EINSTEIN EQUATIONS IN THE GEOMETRY OF SECOND ORDER

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Abstract. In [7], R. Miron and Gh. Atanasiu wrote the Einstein equations of a metric structure \( G \) on the tangent bundle of order two, \( T^2M \) (previously named "2-osculator bundle" and denoted by \( Osc^2M \)), endowed with a nonlinear connection \( N \) and a linear connection \( D \) such that the 2-tangent structure \( J \) be absolutely parallel to \( D \).

In the present paper, the authors determine the Einstein equations by making use of the concept of \( N \)-linear connection defined by Gh. Atanasiu, [1], this is, a linear connection which is not necessarily compatible with \( J \), but only preserves the distributions generated by the nonlinear connection \( N \).

1. The Tangent Bundle \( T^2M \)

Let \( M \) be a real \( n \)-dimensional manifold of class \( C^\infty \), \( (T^2M, \pi^2, M) \) its second order tangent bundle and let \( \tilde{T^2M} \) be the space \( T^2M \) without its null section. For a point \( u \in T^2M \), let \( (x^a, y^{(1)a}, y^{(2)a}) \) be its coordinates in a local chart.

Let \( N \) be a nonlinear connection, [3, 8-13], and denote its coefficients by \( \left(N^a_{\ b}, N^n_{\ 2\ b}\right), \ a, \ b = 1, \ldots, n \). Then, \( N \) determines the direct decomposition

\[
T_uT^2M = N_0(u) \oplus N_1(u) \oplus V_2(u), \ \forall u \in T^2M. \tag{1}
\]
The adapted basis to (1) is \((\delta_a, \delta_1a, \delta_2a)\) and its dual basis is \((dx^a, \delta y^{(1)a}, \delta y^{(2)a})\), where

\[
\begin{align*}
\delta_a &= \frac{\delta}{\delta x^a} = \frac{\partial}{\partial x^a} - N_c^a \frac{\partial}{\partial y^{(1)c}} - \frac{N_c^a}{2} \frac{\partial}{\partial y^{(2)c}} \\
\delta_1a &= \frac{\delta}{\delta y^{(1)a}} = \frac{\partial}{\partial y^{(1)a}} - \frac{N_c^a}{1} \frac{\partial}{\partial y^{(2)c}} \\
\delta_2a &= \frac{\partial}{\partial y^{(2)a}},
\end{align*}
\]

respectively,

\[
\begin{align*}
\delta y^{(1)a} &= dy^{(1)a} + M_1^c dx^c \\
\delta y^{(2)a} &= dy^{(2)a} + M_1^c dy^{(1)c} + M_2^a dx^c,
\end{align*}
\]

where \(M_1^c, M_2^a\) are the dual coefficients of the nonlinear connection \(N\).

Then, a vector field \(X \in \mathcal{X}(T^2M)\) is represented in the local adapted basis as

\[
X = X^{(0)a} \delta_a + X^{(1)a} \delta_1a + X^{(2)a} \delta_2a,
\]

with the three right terms (called d-vector fields) belonging to the distributions \(N, N_1\) and \(V_2\) respectively.

A 1-form \(\omega \in \mathcal{X}^*(T^2M)\) will be decomposed as

\[
\omega = \omega^{(0)}_a dx^a + \omega^{(1)}_a dy^{(1)a} + \omega^{(2)}_a dy^{(2)a}.
\]

Similarly, a tensor field \(T \in T^r_s(T^2M)\) can be split with respect to (1) into components, which will be called d-tensor fields.

The \(\mathcal{F}(T^2M)\)-linear mapping \(J : \mathcal{X}(T^2M) \to \mathcal{X}(T^2M)\) given by

\[
J(\delta_a) = \delta_{1a}, J(\delta_1a) = \delta_{2a}, J(\delta_2a) = 0
\]

is called the 2-tangent structure on \(T^2M\), [8-13].

2. N-linear connections. d-tensors of curvature

An N-linear connection \(D\), [1], is a linear connection on \(T^2M\), which preserves by parallelism the distributions \(N, N_1\) and \(V_2\). Let us notice that an N-linear connection, in the sense of the definition above, is not necessarily compatible to the
2-tangent structure $J$ (an $N$-linear connection which is also compatible to $J$ is called, [1], a \textit{JN-linear connection}).

An $N$-linear connection is locally given by its coefficients

$$D\Gamma (N) = \left( \begin{array}{c} L_{(00)}^a_{\ bc}, L_{(10)}^a_{\ bc}, L_{(20)}^a_{\ bc}, C_{(01)}^a_{\ bc}, C_{(11)}^a_{\ bc}, C_{(21)}^a_{\ bc}, C_{(02)}^a_{\ bc}, C_{(12)}^a_{\ bc}, C_{(22)}^a_{\ bc} \end{array} \right),$$

where

$$\begin{aligned} D_{\delta_1 \delta_2} \delta_b &= L_{(00)}^a_{\ bc} \delta_a, \quad D_{\delta_1 \delta_{1b}} &= L_{(10)}^a_{\ bc} \delta_{1a}, \quad D_{\delta_2 \delta_{2b}} = L_{(20)}^a_{\ bc} \delta_{2a} \\
D_{\delta_1 \delta_1} \delta_b &= C_{(01)}^a_{\ bc} \delta_a, \quad D_{\delta_1 \delta_{1b}} &= C_{(11)}^a_{\ bc} \delta_{1a}, \quad D_{\delta_2 \delta_{2b}} = C_{(21)}^a_{\ bc} \delta_{2a} \\
D_{\delta_2 \delta_2} \delta_b &= C_{(02)}^a_{\ bc} \delta_a, \quad D_{\delta_2 \delta_{1b}} &= C_{(12)}^a_{\ bc} \delta_{1a}, \quad D_{\delta_2 \delta_{2b}} = C_{(22)}^a_{\ bc} \delta_{2a} . 
\end{aligned}$$

In the particular case when $D$ is $J$-compatible, we have

$$\begin{aligned} L_{(00)}^a_{\ bc} &= L_{(10)}^a_{\ bc} = L_{(20)}^a_{\ bc} = L_{(00)}^a_{\ bc} \\
C_{(01)}^a_{\ bc} &= C_{(11)}^a_{\ bc} = C_{(21)}^a_{\ bc} = C_{(11)}^a_{\ bc} \\
C_{(02)}^a_{\ bc} &= C_{(12)}^a_{\ bc} = C_{(22)}^a_{\ bc} = C_{(22)}^a_{\ bc} . \end{aligned}$$

For an $N$-linear connection, let

$$\begin{aligned} D_H^h \ Y &= D_X Y^H, \quad D_V^h Y = D_X v_1 Y^H, \quad D_V^2 Y = D_X v_2 Y^H \\
D_H^\beta \ Y &= D_X Y^V_\beta, \quad D_V^h Y = D_X v_1 Y^V_\beta, \quad D_V^2 Y = D_X v_2 Y^V_\beta, \quad \beta = 1, 2. \end{aligned}$$

$D_H$, $D_V^h$, $D_V^2$ are called respectively, $h_\alpha$, $v_1\alpha$- and $v_2\alpha$-covariant derivatives, $\alpha = 0, 1, 2$. In local coordinates, for a $d$-tensor field

$$T = T_{b_1 \ldots b_r}^{a_1 \ldots a_r} \left( x, y^{(1)}, y^{(2)} \right) \delta_{a_1} \otimes \ldots \otimes \delta_{2a_r} \otimes dx^{b_1} \otimes \ldots \otimes dy^{(2)b_r} .$$

we have

$$D_H^h T = X^{(0)m} T_{b_1 \ldots b_r}^{a_1 \ldots a_r} \delta_{a_1} \otimes \ldots \otimes \delta_{2a_r} \otimes dx^{b_1} \otimes \ldots \otimes dy^{(2)b_r} ,$$

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is completely determined by its components (which are d-tensors) \( R(\delta_{ij}, \delta_{jk}) \delta_{ij} \).

Namely, the 2-forms of curvature of an \( \alpha \) are d-tensors, named the \textit{d-tensors of curvature} of the \( \alpha \)-linear connection \( D \).

The curvature of the \( \alpha \)-linear connection \( D \),

\[
R(X, Y) Z = D_X D_Y Z - D_Y D_X Z - D_{[X, Y]} Z
\]

is completely determined by its components (which are d-tensors) \( R(\delta_{ij}, \delta_{jk}) \delta_{ij} \).

Namely, the 2-forms of curvature of an \( \alpha \)-linear connection are, \([1]\),

\[
\Omega^a_b = \frac{1}{2} R^a_{bcd} dx^c \wedge dx^d + P^a_{bcd} dy^{(1)c} \wedge dy^{(2)d} + \frac{1}{2} S^a_{bcd} dy^{(2)c} \wedge dy^{(2)d}, \tag{8}
\]

\( \alpha = 0, 1, 2 \), where the coefficients \( R^a_{bcd}, P^a_{bcd}, S^a_{bcd} \) are d-tensors, named the d-tensors of curvature of the \( \alpha \)-linear connection \( D \). For a JN-linear connection, there holds

\[
\Omega^a_b = \Omega^a_b = \Omega^a_b,
\]

this is,

\[
R^a_{bcd} = R^a_{bcd} = R^a_{bcd} = R^a_{bcd},
\]

\[
P^a_{bcd} = P^a_{bcd} = P^a_{bcd} = P^a_{bcd} \tag{9}
\]
The detailed expressions of the d-tensors of curvature can be found in [1].

3. Metric structures on $T^2 M$

A Riemannian metric on $T^2 M$ is a tensor field $G$ of type $(0, 2)$, which is nondegenerate in each $u \in T^2 M$ and is positively defined on $T^2 M$.

In this paper, we shall consider metrics in the form

$$G = g_{ab} dx^a \otimes dx^b + g_{ab} \delta y^{(1)a} \otimes \delta y^{(1)b} + g_{ab} \delta y^{(2)a} \otimes \delta y^{(2)b},$$

where $g_{ab} = g_{ab}(x, y^{(1)}, y^{(2)}); \text{ this is, such that the distributions } N, N_1 \text{ and } V_2 \text{ generated by the nonlinear connection } N \text{ be orthogonal with respect to } G.$

An $N$-linear connection $D$ is called a metrical $N$-linear connection if $D X G = 0, \forall X \in \mathcal{X}(T^2 M)$.

This means

$$g_{ab|\alpha c} = g_{ab} \big|_{\alpha c} = 0, \alpha = 0, 1, 2, \beta = 1, 2.$$

The existence of metrical $N$–linear connections is proved in [2].

4. The Ricci tensor $Ric(D)$

Let us notice that, if $D$ is not $J$- compatible, we could expect that the components of the Ricci tensor look in a more complicated way that the ones in the Miron-Atanasiu theory, [7].

Indeed, if we consider the Ricci tensor $Ric(D), [14]$, as the trace of the linear operator

$$V \mapsto R(V, X) Y, \forall V = V^{(0)a} \delta_a + V^{(1)a} \delta_{1a} + V^{(2)a} \delta_{2a} \in \mathcal{X}(T^2 M),$$
then we have:

\[ Ric(D)(X,Y) = \text{trace}(V \mapsto R(V^H,X)Y + R(V^V,X)Y + R(V^V,X)Y). \] (12)

By a straightforward calculus, we obtain:

**Theorem 4.1.** The Ricci tensor \( Ric(d) \) has the following components:

\[
\begin{align*}
Ric(D) \left( \frac{\delta}{\delta x^b}, \frac{\delta}{\delta x^a} \right) &= R^{c}_{(00)a} = R_{ab};
Ric(D) \left( \frac{\delta}{\delta y^{(1)b}}, \frac{\delta}{\delta x^a} \right) &= -P^{c}_{(10)a} = -\frac{2}{10} P_{ab};
Ric(D) \left( \frac{\delta}{\delta y^{(2)b}}, \frac{\delta}{\delta x^a} \right) &= -P^{c}_{(20)a} = -\frac{2}{20} P_{ab};
Ric(D) \left( \frac{\delta}{\delta x^b}, \frac{\delta}{\delta y^{(1)a}} \right) &= P^{c}_{(11)a} = \frac{1}{11} P_{ab};
Ric(D) \left( \frac{\delta}{\delta y^{(1)b}}, \frac{\delta}{\delta y^{(1)a}} \right) &= S^{c}_{(11)a} = S_{ab};
Ric(D) \left( \frac{\delta}{\delta y^{(2)b}}, \frac{\delta}{\delta y^{(1)a}} \right) &= -Q^{c}_{(21)a} = -\frac{2}{21} Q_{ab};
Ric(D) \left( \frac{\delta}{\delta x^b}, \frac{\delta}{\delta y^{(2)a}} \right) &= P^{c}_{(22)a} = \frac{1}{22} P_{ab};
Ric(D) \left( \frac{\delta}{\delta y^{(1)b}}, \frac{\delta}{\delta y^{(2)a}} \right) &= Q^{c}_{(22)a} = \frac{1}{22} Q_{ab};
Ric(D) \left( \frac{\delta}{\delta y^{(2)b}}, \frac{\delta}{\delta y^{(2)a}} \right) &= S^{c}_{(22)a} = S_{ab};
\end{align*}
\]

The Ricci scalar \( Sc(D) \) is, thus,

\[ Sc(D) = g^{ab} R_{ab} + g^{ab}_{(1)} S_{ab} + g^{ab}_{(2)} S_{ab}, \] (13)

where \( g^{ab}_{(1)}, g^{ab}_{(2)}, g^{ab}_{(2)} \) are the coefficients of the inverse matrix of \( G \).
In the particular case of a $JN$-linear connection, taking into account $(8')$, with the notations in [7], we have

$$\frac{1}{(\beta\beta)} P_{ab} = \frac{1}{(\beta)} P_{ab}, \quad \frac{2}{(\beta\beta)} P_{ab} = P_{ab}, \quad \frac{1}{(22)} Q_{ab} = P_{(21)ab} (= Q_{ab}), \quad \frac{2}{(21)} Q_{ab} = P_{(21)ab} (= Q_{ab}). \quad (14)$$

5. Einstein equations

The Einstein equations associated to the metrical $N$-linear connection $D$ are

$$\text{Ric}(D) - \frac{1}{2} \text{Sc}(D) G = \kappa T, \quad (15)$$

where $\kappa$ is a constant and $T$ is the energy-momentum tensor, given by its components

$$T_{(\alpha\beta)} = T(\delta_{\beta\alpha}, \delta_{\alpha\alpha})$$

Expressing the above relation in the adapted frame (2), we obtain

**Theorem 5.1.** The Einstein equations associated to the metrical $N$-linear connection $D$ are

$$R_{ab} - \frac{1}{2} \text{Sc}(D) g_{ab} = \kappa T_{(00)};$$

$$\frac{1}{(\beta\beta)} P_{ab} = \kappa T_{(\beta\beta)}; \quad \beta = 1, 2;$$

$$\frac{2}{(\beta\beta)} P_{ab} = -\kappa T_{(\beta\beta)}; \quad \beta = 1, 2;$$

$$S_{ab} - \frac{1}{2} \text{Sc}(D) g_{ab} = \kappa T_{(\beta\beta)}; \quad \alpha = 1, 2;$$

$$\frac{1}{(22)} Q_{ab} = \kappa T_{(21)};$$

$$\frac{2}{(21)} Q_{ab} = -\kappa T_{(12)}.$$

In the case when $D$ is a $JN$-linear connection, one obtains the result in [7].

In order to avoid confusions when raising and lowering indices, because of the fact that the components $g_{ab}, g^{ab}, g_{ab}^{(1)}, g_{ab}^{(2)}$ are different, we will denote in the following by $i, j, ...$ the indices corresponding to the horizontal distribution, by $a, b, ...$ those corresponding to $N_1$, and by $p, q, ...$ those corresponding to $V_2$. Thus, if we impose
the condition that the divergence of the energy-momentum tensor vanish, in the adapted frame we will obtain

**Theorem 5.2.** The law of conservation on $T^2M$ endowed with the metrical $N$-linear connection $D$ is given by

$$
\left( R^i_j - \frac{1}{2} \mathcal{S}(D) \delta^i_j \right)_{(1)} + \frac{1}{2} \mathcal{P}^a_{(11)} \delta^1_{a} - \frac{1}{2} \mathcal{P}^a_{(10)} \delta^1_{a} + \frac{1}{2} \mathcal{P}^p_{(22)} \delta^2_{p} - \frac{1}{2} \mathcal{P}^p_{(20)} \delta^2_{p} = 0;
$$

$$
\frac{1}{2} \mathcal{P}^i_{(11)} \delta^1_{i} - \frac{1}{2} \mathcal{P}^i_{(10)} \delta^1_{i} + \left( \mathcal{S}^a_{(1)} - \frac{1}{2} \mathcal{S}(D) \delta^a_b \right)_{(1)} = 0;
$$

$$
\frac{1}{2} \mathcal{P}^i_{(22)} \delta^2_{i} - \frac{1}{2} \mathcal{P}^i_{(20)} \delta^2_{i} + \left( \mathcal{S}^a_{(2)} - \frac{1}{2} \mathcal{S}(D) \delta^a_b \right)_{(2)} = 0.
$$

In the same way, one can deduce the Maxwell equations associated to the metrical $N$-linear connection $D$.

**References**


[2] Atanasiu, Gh., *The homogeneous prolongation to the second order tangent bundle of a Riemannian metric* (to appear);


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