \(x \in C([-T, T], X).

Let \(A : C([-T, T], X) \rightarrow C([-T, T], X)\) be defined by
\[
Ax(t) = g(t, hx(t), x(t), x(0)) + \int_{-\theta t}^{\theta t} K(t, s, x(s))ds
\] (2)

Let \(\lambda \in X\) and \(X_\lambda = \{x \in C([-T, T], X) \mid x(0) = \lambda\}\). Then \(C([-T, T], X) = \bigcup_{\lambda \in X} X_\lambda\) is a partition of \(C([-T, T], X)\). From \(c_5\) we have that \(X_\lambda \in I(A)\).

For studying of data dependence we consider the following equations
\[
x(t) = g_1(t, hx(t), x(t), x(0)) + \int_{-\theta t}^{\theta t} K_1(t, s, x(s))ds
\] (3)
\[
x(t) = g_2(t, hx(t), x(t), x(0)) + \int_{-\theta t}^{\theta t} K_2(t, s, x(s))ds
\] (4)

**Theorem 2.1.** We consider the equation (1) under following conditions:

(i) The conditions \(c_1 - c_5\) are satisfied.

(ii) The operators \(h(\cdot), g(t, \cdot, \cdot, \cdot), K(t, s, \cdot, \cdot)\) are monotone increasing.

(iii) There exists \(\eta_1, \eta_2 > 0\) such that
\[
\|g_1(t, u, v, w) - g_2(t, u, v, w)\| < \eta_1,
\]
\[
\|K_1(t, s, u) - K_2(t, s, u)\| \leq \eta_2
\]
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for all \( t \in [-T, T] \), \( u, v, w \in X \). Then:

(a) For all \( x, y \) solutions of (1) with \( x(0) \leq y(0) \) we have \( x(t) \leq y(t) \), for all \( t \in [-T, T] \).

(b) \( H(S_1, S_2) \leq \frac{\eta_1 + 2\eta_2 T}{(1 - l_1 l - l_2 - \frac{\eta_3}{\tau})} \), where \( S_1, S_2 \) is the solutions set of (3), (4).

**Proof** We denote with \( A_{\lambda} \) the restriction of the operator \( A \) at \( X_{\lambda} \). First we show that \( A_{\lambda} \) is a contraction map on \( X_{\lambda} \). From \( c_1 - c_5 \) we have that

\[
\|A_{\lambda}x(t) - A_{\lambda}y(t)\| \leq (l_1 l + l_2) \|x(t) - y(t)\| + \int_{-\theta t}^{\theta t} \|K(t, s, x(s)) - K(t, s, y(s))\| ds \leq \eta_1 + 2\eta_2 T \|
\]

So \( A \) is \( c \)-WPO with

\[
c = \frac{1}{1 - l_1 l - l_2 - \frac{\eta_3}{\tau}}
\]

Using the theorem 1.2 we obtain (b).

For proof of (a) let be \( x, y \) solutions for (1) with \( x(0) \leq y(0) \). Then \( x \in X_{x(0)}, y \in X_{y(0)} \). We define

\[
\tilde{x}(t) = x(0), t \in [0, b]
\]

\[
\tilde{y}(t) = y(0), t \in [0, b]
\]

We have

\[
\tilde{x}(0) \in X_{x(0)}, \tilde{y}(0) \in X_{y(0)}, \tilde{x}(0) \leq \tilde{y}(0).
\]

From lemma 1.1 we obtain that the operator \( A^\infty \) is increasing. It follows that

\[
A^\infty(\tilde{x}(0)) \leq A^\infty(\tilde{y}(0))
\]

i.e \( x \leq y \)

Next we define \( \varphi \) -contraction notion and use this for estimate distance between two weakly Picard operators.

Let \( \varphi : R_+ \rightarrow R_+ \).
Definition 2.1. [5] \( \phi \) is a strict comparison function if \( \phi \) satisfies the following:

i) \( \phi \) is continuous.

ii) \( \phi \) is monotone increasing.

iii) \( \phi^n(t) \to 0 \), for all \( t > 0 \).

iv) \( t^{-\phi(t)} \to \infty \), for \( t \to \infty \).

Let \((X, d)\) be a metric space and \( f : X \to X \) an operator.

Definition 2.2. [5] The operator \( f \) is called a strict \( \phi \)-contraction if:

(i) \( \phi \) is a strict comparison function.

(ii) \( d(f(x), f(y)) \leq \phi(d(x, y)) \), for all \( x, y \in X \).

Theorem 2.2. [5] Let \((X, d)\) be a complete metric space, \( \phi : \mathbb{R}_+ \to \mathbb{R}_+ \) a strict comparison and \( f, g : X \to X \) two orbitally continuous operators. We suppose that:

(i) \( d(f(x), f^2(x)) \leq \phi(d(x, f(x))) \) for any \( x \in X \) and \( d(g(x), g^2(x)) \leq \phi(d(x, g(x))) \) for any \( x \in X \).

(ii) there exists \( \eta > 0 \) such that \( d(f(x), g(x)) \leq \eta \), for any \( x \in X \).

Then:

(a) \( f, g \) are weakly Picard operators.

(b) \( H(F_f, F_g) \leq \tau_\eta \) where \( \tau_\eta = \sup\{ t \mid t - \phi(t) \leq \eta \} \).

Theorem 2.3. We suppose that condition \((c_5)\) is verified and the following conditions are satisfied:

\((H_1)\) there exists \( \varphi \) a strict comparison function such that

\[ (i) \|h_x(t) - h_y(t)\| \leq \|x(t) - y(t)\|, \]

for all \( x, y \in C([-T, T], X), t \in [-T, T] \).

\[ (ii) g(t, u_1, v_1, w) - g(t, u_2, v_2, w) \| \leq a_\varphi(\|u_1 - u_2\|) + b_\varphi(\|v_1 - v_2\|), \]

for all \( t \in [-T, T], u_i, v_i, w \in X, i = 1, 2 \)

\[ (iii) \|K(t, s, u) - K(t, s, u_1)\| \leq l(t, s)\varphi(\|u - u_1\|), \]
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for all \( t, s \in [-T, T] \), \( u, u_1, \in X \), where \( l(t, \cdot) \in L^1[-T, T] \).

\((H_2)\) There exists \( \eta_1, \eta_2 > 0 \) such that

\[
\|g_1(t, u, v, w) - g_2(t, u, v, w)\| \leq \eta_1,
\]

\[
\|K_1(t, s, u) - K_2(t, s, u)\| \leq \eta_2
\]

for all \( t \in [-T, T] \), \( u, v, w \in X \).

\((H_3)\)

\[
a + b + \max_{t \in [-T, T]} \int_{-T}^{T} l(t, s) ds \leq 1
\]

Then:

(i) the equation (1) has at least solution.

(ii) \( H(S_1, S_2) \leq \tau_\eta \) where \( \eta = \eta_1 + 2T\eta_2 \), \( S_1, S_2 \) is the solutions set of (3), (4).

**Proof** Let be \( A_1, A_2 : C([-T, T], X) \longrightarrow C([-T, T], X) \),

\[
A_1x(t) = g_1(t, hx(t), x(t), x(0)) + \int_{-\theta t}^{\theta t} K_1(t, s, x(s)) ds
\]

\[
A_2x(t) = g_2(t, hx(t), x(t), x(0)) + \int_{-\theta t}^{\theta t} K_2(t, s, x(s)) ds.
\]

From

\[
\| A_1x(t) - A_2^2x(t) \| \leq \| g_1(t, hx(t), x(t), x(0)) - g_1(t, hA_1x(t), A_1x(t), A_1x(0)) \| +
\]

\[
+ \int_{-\theta t}^{\theta t} \| K_1(t, s, x(s)) - K_1(t, s, A_1x(s)) \| ds
\]

\[
\leq a\varphi(\|hx(t) - hA_1x(t)\|) + b\varphi(\|x(t) - A_1x(t)\|) +
\]

\[
+ \int_{-\theta t}^{\theta t} l(t, s) \varphi(\|x(s) - A_1x(s)\|) ds \leq a\varphi(\|x(t) - A_1x(t)\|) + b\varphi(\|x(t) - A_1x(t)\|) +
\]

\[
\int_{-\theta t}^{\theta t} l(t, s) \varphi(\|x(s) - A_1x(s)\|) ds \leq (a + b + \max_{t \in [-T, T]} \int_{-T}^{T} l(t, s) ds) \varphi(\|x - A_1x\|_C) \leq 71
\]
we have that 
\[ \| A_i x - A_i^2 x \|_C \leq \varphi(\| x - A_i x \|_C), \ i = 1, 2. \]

Here \( \| \cdot \|_C \) is the Chebyshev norm on \( C([-T, T], X) \).

We note that \( \| A_1 x - A_2 x \|_C \leq \eta_1 + 2T\eta_2 \). From this, using the theorem 2.2
we have the conclusions.

References


Department of Mathematics, University "Lucian Blaga" Sibiu, Romania

E-mail address: olaruim@yahoo.com