EXTREMAL PROBLEMS OF TURÁN TYPE

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Abstract. We give estimations of certain weighted $L^2$-norms of the $k$-th derivative of polynomials which have a curved majorant. They are obtained as applications of special quadrature formulae.

1. Introduction

The following problem was raised by P. Turán. Let $\varphi(x) \geq 0$ for $-1 \leq x \leq 1$ and consider the class $P_n,\varphi$ of all polynomials of degree $n$ such that $|p_n(x)| \leq \varphi(x)$ for $-1 \leq x \leq 1$.

How large can $\max_{[-1,1]} |p_n^{(k)}(x)|$ be if $p_n$ is arbitrary in $P_n,\varphi$?

The aim of this paper is to consider the solution in the weighted $L^2$-norm for the majorant

$$\varphi(x) = \frac{\alpha - \beta x}{\sqrt{1-x^2}}, 0 \leq \beta \leq \alpha.$$  

Let us denote by

$$x_i = \cos \frac{(2i-1)\pi}{2n}, i = 1, 2, ..., n,$$

the zeros of $T_n(x) = \cos n\theta, x = \cos \theta,$  \hspace{1cm} (1.1)

the Chebyshev polynomial of the first kind,

$$y_i^{(k)}$$ the zeros of $U_n^{(k)}(x), U_{n-1}(x) = \sin n\theta/\sin \theta, x = \cos \theta,$  \hspace{1cm} (1.2)

the Chebyshev polynomial of the second kind and

$$G_{n-1}(x) = \alpha U_{n-1}(x) - \beta U_{n-2}(x), 0 \leq \beta \leq \alpha.$$  \hspace{1cm} (1.3)
Let $\Pi_{\alpha,\beta}$ be the class of all polynomials $p_{n-1}$, of degree $\leq n - 1$ such that

$$|p_{n-1}(x_i)| \leq \frac{\alpha - \beta x_i}{\sqrt{1 - x_i^2}}, \quad i = 1, 2, \ldots, n,$$

(1.4)

where the $x_i$'s are given by (1.1) and $0 \leq \beta \leq \alpha$.

2. Results

**Theorem 2.1.** If $p_{n-1} \in \Pi_{\alpha,\beta}$ then we have

$$\int_{-1}^{1} \frac{1}{\sqrt{1 - x^2}} \left[p'_{n-1}(x)\right]^2 \, dx$$

(2.1)

$$\leq \frac{2\pi n (n - 1) \left[(\alpha^2 + \beta^2) (n - 2) \left(n^2 - 2n + 2\right) + 5\beta^2 (n^2 - n + 1)\right]}{15}$$

with equality for $p_{n-1} = G_{n-1}$.

Two cases are of special interest:

**I. Case** $\alpha = \beta = 1$, $\varphi(x) = \sqrt{\frac{1 - x}{1 + x}}$.

$$G_{n-1}(x) = V_{n-1}(x) = \frac{\cos\left((n - \frac{1}{2}) \arccos x\right)}{\cos\left(\frac{1}{2} \arccos x\right)}.$$

Note that $P_{n-1,\varphi} \subset \Pi_{1,1}$, $V_{n-1} \notin P_{n-1,\varphi}$, $V_{n-1} \in \Pi_{1,1}$.

**Corollary 2.2.** If $p_{n-1} \in \Pi_{1,1}$ then we have

$$\int_{-1}^{1} \frac{1}{\sqrt{1 - x^2}} \left[p'_{n-1}(x)\right]^2 \, dx \leq \frac{2\pi n (n - 1) \left(2n - 1\right) \left(n^2 - n + 3\right)}{15}$$

(2.2)

with equality for $p_{n-1} = V_{n-1}$.

**II. Case** $\alpha = 1$, $\beta = 0$, $\varphi(x) = \frac{1}{\sqrt{1 - x^2}}$, $G_{n-1} = U_{n-1}$.

Note that $P_{n-1,\varphi} \subset \Pi_{1,0}$, $U_{n-1} \in P_{n-1,\varphi}$, $U_{n-1} \in \Pi_{1,0}$.

**Corollary 2.3.** If $p_{n-1} \in \Pi_{1,0}$ then we have

$$\int_{-1}^{1} \frac{1}{\sqrt{1 - x^2}} \left[p'_{n-1}(x)\right]^2 \, dx \leq \frac{2\pi n \left(n^2 - 1\right)}{15}$$

(2.3)

with equality for $p_{n-1} = U_{n-1}$.

In this second case we have a more general result:
EXTREMAL PROBLEMS OF TURÁN TYPE

Theorem 2.4. If \( p_{n-1} \in \Pi_{1,0} \) and \( 0 \leq b \leq a \) then we have

\[
\frac{1}{1-a-bx} \frac{1}{1-x^2} \left[ p_{n-1}^{(k+1)}(x) \right]^2 dx = \frac{\pi a (n+k+1)!}{(n-k-2)!} \left[ \frac{2 \left( n^2 - (k+2)^2 \right) (a^2 + 3b^2)}{(2k+1) (2k+3) (2k+5)} + \frac{2 (k+1) a^2 + 3b^2}{(2k+1) (2k+3)} \right]
\]

for \( k = 0, \ldots, n-2 \), with equality for \( p_{n-1} = U_{n-1} \).

Setting \( a = 1, b = 1 \) one obtains the following

Corollary 2.5. If \( p_{n-1} \in \Pi_{1,1} \) then we have

\[
\frac{1}{1-x} \frac{1}{1-x^2} \left[ p_{n-1}^{(k+1)}(x) \right]^2 dx \leq \frac{\pi (n+k+1)!}{(n-k-2)!} \times \frac{8 \left( n^2 - (k+2)^2 \right) + (2k+5)^2}{(2k+1) (2k+3) (2k+5)}
\]

for \( k = 0, \ldots, n-2 \), with equality for \( p_{n-1} = U_{n-1} \).

Setting \( a = 1, b = 0 \) one obtains the following

Corollary 2.6. If \( p_{n-1} \in \Pi_{1,0} \) then we have

\[
\frac{1}{1-x} \frac{1}{1-x^2} \left[ p_{n-1}^{(k+1)}(x) \right]^2 dx \leq \frac{2\pi (n+k+1)!}{(n-k-2)!} \times \frac{n^2 + k^2 + 3k + 1}{(2k+1) (2k+3) (2k+5)}
\]

for \( k = 0, \ldots, n-2 \), with equality for \( p_{n-1} = U_{n-1} \).

3. Lemmas

Here we state some lemmas which help us in proving our theorems.

Lemma 3.1. Let \( p_{n-1} \) be such that \( |p_{n-1}(x_i)| \leq \frac{\alpha - \beta x_i}{\sqrt{1-x_i^2}}, i = 1, 2, \ldots, n \), where the \( x_i \)'s are given by (1.1). Then we have

\[
|p_{n-1}'(y_j)| \leq |G_{n-1}'(y_j)|, \quad k = 0, 1, \ldots, n-1, \text{ and}
\]

\[
|p_{n-1}'(1)| \leq |G_{n-1}'(1)|, \quad |p_{n-1}'(-1)| \leq |G_{n-1}'(-1)|.
\]

113
Lemma 3.3. \( k \) then for 

By the Lagrange interpolation formula based on the zeros of \( T_n \) and using \( T_n''(x_i) = \frac{(-1)^{i+1}n!}{(1-x_i^2)^{i+2}} \), we can represent any polynomial \( p_{n-1} \) by \( p_{n-1}(x) = \frac{1}{n} \sum_{i=1}^{n} \frac{T_n(x)}{x-x_i} (-1)^{i+1} (1-x_i^2)^{1/2} p_{n-1}(x_i). \)

From \( G_{n-1}(x) = (-1)^{i+1} \frac{\alpha-\beta x_i}{\sqrt{1-x_i^2}} \), we have \( G_{n-1}(x) = \frac{1}{n} \sum_{i=1}^{n} \frac{T_n(x)}{x-x_i} (\alpha-\beta x_i) \).

Proof. Let \( n \) and \( k \) be such that \( |p_{n-1}(x_i)| \leq \frac{1}{\sqrt{1-x_i^2}}, i = 1, 2, ..., n, \) where the \( x_i \)'s are given by (1.1). Then we have

\[
\left| p_{n-1}^{(k+1)}(y_i^{(k)}) \right| \leq \left| U_{n-1}^{(k+1)}(y_i^{(k)}) \right|, \text{ whenever } U_{n-1}^{(k)}(y_i^{(k)}) = 0, \quad (3.3)
\]

\[
k = 0, 1, ..., n-1, \text{ and }
\[
\left| p_{n-1}^{(k+1)}(1) \right| \leq \left| U_{n-1}^{(k+1)}(1) \right|, \left| p_{n-1}^{(k+1)}(-1) \right| \leq \left| U_{n-1}^{(k+1)}(-1) \right|. \quad (3.4)
\]

Proof. For \( \alpha = 1, \beta = 0, \) \( G_{n-1} = U_{n-1} \) and (3.1) give \( |p_{n-1}'(y_j)| \leq |U_{n-1}'(y_j)| \) and (3.2) \( |p_{n-1}'(1)| \leq |U_{n-1}'(1)|, \left| p_{n-1}'(-1) \right| \leq |U_{n-1}'(-1)| . \)

Now the proof ends by applying Duffin-Schaeffer Lemma. \( \square \)

We need the following quadrature formulae:

Lemma 3.4. For any given \( n \) and \( k, 0 \leq k \leq n-1, \) let \( y_i^{(k)}, i = 1, ..., n - k - 1, \)
be the zeros of $U_{n-1}^{(k)}$.

Then the quadrature formulae

$$\int_{-1}^{1} (1-x^2)^{k-1/2} f(x) \, dx = A_0 [f(-1) + f(1)] + \sum_{i=1}^{n-k-1} s_i f\left(y_i^{(k)}\right),$$

$$A_0 = \frac{2^{2k-1} (2k + 1) \Gamma (k + 1/2)^2 (n - k - 1)!}{(n + k)!}, s_i > 0$$

and

$$\int_{-1}^{1} (1-x^2)^{k-1/2} f(x) \, dx = B_0 [f(-1) + f(1)] + C_0 [f'(-1) - f'(1)] + \sum_{i=1}^{n-k-2} v_i f\left(y_i^{(k+1)}\right)$$

$$C_0 = \frac{2^{2k} (2k + 3) \Gamma (k + 3/2)^2 (n - k - 2)!}{(n + k + 1)!},$$

$$B_0 = C_0 \frac{2 \left(n^2 - (k + 2)^2\right)(2k + 3) + 4 (k + 1)(2k + 5)}{(2k + 1)(2k + 5)}$$

have algebraic degree of precision $2n - 2k - 1$.

For $r(x) = (a - bx)^3$, $0 \leq b \leq a$ the formulae

$$\int_{-1}^{1} r(x) (1-x^2)^{k-1/2} f(x) \, dx = A_1 f(-1) + B_1 f(1)$$

$$A_1 = \frac{2^{2k-1} (2k + 1) \Gamma (k + 1/2)^2 (n - k - 1)! (a+b)^3}{(n + k)!},$$

$$B_1 = \frac{2^{2k-1} (2k + 1) \Gamma (k + 1/2)^2 (n - k - 1)! (a-b)^3}{(n + k)!}$$
and

\[
\int_{-1}^{1} r(x) (1 - x^2)^{k-1/2} f(x) \, dx = C_1 f(-1) + D_1 f(1) + C_2 f'(-1) - D_2 f'(1) + \sum_{i=1}^{n-k-2} v_i r(y_i^{(k+1)}) f(y_i^{(k+1)}),
\]

\[
C_1 = B_0 (a + b)^3 - 3C_0 b (a + b)^2, \quad D_1 = B_0 (a - b)^3 + 3C_0 b (a - b)^2,
\]

\[
C_2 = C_0 (a + b)^3, \quad D_2 = C_0 (a - b)^3,
\]

have algebraic degree of precision \(2n - 2k - 4\).

**Proof.** The first quadrature formula (3.5) is the Bouzitat quadrature formula of the second kind [3, formula (4.8.1)], for the zeros of \(U_{n-1}^{(k)} = cP_{n-k-1}^{(k+\frac{1}{2}, k+\frac{1}{2})}\).

Setting \(\alpha = \beta = k - 1/2, m = n - k - 1\) in [3, formula (4.8.5)] we find \(A_0\) and \(s_i > 0\) (cf. [3, formula (4.8.4)]).

If in the above quadrature formula (3.6), we put

\[
f(x) = (1 - x)^n (1 + x)^{2k+1} P_{n-k-2}^{(k+\frac{1}{2}, k+\frac{1}{2})}(x),
\]

\[
U_{n-1}^{(k+1)}(x) = cP_{n-k-2}^{(k+\frac{1}{2}, k+\frac{1}{2})}(x),
\]

we obtain \(C_0\), and for

\[
f(x) = (1 + x)^n P_{n-k-2}^{(k+\frac{1}{2}, k+\frac{1}{2})}(x)
\]

we find \(B_0\).

If in formula (3.5) we replace \(f(x)\) with \(r(x) f(x)\) we get (3.7) and if in formula (3.6) we replace \(f(x)\) with \(r(x) f(x)\) we get (3.8).

**4. Proof of the Theorems**

**Proof of Theorem 2.1**

Setting \(k = 0\) in (3.5) we find the formula

\[
\int_{-1}^{1} \frac{f(x)}{\sqrt{1 - x^2}} \, dx = \frac{\pi}{2n} [f(-1) + f(1)] + \frac{\pi}{n} \sum_{i=1}^{n-1} f(y_i)
\]
According to this quadrature formula and using (3.1) and (3.2) we have
\[
\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} [p'_{n-1} (x)]^2 \, dx = \frac{\pi}{n} \left( \frac{n}{2} \right)^2 + \frac{\pi}{n} \left( \frac{n}{2} \right) \left( \frac{n}{2} + 1 \right) + \sum_{i=1}^{n-1} \left( p_{n-1} (y_i) \right)^2 \\
\leq \frac{\pi}{2n} \left( G'_{n-1} (1) \right)^2 + \frac{\pi}{2n} \sum_{i=1}^{n-1} \left( G'_{n-1} (y_i) \right)^2 \\
= \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \left[ G'_{n-1} (x) \right]^2 \, dx.
\]

Using the following formula (k = 0 in (3.6))
\[
\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} f(x) \, dx = \frac{\pi}{n} \left( \frac{n}{2} \right)^2 \left[ f(-1) + f(1) \right] + \frac{\pi}{4n} \sum_{i=1}^{n-1} \left( f'(y_i) \right) + \sum_{i=1}^{n-2} v_i f (y_i)
\]
we find
\[
\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \left[ U'_{n-1} (x) \right]^2 \, dx = \frac{2\pi n (n^2-1)}{15},
\]
and
\[
\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \left[ G'_{n-1} (x) \right]^2 \, dx = \frac{2\pi n (n+1)(n^2+1)-5\beta^2(n^2-n+1)}{15}.
\]

**Proof of Theorem 2.4**

According to the quadrature formula (3.7), positiveness of s_i’s, and using (3.3) and (3.4) we have
\[
\int_{-1}^{1} \left( a - bx \right)^3 (1-x^2)^{k-1/2} \left[ p_{n-1}^{(k+1)} (x) \right]^2 \, dx
\]
\[
= A_1 \left[ p_{n-1}^{(k+1)} (-1) \right]^2 + B_1 \left[ p_{n-1}^{(k+1)} (1) \right]^2 + \sum_{i=1}^{n-k-1} s_i \left[ y_i^{(k)} \right]^2 + \sum_{i=1}^{n} s_i r \left[ y_i^{(k)} \right]^2
\]
\[
\leq A_1 \left[ U_{n-1}^{(k+1)} (-1) \right]^2 + B_1 \left[ U_{n-1}^{(k+1)} (1) \right]^2 + \sum_{i=1}^{n-k-1} s_i \left[ y_i^{(k)} \right]^2 + \sum_{i=1}^{n} s_i r \left[ y_i^{(k)} \right]^2
\]
\[
= \int_{-1}^{1} \left( a - bx \right)^3 (1-x^2)^{k-1/2} \left[ U_{n-1}^{(k+1)} (x) \right]^2 \, dx
\]

In order to complete the proof we apply formula (3.8) to \( f = \left[ U_{n-1}^{(k+1)} (x) \right]^2 \).

Having in mind \( U_{n-1}^{(k+1)} \left( y_i^{(k+1)} \right) = 0 \) and the following relations deduced from [1]
\[
U_{n-1}^{(k+1)} (1) = \frac{n(n^2-1)^2}{1,5,8,13, \ldots, (2k+1)^2}, \quad U_{n-1}^{(k+2)} (1) = \frac{n^2-2nk+2}{2k-1} U_{n-1}^{(k+1)} (1),
\]
\[
U_{n-1}^{(k+1)} (-1) = -U_{n-1}^{(k+1)} (1) U_{n-1}^{(k+2)} (-1) = -U_{n-1}^{(k+1)} (1) U_{n-1}^{(k+2)} (1),
\]
we find
\[
\int_{-1}^{1} \left( a - bx \right)^3 (1-x^2)^{k-1/2} \left[ p_{n-1}^{(k+1)} (x) \right]^2 \, dx = C_1 \left[ U_{n-1}^{(k+1)} (-1) \right]^2 + D_1 \left[ U_{n-1}^{(k+1)} (1) \right]^2
\]
\[
+ 2C_2 U_{n-1}^{(k+1)} (-1) U_{n-1}^{(k+2)} (-1) - 2D_2 U_{n-1}^{(k+1)} (1) U_{n-1}^{(k+2)} (1)
\]
\[
= \frac{\pi n (n+k+1)!}{(n-k-2)!} \left[ 2 \left( n^2 - (k+2)^2 \right)^2 & (a^2 + 3b^2) \right] + \frac{2(k+1)^2 \pi^2 + 3b^2}{(2k+1)(2k+3)(2k+5)}.\]

**References**


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