PROPERTIES OF SOME NEW SEMINORMED SEQUENCE SPACES DEFINED BY A MODULUS FUNCTION

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Abstract. In this paper we introduce the sequence spaces \( \hat{c}_0(p,f,q,s) \), \( \hat{c}(p,f,q,s) \) and \( \hat{m}(p,f,q,s) \) using a modulus function \( f \) and defined over a seminormed space \( (X,q) \) seminormed by \( q \). We study some properties of these sequence spaces and obtain some inclusion relations.

1. Introduction

Let \( m, c \) and \( c_0 \) be the Banach spaces of bounded, convergent and null sequences \( x = (x_k) \) with the usual norm \( \|x\| = \sup_{k \geq 0} |x_k| \). Let \( D \) be the shift operator on \( s \), that is, \( Dx = (x_k)_{k=1}^{\infty}, D^2 x = (x_k)_{k=2}^{\infty} \) and so on. It may be recalled that a Banach limit (see Banach [1]) \( L \) is a nonnegative linear functional on \( m \) such that \( L \) is invariant under shift operator (that is, \( L(Dx) = L(x) \) for \( x \in m \)) and \( L(e) = 1 \), where \( e = (1,1,...) \). A sequence \( x \in m \) is almost convergent (see Lorentz [8]) if all Banach limits of \( x \) coincide. Let \( \hat{c} \) denote the space of almost convergent sequences. It is proved by Lorentz [8] that

\[ \hat{c} = \left\{ x : \lim_{m \to \infty} t_{m,n}(x) \text{ exists uniformly in } n \right\} \]

where

\[ t_{m,n}(x) = \frac{1}{m+1} \sum_{i=0}^{m} D^i x_n, \quad (D^0 = 1). \]

Several authors including Duran [5], King [7] and Nanda ([12], [13]) have studied almost convergent sequences.

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The notion of a modulus function was introduced by Nakano [11] in 1953. We recall that a modulus \( f \) is a function from \([0, \infty)\) to \([0, \infty)\) such that (i) \( f(x) = 0 \) if and only if \( x = 0 \), (ii) \( f(x+y) \leq f(x) + f(y) \), for all \( x \geq 0, y \geq 0 \), (iii) \( f \) is increasing, (iv) \( f \) is continuous from the right at 0.

Since \(|f(x) - f(y)| \leq f(|x - y|)\), it follows from condition (iv) that \( f \) is continuous on \([0, \infty)\). Furthermore, we have \( f(nx) \leq nf(x) \) for all \( n \in \mathbb{N} \), from condition (ii), and so

\[
 f(x) = f\left(nx \frac{1}{n}\right) \leq nf\left(\frac{x}{n}\right),
\]

hence

\[
\frac{1}{n}f(x) \leq f\left(\frac{x}{n}\right) \text{ for all } n \in \mathbb{N}.
\]

A modulus may be bounded or unbounded. For example, \( f(x) = x^p \), \((0 < p \leq 1)\) is unbounded and \( f(x) = \frac{x}{1+x} \) is bounded. Maddox [10] and Ruckle [14] used a modulus function to construct some sequence spaces.

After then some sequence spaces, defined by a modulus function, were introduced and studied by Bhardwaj [2], Bilgin [3], Connor [4], Esi [6], and many others.

**Definition 1.1.** Let \( q_1, q_2 \) be seminorms on a vector space \( X \). Then \( q_1 \) is said to be stronger than \( q_2 \) if whenever \((x_n)\) is a sequence such that \( q_1(x_n) \to 0 \), then also \( q_2(x_n) \to 0 \). If each is stronger than the other \( q_1 \) and \( q_2 \) are said to be equivalent (one may refer to Wilansky [15]).

**Lemma 1.1.** Let \( q_1 \) and \( q_2 \) be seminorms on a linear space \( X \). Then \( q_1 \) is stronger than \( q_2 \) if and only if there exists a constant \( M \) such that \( q_2(x) \leq Mq_1(x) \) for all \( x \in X \) (see for instance Wilansky [15]).

Let \( p = (p_n) \) be a sequence of strictly positive real numbers and \( X \) be a seminormed space over the field \( \mathbb{C} \) of complex numbers with the seminorm \( q \). We
define the sequence spaces as follows:

\[ \hat{c}_0 (p, f, q, s) = \left\{ x \in X : \lim_{m \to \infty} m^{-s} [(f (q (t_{m,n} (x))))]^p_m = 0 \text{ uniformly in } n \right\}, \]

\[ \hat{c} (p, f, q, s) = \left\{ x \in X : \lim_{m \to \infty} m^{-s} [(f (q (t_{m,n} (x - \ell e))))]^p_m = 0 \text{ for some } \ell, \text{ uniformly in } n \right\}, \]

\[ \hat{m} (p, f, q, s) = \left\{ x \in X : \sup_{m,n} m^{-s} [(f (q (t_{m,n} (x))))]^p_m < \infty \right\}. \]

where \( f \) is a modulus function.

The following inequalities will be used throughout the paper. Let \( p = (p_m) \) be a bounded sequence of strictly positive real numbers with \( 0 < p_m \leq \sup p_m = H \), \( C = \max (1, 2^H - 1) \), then

\[ |a_m + b_m|^p_m \leq C \{ |a_m|^p_m + |b_m|^p_m \}, \tag{1.1} \]

where \( a_m, b_m \in \mathbb{C} \).

2. Main results

**Theorem 2.1.** Let \( p = (p_m) \) be a bounded sequence, then \( \hat{c}_0 (p, f, q, s) \), \( \hat{c} (p, f, q, s) \), \( \hat{m} (p, f, q, s) \) are linear spaces.

**Proof.** We give the proof for \( \hat{c}_0 (p, f, q, s) \) only. The others can be treated similarly. Let \( x, y \in \hat{c}_0 (p, f, q, s) \). For \( \lambda, \mu \in \mathbb{C} \), there exist positive integers \( M_\lambda \) and \( N_\mu \) such that \( |\lambda| \leq M_\lambda \) and \( |\mu| \leq N_\mu \). Since \( f \) is subadditive and \( q \) is a seminorm

\[ m^{-s} [f (q (t_{m,n} (q (t_{m,n} (x)))))]^p_m \leq C (M_\lambda)^H m^{-s} [f (q (t_{m,n} (x))))]^p_m + C (N_\mu)^H m^{-s} [f (q (t_{m,n} (y))))]^p_m \rightarrow 0, \text{ uniformly in } n. \]

This proves that \( \hat{c}_0 (p, f, q, s) \) is a linear space.

**Theorem 2.2.** The space \( \hat{c}_0 (p, f, q, s) \) is a paranormed space, paranormed by

\[ g (x) = \sup_{m,n} m^{-s} [(f (q (t_{m,n} (x))))]^p_m \]^{\frac{1}{p_m}}, \]

where \( M = \max (1, \sup p_m) \). The spaces \( \hat{c} (p, f, q, s) \), \( \hat{m} (p, f, q, s) \) are paranormed by \( g \), if \( \inf p_m > 0. \)
Proof. Omitted.

Theorem 2.3. Let $f$ be modulus function, then

(i) $\hat{c}_0 (p, f, q, s) \subseteq \hat{m} (p, f, q, s)$,

(ii) $\hat{c} (p, f, q, s) \subseteq \hat{\mathbf{m}} (p, f, q, s)$.

Proof. We prove the second inclusion, since the first inclusion is obvious. Let $x \in \hat{c} (p, f, q, s)$, by definition of a modulus function (the inequality (ii)), we have

$$m^{-s} [f (q (t_{m,n} (x)))]^p \leq C m^{-s} [f (q (t_{m,n} (x - \ell)))]^p + C m^{-s} [f (q (\ell))]^p.$$  

Then there exists an integer $K_\ell$ such that $q (\ell) \leq K_\ell$. Hence, we have

$$m^{-s} [f (q (t_{m,n} (x)))]^p \leq C m^{-s} [f (q (t_{m,n} (x - \ell)))]^p + C m^{-s} \max (1, [(K_\ell) f (1)]^H),$$

so $x \in \hat{m} (p, f, q, s)$.

Theorem 2.4. Let $f, f_1, f_2$ be modulus functions $q, q_1, q_2$ seminorms and $s, s_1, s_2 \geq 0$. Then

(i) If $s > 1$ then $Z (f_1, q, s) \subseteq Z (f \circ f_1, q, s)$,

(ii) $Z (p, f_1, q, s) \cap Z (p, f_2, q, s) \subseteq Z (p, f_1 + f_2, q, s)$,

(iii) $Z (p, f, q_1, s) \cap Z (p, f, q_2, s) \subseteq Z (p, f, q_1 + q_2, s)$,

(iv) If $q_1$ is stronger than $q_2$ then $Z (p, f, q_1, s) \subseteq Z (p, f, q_2, s)$,

(v) If $s_1 \leq s_2$ then $Z (p, f, q, s_1) \subseteq Z (p, f, q, s_2)$,

(vi) If $q_1 \equiv (equivalent to) q_2$, then $Z (p, f, q_1, s) = Z (p, f, q_2, s)$, where $Z = \hat{m}, \hat{c}$ and $\hat{c}_0$.

Proof. (i) We prove this part for $Z = \hat{c}$ and the rest of the cases will follow similarly. Let $x \in \hat{c} (p, f, q, s)$, so that

$$S_m = m^{-s} [f_1 (q (t_{m,n} (x - \ell)))] \to 0.$$

Let $\varepsilon > 0$ and choose $\delta$ with $0 < \delta < 1$ such that $f (c) < \varepsilon$ for $0 \leq t \leq \delta$. Now we write
\[I_1 = \{ m \in \mathbb{N} : f_1(q(t_{m,n}(x-\ell))) \leq \delta \}\]
\[I_2 = \{ m \in \mathbb{N} : f_1(q(t_{m,n}(x-\ell))) > \delta \} .\]

For \( f_1(q(t_{m,n}(x-\ell))) > \delta , \)

\[f_1(q(t_{m,n}(x-\ell))) < f_1(q(t_{m,n}(x-\ell))) \delta^{-1} < 1 + [f_1(q(t_{m,n}(x-\ell))) \delta^{-1}]]\]

where \( m \in I_2 \) and \([|u|]\) denotes the integer part of \( u \). By the definition of \( f \) we have

\[f(f_1(q(t_{m,n}(x-\ell)))) \leq (1 + [f_1(q(t_{m,n}(x-\ell))) \delta^{-1}]) f(1) \leq 2 f(1) f_1(q(t_{m,n}(x-\ell))) \delta^{-1}. \quad (2.1)\]

For \( f_1(q(t_{m,n}(x-\ell))) \leq \delta , \)

\[f(f_1(q(t_{m,n}(x-\ell)))) < \varepsilon \quad (2.2)\]

where \( m \in I_1 \). By (2.1) and (2.2) we have

\[m^{-s} [f(f_1(q(t_{m,n}(x-\ell))))] \leq m^{-s} \varepsilon + \left[2 f(1) \delta^{-1}\right] S_m \rightarrow 0 \text{ as } m \rightarrow \infty, \text{uniformly} \ n.\]

Hence \( \hat{c}(p, f_1, q, s) \subseteq \hat{c}(p, f \circ f_1, q, s). \)

(ii) The proof follows from the following inequality

\[m^{-s} [(f_1 + f_2)(q(t_{m,n}(x)))]^{pm} \leq C m^{-s} [f_1(q(t_{m,n}(x)))]^{pm} + C m^{-s} [f_2(q(t_{m,n}(x)))]^{pm}.\]

(iii), (iv) (v) and (vi) follow easily.

**Corollary 2.1.** Let \( f \) be a modulus function, then we have

(i) If \( s > 1, Z(p,q,s) \subseteq Z(p,f,q,s), \)

(ii) \( Z(p,f,q) \subseteq Z(p,q,s), \)

(iii) \( Z(p,q) \subseteq Z(p,q,s), \)

(iv) \( Z(f,q) \subseteq Z(f,q,s) \)
where $Z = \hat{m}$, $\hat{c}$ and $\hat{c}_0$.

The proof is straightforward.

**Theorem 2.5.** For any two sequences $p = (p_k)$ and $r = (r_k)$ of positive real numbers and for any two seminorms $q_1$ and $q_2$ on $X$ we have $Z(p, f, q_1, s) \cap Z(r, f, q_2, s) \neq \emptyset$.

**Proof.** The proof follows from the fact that the zero element $\theta \in \hat{c}_0$ belongs to each of the classes of sequences involved in the intersection.

**Theorem 2.6.** For any two sequences $p = (p_m)$ and $r = (r_m)$, we have $\hat{c}_0(p, f, q, s) \subseteq \hat{c}_0(r, f, q, s)$ if and only if $\liminf \frac{r_m}{p_m} > 0$.

**Proof.** If we take $y_m = f(q(t_{m,n}(x)))$ for all $m \in \mathbb{N}$, then using the same technique of lemma 1 of Maddox [9], it is easy to prove the theorem.

**Theorem 2.7.** For any two sequences $p = (p_m)$ and $r = (r_m)$, we have $\hat{c}_0(p, f, q, s) = \hat{c}_0(r, f, q, s)$ if and only if $\liminf \frac{p_m}{r_m} > 0$ and $\liminf \frac{r_m}{p_m} > 0$.

**Theorem 2.8.** Let $0 < p_m \leq r_m \leq 1$. Then $\hat{m}(r, f, q, s)$ is closed subspace of $\hat{m}(p, f, q, s)$.

**Proof.** Let $x \in \hat{m}(r, f, q, s)$. Then there exists a constant $B > 1$ such that

$$k^{-s} |f(t_{m,n}(x))|^{p_m/M} \leq B \quad \text{for all } m, n$$

and so

$$k^{-s} |f(t_{m,n}(x))|^{p_m/M} \leq B \quad \text{for all } m, n.$$ 

Thus $x \in \hat{m}(p, f, q, s)$. To show that $\hat{m}(r, f, q, s)$ is closed, suppose that $x^i \in \hat{m}(r, f, q, s)$ and $x^i \to x \in \hat{m}(p, f, q, s)$. Then for every $0 < \varepsilon < 1$, there exists $N$ such that for all $m, n$

$$k^{-s} |f(t_{m,n}(x^i - x))|^{p_m/M} \leq B \quad \text{for all } i > N.$$ 

Now

$$k^{-s} |f(t_{m,n}(x^i - x))|^{r_m/M} < k^{-s} |f(t_{m,n}(x^i - x))|^{p_m/M} < \varepsilon \quad \text{for all } i > N.$$ 

Therefore $x \in \hat{m}(r, f, q, s)$. This completes the proof.
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References


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